Vol 15 No 9, September 2006
1009-1963/2006/15(09)/1927-05

Chinese Physics

© 2006 Chin. Phys. Soc.
and IOP Publishing Ltd

Homotopic method of solving a class of EI Niño/La Niña-Southern Oscillation sea–air oscillator*

Mo Jia-Qi(莫嘉琪)[a)b)†], Wang Hui(王 辉)[c)], and Lin Wan-Tao(林万涛)[d)]

[a)] *Anhui Normal University, Wuhu* 241000, *China*
[b)] *Division of Computational Science, E-Institutes of Shanghai Universities, at Shanghai Jiaotong University, Shanghai* 200240, *China*
[c)] *Chinese Academy of Meteorological Sciences, Beijing* 100081, *China*
[d)] *LASG, Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing* 100029, *China*

(Received 17 February 2006; revised manuscript received 22 May 2006)

The EI Niño/La Niña-Southern Oscillation (ENSO) is an interannual phenomenon involved in the tropical Pacific Ocean–atmosphere interactions. In this paper, an asymptotic method of solving the nonlinear equation for the ENSO model is created. And based on a class of oscillator of the ENSO model, the approximate solution of a corresponding problem is studied by employing the method of homotopic mapping. It is proved from the results that the homotopic method can be used for analysing the sea surface temperature anomaly in the equatorial eastern Pacific and the thermocline depth anomaly of the atmosphere–ocean oscillation for the ENSO model.

Keywords: nonlinear, homotopic theory, El Niño/La Niño-Southern Oscillation model

PACC: 0230, 0200

1. Introduction

The El Niño/ La Niño-Southern Oscillation (ENSO) is an interannual phenomenon involved in the tropical Pacific Ocean–atmosphere interactions. It is a very attractive object of study in the international academic circles.[1−7] Lin *et al* also considered some problems of this class in atmospheric physics, ocean weather, dynamic system and so on.[8−13] The oscillatory nature of ENSO requires both positive and negative ocean–atmosphere feedbacks. An initial positive sea surface temperature (SST) anomaly in the equatorial eastern Pacific can reduce the zonal SST gradient, and hence the strength of the southern oscillation circulation, resulting in weaker trade winds around the equator. The weaker trade winds in turn bring about ocean circulation changes that reinforce the SST anomaly. This positive ocean–atmosphere feedback or coupled ocean–atmosphere instability in the equatorial Pacific leads to a never-ending warm state occurring.

Recently a large number of scholars have investigated the relevant nonlinear problems. Approximate methods have been developed and improved, including the mean method, boundary layer method, matched asymptotic expansion method, and multiple scale method.[14−18] Using the perturbation method, Mo *et al* considered a class of nonlinear perturbation problems.[19−28] In this paper we consider a class of ENSO model by using a simple and special method for homotopic theory.[29,30]

The conceptual oscillator model should consider the variations of both the eastern and western Pacific anomaly patterns. As will be shown below, the oscillator model can be reduced to a simple oscillator model by further making simplification and assumptions.

The western Pacific oscillator[4,5] emphasizes the role of the western Pacific anomaly patterns in ENSO. In particular, off-equatorial SST and SLP variations initiate equatorial wind anomalies in the western Pacific. These wind anomalies force the ocean to progress eastward to affect anomalies in the equatorial eastern Pacific. If all delay parameters are set to be zero,[6,7] we can understand why the slow ocean dynamical adjustment does not need the role of wave propagation.

*Project supported by the National Natural Science Foundation of China (Grant Nos 90111011 and 10471039), the State Key Program for Basics Research of China (Grant Nos 2003CB415101-03 and 2004CB418304), the Key Project of the Chinese Academy of Sciences (Grant No KZCX3-SW-221) and partly by E-Institutes of Shanghai Municipal Education Commission of China (Grant No N.E03004).
†E-mail: mojiaqi@mail.ahnu.edu.cn

