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Stability with respect to partial variables for Birkhoff systems*

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In this paper, the stability with respect to partial variables for the Birkhoff system is studied. By transplanting the results of the partial stability for general systems to the Birkhoff system and constructing a suitable Liapunov function, the partial stability of the system can be achieved. Finally, two examples are given to illustrate the application of the results.

Keywords: Birkhoff system, partial stability, Liapunov function
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1. Introduction

The proposition on the stability with respect to partial variables was first raised by Liapunov, and elaborated by Malkin later. But its proof had not been given until Rumjantsev first proved the stability theorem and the asymptotic stability theorem with respect to partial variables.[1,2] The stability with respect to partial variables is from Liapunov stability, and its research scope has been extended.[1] The Birkhoff system is a kind of constrained mechanical system, and there have been some results of research on the stability of the system.[3−5] Recently, the conserved quantity and stability for the Birkhoff system were studied in Ref.[6], and the stability for the equilibrium of rotational relativistic Birkhoff systems was discussed in Ref.[7]. But the research on stability with respect to partial variables for the Birkhoff system has not been reported yet. In this paper, we study the stability with respect to partial variables for the Birkhoff system. It is clear that the results of the stability with respect to partial variables for general systems are applicable to the Birkhoff system. But it is very difficult to find a suitable Liapunov function for the Birkhoff system.

2. Birkhoff equations

The Birkhoff equations have the form

$$\Big(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\Big)\dot{a}^\nu-\frac{\partial B}{\partial a^\mu}-\frac{\partial R_\mu}{\partial t}=0\,,\quad(\mu,\nu=1,\ldots,2n)\,,\tag{1}$$

where the function $B=B(t,\boldsymbol{a})$ is called the Birkhoffian, and $R_\mu=R_\mu(t,\boldsymbol{a})$ are called Birkhoff functions. Now suppose that Eqs.(1) are nonsingular, i.e.

$$\det\Big(\frac{\partial R_\nu}{\partial a^\mu}-\frac{\partial R_\mu}{\partial a^\nu}\Big)\neq 0\,.\tag{2}$$

Then from Eqs.(1) all $\dot{a}^\mu$ can be solved as

$$\dot{a}^\mu=\omega^{\mu\nu}\Big(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}\Big)\,,\tag{3}$$

which can be written as

$$\dot{a}^\mu=\sigma_\mu(t,\boldsymbol{a})\,,\tag{4}$$

where

$$\sigma_\mu=\omega^{\mu\nu}\Big(\frac{\partial B}{\partial a^\nu}+\frac{\partial R_\nu}{\partial t}\Big).\tag{5}$$

3. Stability with respect to partial variables

Suppose that

$$\sigma_\mu(t,\boldsymbol{0})=0\,,\tag{6}$$

and then take the null solution $\boldsymbol{a}=\boldsymbol{0}$ of the system (4) as the undisturbed motion. In order to discuss

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the stability with respect to partial variables for the system (4), we take

$$\begin{aligned} &\boldsymbol{x} = (a^1, \ldots, a^m)^{\mathrm{T}}, \\ &\boldsymbol{y} = (a^{m+1}, \ldots, a^{2n})^{\mathrm{T}}, \end{aligned} \quad (m \leq 2n). \tag{7}$$

Definition 1 The undisturbed motion $\boldsymbol{a} = \boldsymbol{0}$ of the system (4) is said to be stable with respect to partial variables $\boldsymbol{x}$, if for any given positive number $\varepsilon > 0$ there exists such a number $\delta = \delta(t_0, \varepsilon)$ that

$$\|\boldsymbol{x}(t; t_0, \boldsymbol{a}_0)\| < \varepsilon, \quad \text{for all} \quad t \geq t_0,$$

when $\|\boldsymbol{a}_0\| < \delta$. Otherwise, the undisturbed motion $\boldsymbol{a} = \boldsymbol{0}$ of system (4) is said to be unstable with respect to partial variables $\boldsymbol{x}$.

Definition 2 The undisturbed motion $\boldsymbol{a} = \boldsymbol{0}$ of the system (4) is said to be asymptotically stable with respect to partial variables $\boldsymbol{x}$, if it is stable with respect to $\boldsymbol{x}$ and there exists such a number $\delta > 0$ that

$$\lim_{t \to \infty} \|\boldsymbol{x}(t; t_0, \boldsymbol{a}_0)\| = 0,$$

when $\|\boldsymbol{a}_0\| < \delta$.

Clearly, the results of the stability with respect to partial variables for general systems can be transplanted to the Birkhoff system. The corresponding results are embodied in the following two theorems.

Theorem 1 If there exists a positive (negative) definite function $V(t, \boldsymbol{a})$ about partial variables $\boldsymbol{x}$, and its derivative is $\dot{V} \leq 0 (\geq 0)$ along the curve of Eqs.(4) or identically vanishing, then the undisturbed motion $\boldsymbol{a} = \boldsymbol{0}$ of the system (4) is stable with respect to partial variables $\boldsymbol{x}$.

