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The quadratic-form identity for constructing Hamiltonian structures of the Guo hierarchy

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The trace identity is extended to the quadratic-form identity. The Hamiltonian structures of the multi-component Guo hierarchy, integrable coupling of Guo hierarchy and (2+1)-dimensional Guo hierarchy are obtained by the quadratic-form identity. The method can be used to produce the Hamiltonian structures of the other integrable couplings or multi-component hierarchies.

Keywords: Hamiltonian structure, Guo's hierarchy, quadratic-form identity
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1. Introduction

The theory of integrable Hamiltonian systems with infinite dimensions has rapidly developed since the late 1960s. In 1988, Tu[1] proposed an efficient approach to searching for an integrable Hamiltonian hierarchy. Nowadays, using the Tu scheme, a lot of integrable systems with physical significance have been obtained.[2−5] At present, a large number of multi-component integrable systems such as the multi-component TD hierarchy, Burgers hierarchy, etc., have been presented.[6−10] However, the Hamiltonian structures of these multi-component integrable systems have not been deeply investigated because the trace identity is not suitable for them. In this paper, we extended the trace identity to the quadratic-form identity. By using this identity, the Hamiltonian structures of the multi-component Guo hierarchy, integrable coupling of Guo hierarchy and (2+1)-dimensional Guo hierarchy are obtained. The method proposed in this paper can also be used to solve some other problems of this kind.

Set V_s to be an s-dimensional linear space with basis $e_1, e_2, \ldots, e_s$, and let

$$a = \sum_{i=1}^{s} a_i e_i = (a_1, a_2, \ldots, a_s)^{\mathrm{T}},$$

$$b = \sum_{i=1}^{s} b_i e_i = (b_1, b_2, \ldots, b_s)^{\mathrm{T}}$$

be two elements of V_s and define a commutation operation as

$$\begin{aligned}[a, b] &= \sum_{i,j=1}^{s} a_i b_j [e_i, e_j] \\ &= \sum_{i=1}^{s} c_i e_i = c = (c_1, c_2, \ldots, c_s)^{\mathrm{T}}, \end{aligned} \tag{1}$$

which makes V_s become a Lie algebra.

A corresponding loop algebra $\tilde{V}_s$ is presented respectively with the following basis and commutation operation:

$$\begin{aligned} &e_i(m) = e_i \lambda^m, \qquad [e_i(m), e_j(n)] = [e_i, e_j]\lambda^{m+n}, \\ &i, j = 1, 2, \ldots, s; \quad m, n = 0, \pm 1, \pm 2, \ldots. \end{aligned} \tag{2}$$

The linear isospectral problem that is constructed by $\tilde{V}_s$ can be taken as

$$\begin{cases} \psi_x = [U, \psi], & U, V, \psi \in \tilde{V}_s, \\ \psi_t = [V, \psi], & \lambda_t = 0, \end{cases} \tag{3}$$

from which, the zero-curvature equation

$$V_x = [U, V] \tag{4}$$

is easy to derive, and we should give the following definitions:

$$\begin{aligned} &\mathrm{rank}(U) = \mathrm{rank}\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big) = \alpha = \text{const.}, \\ &\mathrm{rank}(V) = \mathrm{rank}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big) = \xi = \text{const.} \end{aligned} \tag{5}$$

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Let the two arbitrary solutions V_1 and V_2 of Eq.(4) with the same rank be linearly related by

$$V_1 = \gamma V_2\,, \qquad \gamma = \text{const.} \tag{6}$$

For $a, b \in \tilde{V}_s$, the s-order matrix $R(b)$ is determined by

$$[a,b]^{\mathrm{T}} = a^{\mathrm{T}} R(b)\,, \tag{7}$$

and constant matrix $F = (f_{ij})_{s\times s}$ is determined by

$$F = F^{\mathrm{T}}\,, \qquad R(b)F = -(R(b)F)^{\mathrm{T}}\,. \tag{8}$$

We introduce a quadratic-form identity functional

$$\{a,b\} = a^{\mathrm{T}} F b\,, \qquad a,b \in \tilde{V}_s\,, \tag{9}$$

and consider a functional

$$W = \{V, U_\lambda\} + \{\Lambda, V_x - [U,V]\}\,, \quad U,V,\Lambda \in \tilde{V}_s, \tag{10}$$