2. An ENSO model

Now we consider the sea–air oscillator of an ENSO model, which may be described by the following equations:

$$\frac{\mathrm{d}T}{\mathrm{d}t} = a\tau_1 + b\tau_2 - \varepsilon_1 T^3, \tag{1}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -c\tau_1 - R_\mathrm{h} h, \tag{2}$$

$$\varepsilon_2 \frac{\mathrm{d}\tau_1}{\mathrm{d}t} = dT - R_{\tau 1}\tau_1, \tag{3}$$

$$\varepsilon_3 \frac{\mathrm{d}\tau_2}{\mathrm{d}t} = eh - R_{\tau 2}\tau_2, \tag{4}$$

where T is the SST anomaly in the equatorial eastern Pacific; h is the thermocline depth anomaly; τ_i, where $i = 1, 2$, are the equatorial zonal wind stress anomalies in the central Pacific and the western Pacific, respectively; the model parameters a, b, c, d, e and ε_i, where $i = 1, 2$, are positive constants; and the R_h, $R_{\tau 1}$, $R_{\tau 2}$ and ε_1 are positive damping coefficients.

It is easy to obtain the solution $(\bar{T}(t), \bar{h}(t))$, $\bar{\tau}_1(t), \bar{\tau}_2(t))$ of the linear system

$$\frac{\mathrm{d}T}{\mathrm{d}t} = a\tau_1 + b\tau_2, \tag{5}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -c\tau_1 - R_\mathrm{h} h, \tag{6}$$

$$\varepsilon_2 \frac{\mathrm{d}\tau_1}{\mathrm{d}t} = dT - R_{\tau 1}\tau_1, \tag{7}$$

$$\varepsilon_3 \frac{\mathrm{d}\tau_2}{\mathrm{d}t} = eh - R_{\tau 2}\tau_2, \tag{8}$$

as

$$\bar{T}(t) = \sum_{i=1}^{4} K_i A_i \exp(\lambda_i t), \tag{9}$$

$$\bar{h}(t) = \sum_{i=1}^{4} K_i B_i \exp(\lambda_i t), \tag{10}$$

$$\bar{\tau}_1(t) = \sum_{i=1}^{4} K_i C_i \exp(\lambda_i t), \tag{11}$$

$$\bar{\tau}_2(t) = \sum_{i=1}^{4} K_i D_i \exp(\lambda_i t). \tag{12}$$

Here λ_i, where $i = 1, 2, 3, 4$, are characteristic roots of the linear system (5)–(8), K_i, where $i = 1, 2, 3, 4$, are arbitrary constants, and A_i, B_i, C_i, D_i, where $i = 1, 2, 3, 4$, satisfy the following linear system:

$$aC_i + bD_i = 0, \tag{13}$$

$$(R_\mathrm{h} + \lambda_i)B_i + cC_i = 0, \tag{14}$$

$$dA_i - (R_{\tau 1} + \lambda_i)C_i = 0, \tag{15}$$

$$eB_i - (R_{\tau 2} + \lambda_i D_i) = 0. \tag{16}$$

3. Homotopic mapping and approximate solution

Introducing a set of[29,30] $H_i(T, h, p)$ with $X \times I \to R$, where $i = 1, 2, 3, 4$, we have

$$H_1(T, h, p) = L_1(T, h) - L_1(u_0, v_0, w_0, z_0) + p(L_1(u_0, v_0, w_0, z_0) + \varepsilon T^3), \tag{17}$$

$$H_i(T, h, \tau_1, \tau_2, p) = L_i(T, H, \tau_1, \tau_2,) - L_i(u_0, v_0, w_0, z_0) + pL_i(u_0, v_0, w_0, z_0), \quad \text{where} \quad i = 1, 2, 3, 4. \tag{18}$$

Here $X = [0, \infty), i = [0, 1], R = (-\infty, +\infty)$; $(u_0, v_0, w_0, z_0)]$ is a zeroth-order approximate solution of the original coupled system (1)–(4), and L_i where $i = 1, 2, 3, 4$, are linear operators satisfying the following equations:

$$L_1(T, h, \tau_1, \tau_2) = \frac{\mathrm{d}T}{\mathrm{d}t} - a\tau_1 - b\tau_2, \quad L_2(T, h, \tau_1, \tau_2) = \frac{\mathrm{d}h}{\mathrm{d}t} + c\tau_1 + R_\mathrm{h} h,$$

$$L_3(T, h, \tau_1, \tau_2) = \varepsilon_2 \frac{\mathrm{d}\tau_1}{\mathrm{d}t} - dT - R_{\tau 1}\tau_1, \quad L_4(T, h, \tau_1, \tau_2) = \varepsilon_3 \frac{\mathrm{d}\tau_2}{\mathrm{d}t} - eh + R_{\tau 2}\tau_2,$$