Theorem 2 If there exists a positive definite function $V(t, \boldsymbol{x})$ about partial variables $\boldsymbol{x}$, it has the infinitesimal upper bound and its derivative $\dot{V}$ along the curve of Eqs.(4) is negative definite, then the undisturbed motion $\boldsymbol{a} = \boldsymbol{0}$ of the system (4) is asymptotically stable with respect to partial variables $\boldsymbol{x}$.

The key issue and the difficulty in applying the above theorems consist in how to construct a suitable Liapunov function $V(t, \boldsymbol{a})$, and no general method of constructing the function. For an autonomous Birkhoff system, the Birkhoffian, if it is positive definite, can be taken as the Liapunov function in order to study the stability with respect to all variables.[5] For discussing the stability with respect to partial variables, the Birkhoffian is also useful for constructing the Liapunov function.

4. Illustrative examples

The following are two examples to illustrate the above results.

Example 1 The Birkhoffian and the Birkhoff functions of a four-dimensional Birkhoff system are respectively

$$\begin{aligned} &B = \frac{1}{2}\{(a^3)^2 + 2a^2a^3 - (a^4)^2\}, \\ &R_1 = a^2 + a^3, \quad R_2 = 0, \quad R_3 = a^4, \quad R_4 = 0. \end{aligned} \tag{8}$$

Birkhoff's equations (1) give

$$\begin{aligned} &-\dot{a}^2 - \dot{a}^3 = 0, \quad \dot{a}^1 - a^3 = 0, \\ &\dot{a}^1 - \dot{a}^4 - (a^2 + a^3) = 0, \quad \dot{a}^3 + a^4 = 0. \end{aligned} \tag{9}$$

This is the well-known Hojman–Urrutia problem.[8,9] Equations (8) can be expressed as

$$\dot{a}^1 = a^3, \quad \dot{a}^2 = a^4, \quad \dot{a}^3 = -a^4, \quad \dot{a}^4 = -a^2. \tag{10}$$

Now, we discuss the stability of its null solution $a^1 = a^2 = a^3 = a^4 = 0$ with respect to partial variables. Take

$$\begin{aligned} V(t, \boldsymbol{a}) = &\frac{1}{2}\{(a^2)^2 + (a^4)^2\} \\ &+ \frac{1}{2}\{(a^3)^2 + 2a^2a^3 - (a^4)^2\}^2 \\ &\geq \frac{1}{2}\{(a^3)^2 + (a^4)^2\}, \end{aligned} \tag{11}$$

which is positive definite with respect to variables a^2 and a^4. Taking the derivative $\dot{V}$ of the above expression according to Eqs.(9), we have

$$\dot{V} = 0. \tag{12}$$

By using Theorem 1, we know that the null solution of Eqs.(9) is stable with respect to partial variables a^2 and a^4.

Example 2 A Birkhoff system of second order is

$$\begin{aligned} &R_1 = \frac{1}{2}a^2\exp(\gamma t), \quad R_2 = \frac{1}{2}a^1\exp(\gamma t), \\ &B = \frac{1}{2}\{(a^1)^2 + (a^2)^2 + \gamma a^1a^2\}\exp(\gamma t), \\ &\quad (\gamma > 0). \end{aligned} \tag{13}$$

Birkhoff's equations (1) show

$$\begin{aligned} &-\dot{a}^2\exp(\gamma t) - (a^1 + \gamma a^2)\exp(\gamma t) = 0, \\ &\dot{a}^1\exp(\gamma t) - a^2\exp(\gamma t) = 0, \end{aligned}$$

i.e.

$$\dot{a}^1 = a^2, \qquad \dot{a}^2 = -a^1 - \gamma a^2. \tag{14}$$

We now study the stability of its null solution $a^1 = a^2 = 0$ with respect to partial variables. Take the Liapunov function as

$$V = \frac{1}{2}\{(a^1)^2 + (a^2)^2\}\,. \tag{15}$$

Clearly, it is positive definite with respect to variable a^2, does not contain t explicitly, and possesses the infinitesimal upper bound. Taking the derivative $\dot{V}$ of the above expression according to Eqs.(14), we have

$$\dot{V} = -\gamma(a^2)^2\,. \tag{16}$$

This is negative definite with respect to a^2. By Theorem 2, the null solution $a^1 = a^2 = 0$ is asymptotically stable with respect to a^2.

5. Summary

The motion stabilization of the Birkhoff system is an important part of the dynamics of the system. The stability with respect to partial variables is a kind of generalization of the general theory of Liapunov stability. The results in this paper are complementary to the study of stability for the Birkhoff system, and the key issue in its application is how to construct a suitable Liapunov function.

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