which has the following variation constraint conditions:

$$\nabla_\Lambda W = V_x - [U,V] = 0\,, \tag{11}$$

$$\nabla_V W = U_\lambda - \Lambda_x + [U,\Lambda] = 0\,. \tag{12}$$

From the expressions (8), (9) and (7), we obtain

$$[\Lambda, V] - V_\lambda = \frac{\gamma}{\lambda} V\,, \quad \gamma = \text{const.}\,, \tag{13}$$

then we obtain

$$\frac{\delta}{\delta u_i}\{V, U_\lambda\} = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\Big(\lambda^r\Big\{V, \frac{\partial U}{\partial u_i}\Big\}\Big)\,, \quad (1 \le i \le l)\,, \tag{14}$$

where γ is a constant to be determined. We call Eq.(14) the quadratic-form identity.

2. Multi-component Guo hierarchy and its Hamiltonian structure

If $G_M = \{a = (a_{ij})_{M\times 3} = (a_1, a_2, a_3)\}$ denotes a set of matrices where M is a positive integer, and $a_i (i = 1,2,3)$ is the ith column of the matrix a, then G_M is a linear space.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_M)^{\mathrm{T}}$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_M)^{\mathrm{T}}$, and define their products as $\alpha * \beta = \beta * \alpha = (\alpha_1\beta_1, \alpha_2\beta_2, \ldots, \alpha_M\beta_M)^{\mathrm{T}}$ and $\alpha/\beta = (\alpha_1/\beta_1, \alpha_2/\beta_2, \ldots, \alpha_M/\beta_M)^{\mathrm{T}}$. If $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in G_M$, then a commutation operation for G_M is defined as

$$[a,b] = (a_3 * b_2 - a_2 * b_3, a_1 * b_3 - a_3 * b_1, a_1 * b_2 - a_2 * b_1)^{\mathrm{T}}. \tag{15}$$

It is easy to see that G_M is a Lie algebra. A corresponding Lie algebra $\tilde{G}_M$ is defined as

$$\tilde{G} = \{a\lambda^n, a \in G_M, [a\lambda^m, b\lambda^n] = [a,b]\lambda^{m+n}, a,b \in G_M\}\,. \tag{16}$$

Taking the following isospectral problem:

$$\psi_x = [U,\psi], U, V, \psi \in \tilde{G}_M\,, \qquad \psi_t = [V,\psi], \lambda_t = 0\,, \tag{17}$$

where

$$U = \Big(\frac{q}{2}, \frac{r}{2\lambda}, \frac{\lambda}{2} I_M\Big)^{\mathrm{T}}\,,$$

$$V = (v_1, \lambda v_2, \lambda v_3)^{\mathrm{T}}\,,$$

$$v_k = \sum_{m\ge 0} (v_{k_1,m}, v_{k_2,m}, \ldots, v_{k_M,m})^{\mathrm{T}} \lambda^{-2m}\,,$$

$$q = (q_1, q_2, \ldots, q_M)^{\mathrm{T}}\,,$$

and $r = (r_1, r_2, \ldots, r_M)^{\mathrm{T}}$, solving

$$V_x = [U,V] \tag{18}$$

gives rise to

$$\begin{cases} v_{2,m+1} = v_{1,mx} + r * v_{3,m}\,, \\ v_{2,mx} = -v_{1,m} + q * v_{3,m}\,, \\ v_{3,m+1x} = q * (v_{1,mx} + r * v_{3,m}) - r * v_{1,m}\,, \\ v_{2,0} = 0\,, \quad v_{3,0} = \alpha\,, \quad v_{1,0} = \alpha q\,, \quad v_{1,1} = -\alpha q_{xx} - \alpha r_x + \dfrac{\alpha}{2} q * q * q\,, \\ v_{2,1} = \alpha(q_x + r)\,, \quad v_{3,1} = \dfrac{\alpha}{2} q * q\,. \end{cases} \tag{19}$$