Firstly, from Eqs.(9)–(12), we can obtain

$$u_0(t) = \sum_{i=1}^{4} K_i A_i \exp(\lambda_i t), \tag{19}$$

$$v_0(t) = \sum_{i=1}^{4} K_i B_i \exp(\lambda_i t), \tag{20}$$

$$w_0(t) = \sum_{i=1}^{4} K_i C_i \exp(\lambda_i t), \tag{21}$$

$$z_0(t) = \sum_{i=1}^{4} K_i D_i \exp(\lambda_i t). \tag{22}$$

Obviously, from Eqs.(17) and (18), the solution

$(T(t), h(t), \tau_1(t), \tau_2(t))$ of coupled system (1)–(4) is the solution of the $H_j(T, h, \tau_1, \tau_2, 1) = 0$ where $j = 1, 2, 3, 4$, as $p \to 1$.

Let

$$T = \sum_{k=0}^{\infty} T_k(t)p^k, \quad h = \sum_{k=0}^{\infty} h_k(t)p^k,$$
$$\tau_1 = \sum_{k=0}^{\infty} \tau_{1k}(t)p^k, \quad \tau_2 = \sum_{k=0}^{\infty} \tau_{2k}(t)p^k. \quad (23)$$

By expanding the expressions (19)–(22) into the power series of p, substituting them into the expressions (17) and (18), and then comparing the coefficients of the same power in p on the right- and left-hand sides of the resulting equations $H_1(T, h, p) = 0$ and $H_i(T, h, \tau_1, \tau_2, p) = 0$, where $i = 2, 3, 4$, we can finally obtain the corresponding solutions as shown below.

From the coefficients of zeroth power in p, we have

$$L_i(T_0, h_0, \tau_1, \tau_2) = L_i(u_0, v_0, w_0, z_0), \quad i = 1, 2, 3, 4.$$

Thus we obtain

$$T_0(t) = \sum_{i=1}^{4} K_i A_i \exp(\lambda_i t),$$
$$h_0(t) = \sum_{i=1}^{4} K_i B_i \exp(\lambda_i t),$$
$$\tau_{10}(t) = \sum_{i=1}^{4} K_i C_i \exp(\lambda_i t),$$
$$\tau_{20}(t) = \sum_{i=1}^{4} K_i D_i \exp(\lambda_i t).$$

From the coefficients of first power in p, we have

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} - a\tau_{11} - b\tau_{21} = -\varepsilon \left(\sum_{i=1}^{4} K_i A_i \exp(\lambda_i t) \right)^3, \quad (24)$$
$$\frac{\mathrm{d}h_1}{\mathrm{d}t} + c\tau_{11} + R_{\mathrm{h}} h_1 = 0, \quad (25)$$
$$\varepsilon_2 \frac{\mathrm{d}\tau_{11}}{\mathrm{d}t} - dT_1 + R_{\tau 1}\tau_{11} = 0, \quad (26)$$
$$\varepsilon_3 \frac{\mathrm{d}\tau_{21}}{\mathrm{d}t} - eh_1 + R_{\tau 2}\tau_{21} = 0. \quad (27)$$

From Eqs.(24)–(27), it is easy to obtain $(T_1(t), h_1(t), \tau_{11}, \tau_{21})$. Their derivation is tedious and is omitted here.