For an arbitrary natural number n, denoting

$$V_+^{(n)} = \sum_{m=0}^{n} \begin{pmatrix} v_{1,m} \\ \lambda v_{2,m} \\ \lambda v_{3,m} \end{pmatrix} \lambda^{2n-2m},$$

$$V_-^{(n)} = \lambda^n V - V_+^{(n)},$$

then Eq.(18) can be written as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}]. \tag{20}$$

The terms on the left-hand side of Eq.(20) are of degree ≥ -1, while the terms on the right-hand side are of degree ≤ 0. Therefore,

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = -\begin{pmatrix} v_{2,n+1} \\ 0 \\ \dfrac{r * v_{1,n}}{\lambda^2} \end{pmatrix}.$$

Denoting

$$V^{(n)} = V_+^{(n)} + \Delta_n, \quad \Delta_n = (0, \frac{r}{\lambda^2 q} * v_{1,n}, 0)^{\mathrm{T}},$$

the following zero-curvature equation:

$$U_t - V_x^{(n)} + [U, V^{(n)}] = 0 \tag{21}$$

gives the integrable system

$$\begin{aligned} u_t = \begin{pmatrix} q \\ r \end{pmatrix}_t &= \begin{pmatrix} \left(\dfrac{r}{q} * v_{1,n}\right)_x \\ v_{1,nx} - \dfrac{r}{q} v_{2,nx} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\partial \dfrac{r}{q} * \\ -\dfrac{r}{q} * \partial & -\partial \end{pmatrix} \begin{pmatrix} v_{2,n} \\ -v_{1,n} \end{pmatrix} \\ &= J \begin{pmatrix} v_{2,n} \\ -v_{1,n} \end{pmatrix}, \end{aligned} \tag{22}$$

where J is a Hamiltonian operator.

From the expression (19),we obtain a recurrence operator

$$L = \begin{pmatrix} -\dfrac{r}{q} * \partial & -\partial - \dfrac{r}{q} * \\ \partial \dfrac{r}{q} * \partial + q * \partial^{-1} r * \partial & \partial^2 + \partial \dfrac{r}{q} * + q * \partial^{-1} q * \partial \end{pmatrix},$$

which meets

$$\begin{pmatrix} v_{2,n+1} \\ -v_{1,n+1} \end{pmatrix} = L \begin{pmatrix} v_{2,n} \\ -v_{1,n} \end{pmatrix}.$$

Therefore, the expression (22) can be written as

$$u_t = \begin{pmatrix} r \\ q \end{pmatrix}_t = JL^n \begin{pmatrix} 0 \\ -\alpha q \end{pmatrix}. \tag{23}$$

When $M > 1$, the hierarchy (21) is a multi-component Guo hierarchy; when $M = 1$, the hierarchy (21) reduces to the Guo system.

From the expression (15), we have

$$[a, b]^{\mathrm{T}} = a^{\mathrm{T}} \begin{pmatrix} 0 & b_3 & b_2 \\ -b_3 & 0 & -b_1 \\ b_2 & -b_1 & 0 \end{pmatrix} = a^{\mathrm{T}} R(b). \tag{24}$$

Solving the matrix equation (8) for F yields

$$F = \begin{pmatrix} -I_M & 0 & 0 \\ 0 & -I_M & 0 \\ 0 & 0 & I_M \end{pmatrix}, \tag{25}$$

$$\{a, b\} = -a_1 * b_1 - a_2 * b_2 + a_3 * b_3. \tag{26}$$

Hence, we obtain by using the quadratic-form identity

$$\frac{\delta}{\delta u_i}\left(\frac{\lambda}{2} v_3 - \frac{r}{2\lambda} * v_2\right) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \begin{pmatrix} \dfrac{1}{2} v_2 \\ -\dfrac{1}{2} v_1 \end{pmatrix}. \tag{27}$$

Comparing the coefficients of λ^{-2n+1} yields

$$\frac{\delta}{\delta u}(v_{3,n+1} - r * v_{2,n}) = (\gamma - 2n) \begin{pmatrix} v_{2,n} \\ -v_{1,n} \end{pmatrix}.$$

Taking $n = 1$ in the above equation gives $\gamma = -1$. Thus, the Hamiltonian structure of the expression (22) is derived as follows:

$$u_t = \begin{pmatrix} r \\ q \end{pmatrix}_t = J \begin{pmatrix} \dfrac{\delta}{\delta r} \\ \dfrac{\delta}{\delta q} \end{pmatrix} H_n = J \frac{\delta H_n}{\delta u}, \tag{28}$$

where

$$H_n = -\frac{1}{2n+1}(r * v_{2,n+1} - v_{3,n+1}).$$

Therefore, the expression (22) can be written as

$$\begin{aligned} u_t = \begin{pmatrix} r \\ q \end{pmatrix}_t &= JL^n \begin{pmatrix} 0 \\ -\alpha q \end{pmatrix} \\ &= J \begin{pmatrix} \dfrac{\delta}{\delta r} \\ \dfrac{\delta}{\delta q} \end{pmatrix} H_n. \end{aligned} \tag{29}$$

We observe that $JL = L^* J$, Hamiltonian functions $H_l (l \geq 1)$ are involutive with each other and each $H_l (l \geq 1)$ is a common conserved density of the expression (22). Therefore, the multi-component Guo hierarchy (22) is Liouville integrable.

3. Integrable coupling of multi-component Guo hierarchy and its Hamiltonian structure

Consider a linear space

$$G_6 = \{a = (a_1, a_2, a_3, a_4, a_5, a_6)^{\mathrm{T}}\}$$

with the commuting relations

$$\begin{aligned}[a,b] = (&a_3 * b_2 - a_2 * b_3, a_1 * b_3 - a_3 * b_2, a_1 * b_2 - a_2 * b_1, a_3 * b_5 - a_5 * b_3 + a_6 * b_2 \\ &- a_6 * b_2, a_1 * b_6 - a_6 * b_1 + a_4 * b_3 - a_3 * b_4, a_1 * b_5 - a_5 * b_1 + a_4 * b_2 - a_2 * b_4)^{\mathrm{T}} . \end{aligned} \tag{30}$$

It is easy to verify that G_6 is a Lie algebra, which leads to the resulting loop algebra

$$\tilde{G}_6 = \{a(\lambda) = a\lambda^m, a \in G_6, [a\lambda^m, b\lambda^n] = [a,b]\lambda^{2m+2n}, a, b \in G_6\} . \tag{31}$$

Take the following isospectral problem:

$$\begin{cases} \psi_x = [U, \psi] , & U, V, \psi \in \tilde{G}_5 , \\ \psi_t = [V, \psi] , & \lambda_t = 0 . \end{cases} \tag{32}$$

According to the definition we can have the following expressions:

$$U = \Big(\frac{u_2}{2}, \frac{u_1}{2\lambda}, \frac{\lambda}{2} I_M, 0, \frac{u_3}{\lambda}, \frac{u_4}{\lambda}\Big)^{\mathrm{T}} , \qquad V = (v_1, \lambda v_2, \lambda v_3, v_4, \lambda v_5, \lambda v_6)^{\mathrm{T}} ,$$

$v_k = \sum\limits_{m \geq 0} (v_{k_1,m}, v_{k_2,m}, \ldots, v_{k_M,m})^{\mathrm{T}} \lambda^{-2m}$, and $u_i = (q_{i1}, q_{i2}, \ldots, q_{iM})^{\mathrm{T}}$, $(i = 1,2,3,4, k = 1,2,3,4,5,6)$.