Then we can obtain the first-order approximate solution of the coupled system (1)–(4):

$$T_{\mathrm{app}}(t) = \sum_{i=1}^{4} K_i A_i \exp(\lambda_i t) + T_1(t), \quad h_{\mathrm{app}}(t) = \sum_{i=1}^{4} K_i B_i \exp(\lambda_i t) + h_1(t), \quad (28)$$
$$\tau_{1\mathrm{app}}(t) = \sum_{i=1}^{4} K_i C_i \exp(\lambda_i t) + \tau_{11}(t), \quad \tau_{2\mathrm{app}}(t) = \sum_{i=1}^{4} K_i D_i \exp(\lambda_i t) + \tau_{21}. \quad (29)$$

Here λ_i, where $i = 1, 2, 3, 4$, are characteristic roots of the linear system (5)–(8), K_i where $i = 1, 2, 3, 4$, are arbitrary constants, and A_i, B_i, C_i, D_i, where $i = 1, 2, 3, 4$, satisfy the linear system (13)–(16) and $(T_1(t), h_1(t), \tau_{11}, \tau_{21})$ is a solution of Eqs.(24)–(27).

Similarly, we can obtain a higher-order approximate solution for the coupled system (1)–(4). It can be seen that it is easy to obtain a solution $(T_j(t), h_j(t), \tau_{1j}(t), \tau_{2j}(t))$ where $j = 1, 2, 3, \ldots$, successively.

4. Accuracy of approximate solution

Substituting the solutions $(T_j(t), h_j(t), \tau_{1j}(t), \tau_{2j}(t))$ where $j = 0, 1, 2, \ldots$, into Eq.(23), and letting $p = 1$, we have

$$\widetilde{T}=\sum_{k=0}^{\infty}T_k(t),\quad \widetilde{h}=\sum_{k=0}^{\infty}h_k(t),\quad \widetilde{\tau_1}=\sum_{k=0}^{\infty}\tau_{1k}(t),\quad \widetilde{\tau_2}=\sum_{k=0}^{\infty}\tau_{2k}(t). \tag{30}$$

We can also prove that the expressions (30) are uniformly convergent in $R\times I$. And then $(\widetilde{T},\widetilde{h},\widetilde{\tau_1},\widetilde{\tau_2})$ is a solution for the nonlinear problem (1)–(4).[27] From the expressions (30), we can obtain an approximate solution for the original reaction diffusion problem.

In order to explain the accuracy of the expressions of the first approximate solution (28) and (29), we now select some particular parameters of the model:[5] $a=1.5\times10^{-2}$ °C·m^2·N^{-1}·yr^{-1}, $b=7.5\times10^{2}$ °C·m^2·N^{-1}·yr^{-1}, $c=1.5\times10^{3}$ m^3·N^{-1}·yr^{-1}, $d=3.6\times10^{-2}$ °C^{-1}·N·m^{-2}yr^{-1}, $e=3.0\times10^{-3}$N·m^{-3}·yr^{-1}, $R_{\rm h}=5.0{\rm yr}^{-1}$, $R_{\tau1}=R_{\tau2}=2.0{\rm yr}^{-1}$ and let $\varepsilon_1=0.1$ °C^{-2}·yr^{-1}, $t\in[0,4]$, $[T,h,\tau_1,\tau_2]=[1,0,0,0]$. Then we have the curves of the SST anomaly T as shown in Fig.1 and those of the thermocline depth anomaly h in Fig.2 to make a comparison between the simulation solution and the homotopic approximate solution.

From Fig.1 and Fig.2, we can see that the first approximate solution obtained using the homotopic method has a good accuracy.

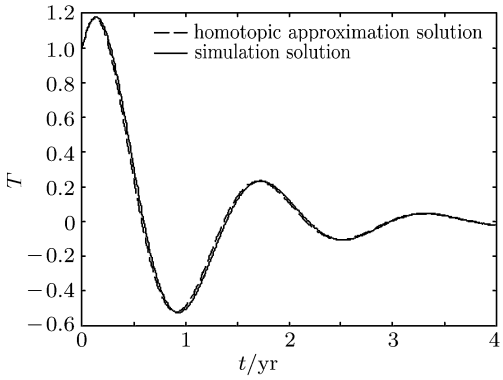


Fig.1. Comparison between simulation solution and homotopic approximate solution.