Solving

$$V_x = [U, V] \tag{33}$$

gives rise to

$$\begin{cases} v_{2,m+1} = v_{1,mx} + u_1 * v_{3,m} , \\ v_{2,mx} = -v_{1,m} + u_2 * v_{3,m} , \\ v_{3,m+1x} = u_2 * (v_{1,mx} + u_1 * v_{3,m}) - u_1 * v_{1,m} , \\ v_{5,m+1} = v_{4,mx} + u_1 * (v_{5,m} + u_3 * v_{3,m}) - u_4 * v_{2,m} , \\ v_{4,m+1} = -(v_{5,m+1x} + u_2 * v_{6,m+1}) - u_4 * v_{1,m} , \\ v_{6,m+1x} = -u_1 * (v_{4,m} + u_2 * v_{5,m+1}) - u_3 * v_{1,m} , \\ v_{2,0} = v_{4,0} = v_{5,0} = v_{6,0} = 0 , \quad v_{3,0} = \alpha , \quad v_{1,0} = \alpha r, \\ v_{1,1} = -\alpha u_{2xx} - \alpha u_{1x} + \frac{\alpha}{2} u_1 * u_1 * u_1 , \quad v_{2,1} = \alpha(u_{2x} + u_1) , \\ v_{3,1} = \dfrac{\alpha}{2} u_2 * u_2 , \quad v_{6,1} = \alpha u_3, \quad v_{5,1} = 0 , \quad v_{4,1} = -\alpha u_2 * u_4 . \end{cases} \tag{34}$$

And for an arbitrary natural number n, denoting

$$V_+^{(n)} = \sum_{m=0}^{n} \begin{pmatrix} v_{1,m} \\ \lambda v_{2,m} \\ \lambda v_{3,m} \\ v_{4,m} \\ \lambda v_{5,m} \\ \lambda v_{6,m} \end{pmatrix} \lambda^{2n-2m} ,$$

$$V_-^{(n)} = \lambda^n V - V_+^{(n)} ,$$

then Eq.(18) can be written as

$$-V_{+x}^{(n)} + [U, V_+^{(n)}] = V_{-x}^{(n)} - [U, V_-^{(n)}] , \tag{35}$$

where the terms on the left-hand side of Eq.(35) are of degree ≥ -1, while the terms on the right-hand side

are of degree ≤ 0. Therefore,

$$-V_{+x}^{(n)}+[U,V_{+}^{(n)}]=\begin{pmatrix} -v_{2,n+1} \\ 0 \\ -\dfrac{u_1*v_{1,n}}{\lambda^2}-v_{5,n+1} \\ \dfrac{v_{5,n+1x}+v_{4,n+1}-u_2*v_{6,n+1}}{\lambda^2} \\ \dfrac{v_{6,n+1x}-u_2*v_{5,n+1}}{\lambda^2} \end{pmatrix}.$$

Denoting $V^{(n)}=V_{+}^{(n)}+\Delta_n$, and

$$\Delta_n=\Big(0,\frac{u_1*v_{1,n}}{\lambda u_2},0,0,\frac{v_{5,n+1}}{\lambda},\frac{u_4}{\lambda u_2}*v_{1,n}\Big)^{\mathrm{T}},$$

the following zero-curvature equation:

$$U_t-V_x^{(n)}+[U,V^{(n)}]=0 \tag{36}$$

gives an integrable system

$$u_t=\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t=\begin{pmatrix} \Big(\dfrac{u_1}{u_2}*v_{1,n}\Big)_x \\ v_{1,nx}-\dfrac{u_1}{u_2}*v_{2,nx} \\ v_{5,n+1x} \\ \Big(\dfrac{u_4}{u_2}*v_{1,n}\Big)_x-v_{6,n+1x} \end{pmatrix}=\begin{pmatrix} 0 & -\partial\dfrac{u_1}{u_2}* & 0 & 0 \\ -\dfrac{u_1}{u_2}\partial & -\partial & 0 & 0 \\ 0 & 0 & \partial & 0 \\ 0 & -\partial\dfrac{u_4}{u_2}* & 0 & -1 \end{pmatrix}\begin{pmatrix} v_{2,n} \\ -v_{1,n} \\ v_{5,n+1} \\ v_{6,n+1} \end{pmatrix}=J\begin{pmatrix} v_{2,n} \\ -v_{1,n} \\ v_{5,n+1} \\ v_{6,n+1} \end{pmatrix}. \tag{37}$$

From the expression (34), we obtain a recurrence operator

$$L=\begin{pmatrix} -\dfrac{u_1}{u_2}\partial & -\partial-\frac{u_1}{u_2}* & 0 & 0 \\ \partial\dfrac{u_1}{u_2}*\partial+u_2*\partial^{-1}u_2\partial & \partial^2+\partial\dfrac{u_1}{u_2}*+u_2\partial^{-1}u_2*\partial & 0 & 0 \\ \dfrac{u_3}{u_2}*\partial-u_4* & \dfrac{u_3}{u_2}*-\partial u_4* & -\partial^2 & \partial u_2*+u_1* \\ \partial^{-1}(u_3*\partial-u_2u_4*) & \partial^{-1}(u_1*u_4*-u_2*\partial u_4*) & \partial^{-1}(u_1*\partial-u_2\partial^2) & \partial^{-1}u_2*\partial u_2* \end{pmatrix},$$

which meets

$$\begin{pmatrix} v_{2,n} \\ -v_{1,n} \\ v_{5,n+1} \\ v_{6,n+1} \end{pmatrix}=L\begin{pmatrix} v_{2,n-1} \\ -v_{1,n-1} \\ v_{5,n} \\ v_{6,n} \end{pmatrix}.$$

Therefore, the expression (37) can be written as

$$u_t=\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}_t=JL^n\begin{pmatrix} 0 \\ -\alpha u_2 \\ \alpha u_3 \\ 0 \end{pmatrix}. \tag{38}$$

When taking $u_3=u_4=0$, the hierarchy (38) reduces to the hierarchy (21). According to the definition of integrable coupling, we know the hierarchy (38) is the integrable coupling of multi-component Guo hierarchy (21).

From expression (30), we have

$$[a,b]^{\mathrm{T}}=a^{\mathrm{T}}\begin{pmatrix} 0 & b_3 & b_2 & 0 & b_6 & b_5 \\ -b_3 & 0 & -b_1 & -b_6 & 0 & -b_4 \\ b_2 & -b_1 & 0 & b_5 & -b_4 & 0 \\ 0 & 0 & 0 & 0 & b_3 & b_2 \\ 0 & 0 & 0 & -b_3 & 0 & -b_1 \\ 0 & 0 & 0 & b_2 & -b_1 & 0 \end{pmatrix}=a^{\mathrm{T}}R(b). \tag{39}$$

Solving the matrix equation (8) for F yields

$$F=\begin{pmatrix} I_M & 0 & 0 & I_M & 0 & 0 \\ 0 & I_M & 0 & 0 & I_M & 0 \\ 0 & 0 & I_M & 0 & 0 & I_M \\ I_M & 0 & 0 & 0 & 0 & 0 \\ 0 & I_M & 0 & 0 & 0 & 0 \\ 0 & 0 & I_M & 0 & 0 & 0 \end{pmatrix}, \tag{40}$$

$$\{a,b\}=(a_1+a_4)*b_1+(a_2+a_5)*b_2+(a_3+a_6)*b_3+a_1*b_4+a_2*b_5+a_3*b_6. \tag{41}$$

Hence, we obtain by using the quadratic-form identity

$$\frac{\delta}{\delta u_i}\Big(-\frac{u_1*(v_2+v_5)}{2\lambda}-\frac{\lambda}{2}(v_3+v_6)-\frac{u_3*v_2+u_4*v_6}{\lambda}\Big)=\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\begin{pmatrix}=\frac{1}{2}(v_2+v_5)\\ \frac{1}{2}(v_4+v_1)\\ v_2\\ v_3\end{pmatrix}. \tag{42}$$

Comparing the coefficients of λ^{-2n+1} yields

$$\frac{\delta}{\delta u}(-v_{3,n+1}-v_{6,n+1}-u_1*(v_{2,n}+v_{5,n})-u_3*v_{2,n}-u_4*v_{6,n})=(\gamma-2n)\begin{pmatrix}v_{2,n}+v_{5,n}\\ v_{1,n}+v_{4,n}\\ 2v_{2,n}\\ 2v_{3,n}\end{pmatrix}.$$