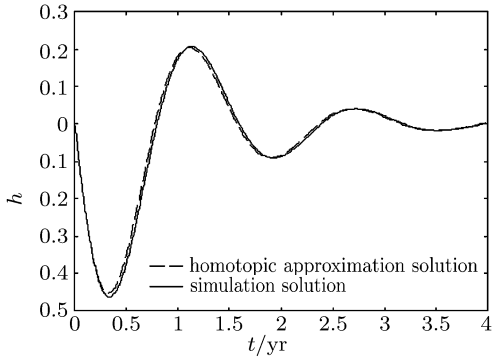


Fig.2. Comparison between simulation solution and homotopic approximate solution.

References

[1] Feng G L, Dong W J, Jia X J and Cao H X 2002 *Acta Phys. Sin.* **51** 1181 (in Chinese)

[2] Liu S K, Fu Z T, Liu S D and Zhao Q 2002 *Acta Phys. Sin.* **51** 10 (in Chinese)

[3] Feng G L, Dai X G, Wang A H and Chou J F 2001 *Acta Phys. Sin.* **50** 606 (in Chinese)

[4] Wang C, Weisberg R H and Virmani J I 1999 *J. Geophys. Res.* **104** 5131

[5] Wang C 2001 *Adv. Atmos. Sci.* **18** 674

[6] Jin F F 1997 *J. Atmos. Sci.* **54** 811

[7] Jin F F 1997 *J. Atmos. Sci.* **54** 830

[8] Lin W T, Ji Z Z, Wang B and Yang X Z 2000 *Chin. Sci. Bulletin* **45** 1358

[9] Lin W T, Ji Z Z and Wang B 2001 *Acta Aerodynamica Sin.* **19** 348 (in Chinese)

[10] Lin W T, Ji Z Z and Wang B 2001 *Adv. Atmos. Sci.* **18** 604

[11] Lin W T, Ji Z Z and Wang B 2002 *Adv. Atmos. Sci.* **19** 699

[12] Lin W T, Ji Z Z, Wang B and Zhang X 2002 *Progr. Nat. Sci.* **12** 1326 (in Chinese)

[13] Lin W T and Mo J Q 2004 *Chin. Sci. Bulletin* **48** II: 5

[14] de Jager E M and Jiang F R 1996 *Theory of Singular Perturbation* (Amsterdam: North-Holland Publishing Co.)

[15] Kelley W G 2001 *J. Math. Anal. Appl.* **255** 678

[16] Hamouda M 2002 *Applicable Anal.* **81** 837

[17] Bell D C and Deng B 2003 *Nonlinear Anal. Real World Appl.* **3** 515

[18] Adams K L, King J R and Tew R H 2003 *J. Engin. Math.* **45** 197

[19] Mo J Q, Zhu J and Wang H 2003 *Progress in Natural Sci.* **13** 768

[20] Mo J Q and Zhu J 2003 *Appl. Math. Mech.* **24** 527

[21] Mo J Q, Lin W T and Zhu J 2004 *Progr. Nat. Sci.* **14** 550

[22] Mo J Q and Lin W T 2004 *Acta Phys. Sin.* **53** 996 (in Chinese)
[23] Mo J Q, Lin W T and Zhu J 2004 *Acta Phys. Sin.* **53** 3245 (in Chinese)
[24] Mo J Q and Lin W T 2005 *Acta Phys. Sin.* **54** 993 (in Chinese)
[25] Mo J Q and Lin W T 2005 *Acta Phys. Sin.* **54** 1081 (in Chinese)
[26] Mo J Q, Lin W T and Wang H 2005 *Acta Phys. Sin.* **54** 3967 (in Chinese)
[27] Mo J Q, Lin Y H and Lin W T 2005 *Acta Phys. Sin.* **54** 3971 (in Chinese)
[28] Mo J Q and Lin W T 2005 *Chin. Phys.* **14** 875
[29] He J H 2002 *Approximate Analytical Methods in Engineering and Sciences* (Shengzhou: Science and Technology Publisher)
[30] Liao S J 2004 *Beyond Perturbation: Introduction to the Homotopy Analysis Method* (New York: CRC Press Co.)