Taking $n=1$ in above equation gives $\gamma=-1$. Thus, Hamiltonian structure of the expression (37) is derived as follows:

$$u_t=\begin{pmatrix}u_{1t}\\ u_{2t}\\ u_{3t}\\ u_{4t}\end{pmatrix}_t=J\begin{pmatrix}\frac{\delta}{\delta u_1}\\ \frac{\delta}{\delta u_2}\\ \frac{\delta}{\delta u_3}\\ \frac{\delta}{\delta u_4}\end{pmatrix}H_n=J\frac{\delta H_n}{\delta u}, \tag{43}$$

where

$$H_n=-\frac{1}{2n+1}(v_{3,n+1}+v_{6,n+1}+u_1*(v_{2,n}+v_{5,n})+u_3*v_{2,n}+u_4*v_{6,n}).$$

Therefore, the expression (37) can be written as

$$u_t=\begin{pmatrix}u_1\\ u_2\\ u_3\\ u_4\end{pmatrix}_t=JL^n\begin{pmatrix}0\\ -\alpha u_2\\ \alpha u_3\\ 0\end{pmatrix}=J\begin{pmatrix}\frac{\delta}{\delta u_1}\\ \frac{\delta}{\delta u_2}\\ \frac{\delta}{\delta u_3}\\ \frac{\delta}{\delta u_4}\end{pmatrix}H_n. \tag{44}$$

4. Multi-component (2+1)-dimensional Guo hierarchy and its Hamiltonian structure

Denoting

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial x}-\frac{\partial}{\partial y},\quad \frac{\partial}{\partial w}=\frac{\partial}{\partial t}-\frac{\partial}{\partial x},$$

the zero curvature equation

$$U_w-V_z+[U,V]=0 \tag{45}$$

may be written in a (2+1)-dimensional form

$$U_t-V_y+[U,V]+V_x-U_x=0, \tag{46}$$

which is regarded as the compatibility of the following Lax pairs. In this section, we try to seek for a multi-component (2+1)-dimensional integrable hierarchy given by Tu's scheme and the generalized zero-curvature equation. In the following, as an example of application, we obtain a multi-component (2+1)-dimensional Guo hierarchy and its Hamiltonian structure.

Consider the following isospectral problem:

$$\begin{cases}\psi_x=[U,\psi], & U,V,\psi\in\tilde{G}_M,\\ \psi_t=[V,\psi], & \lambda_t=0,\end{cases} \tag{47}$$

where

$$U=\Big(\frac{q}{2},\frac{r}{2\lambda},\frac{\lambda}{2}I_M\Big)^{\mathrm{T}},$$

$$V=(v_1,\lambda v_2,\lambda v_3)^{\mathrm{T}},$$

$$v_k=\sum_{m\geq 0}(v_{k_1,m},v_{k_2,m},\ldots,v_{k_M,m})^{\mathrm{T}}\lambda^{-2m},$$

and $q=(q_1,q_2,\ldots,q_M)^{\mathrm{T}}$, $r=(r_1,r_2,\ldots,r_M)^{\mathrm{T}}$.

Solving

$$V_y-V_x=[U,V] \tag{48}$$

gives rise to

$$\begin{cases} v_{2,m+1} = -v_{1,my} + v_{1,mx} + r * v_{3,m}, \\ v_{2,my} - v_{2,mx} = -v_{1,m} + q * v_{3,m}, \\ v_{3,m+1y} - v_{3,m+1x} = q * (v_{1,my} - v_{1,mx} + r * v_{3,m}) - r * v_{1,m}, \\ v_{2,0} = 0, \quad v_{3,0} = \alpha, \quad v_{1,0} = \alpha q, \quad v_{2,1} = \alpha(q_x - q_y + r), \quad v_{3,1} = \frac{\alpha}{2} q * q, \\ v_{1,1} = \alpha(q_{xx} - 2q_{xy} + q_{yy} + r_x - r_y) + \dfrac{\alpha}{2} q * q * q\,. \end{cases} \tag{49}$$

For an arbitrary natural number n, denoting

$$V_+^{(n)} = \sum_{m=0}^{n} \begin{pmatrix} v_{1,m} \\ \lambda v_{2,m} \\ \lambda v_{3,m} \end{pmatrix} \lambda^{2n-2m}\,,$$
$$V_-^{(n)} = \lambda^n V - V_+^{(n)}\,,$$

then Eq.(18) can be written as

$$\begin{aligned} &- V_{+y}^{(n)} + V_{+x}^{(n)} + [U, V_+^{(n)}] \\ &\quad = V_{-x}^{(n)} - V_{-y}^{(n)} - [U, V_-^{(n)}]\,, \end{aligned} \tag{50}$$

where the terms on the left-hand side of Eq.(50) are of degree ≥ -1, while the terms on the right-hand side are of degree ≤ 0. Therefore,

$$-V_{+y}^{(n)} + V_{+x}^{(n)} + [U, V_+^{(n)}] = \begin{pmatrix} -v_{2,n+1} \\ 0 \\ -\dfrac{r * v_{1,n}}{\lambda^2} \end{pmatrix}.$$

Denoting $V^{(n)} = V_+^{(n)} + \Delta_n$, and $\Delta_n = (0, \frac{r}{\lambda^2 q} * v_{1,n}, 0)^{\mathrm{T}}$, the following zero-curvature equation:

$$U_t - V_y^{(n)} + V_x^{(n)} + [U, V^{(n)}] = 0 \tag{51}$$

gives an integrable system

$$\begin{aligned} u_t - u_x &= \begin{pmatrix} r \\ q \end{pmatrix}_t - \begin{pmatrix} r \\ q \end{pmatrix}_x \\ &= \begin{pmatrix} \left(\dfrac{r}{q} * v_{1,n}\right)_x - \left(\dfrac{r}{q} * v_{1,n}\right)_y \\ v_{1,nx} - v_{1,ny} - \dfrac{r}{q} * (v_{2,nx} - v_{2,ny}) \end{pmatrix} \\ &= \begin{pmatrix} 0 & (\partial_y - \partial_x)\dfrac{r}{q}* \\ \dfrac{r}{q} * (\partial_y - \partial_x) & (\partial_y - \partial_x) \end{pmatrix} \begin{pmatrix} v_{2,n} \\ -v_{1,n} \end{pmatrix} \\ &= J \begin{pmatrix} v_{2,n} \\ -v_{1,n} \end{pmatrix}, \end{aligned} \tag{52}$$

where J is a Hamiltonian operator.

From the expression (49), we obtain a recurrence operator

$$L = \begin{pmatrix} \dfrac{r}{q} * (\partial_y - \partial_x) & (\partial_y - \partial_x) + \dfrac{r}{q}* \\ A & B \end{pmatrix},$$

where

$$\begin{aligned} A &= (\partial_x - \partial_y)\frac{r}{q} * (\partial_y - \partial_x) \\ &\quad + q * (\partial_y^{-1} - \partial_x^{-1}) q * (\partial_y - \partial_x)\,, \\ B &= \partial_x^2 - \partial_y^2 + (\partial_x - \partial_y)\frac{r}{q}* \\ &\quad + q * (\partial_y^{-1} - \partial_x^{-1}) q * (\partial_y - \partial_x)\,, \end{aligned}$$

and L meets

$$\begin{pmatrix} v_{2,n} \\ -v_{1,n} \end{pmatrix} = L \begin{pmatrix} v_{2,n-1} \\ -v_{1,n-1} \end{pmatrix}.$$

Therefore, expression (52) can be written as

$$\begin{aligned} u_t = \begin{pmatrix} r \\ q \end{pmatrix}_t &= \begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix} \begin{pmatrix} r \\ q \end{pmatrix} \\ &\quad + JL^n \begin{pmatrix} 0 \\ -\alpha q \end{pmatrix}. \end{aligned} \tag{53}$$

If we take $\partial_x = \partial_x^{-1} = 0$, the system (53) reduces to the Guo hierarchy.

In terms of $\tilde{F}$, we can obtain the symmetric matrix

$$F = \begin{pmatrix} -I_M & 0 & 0 \\ 0 & -I_M & 0 \\ 0 & 0 & I_M \end{pmatrix}, \tag{54}$$

$$\{a, b\} = -a_1 * b_1 - a_2 * b_2 + a_3 * b_3\,. \tag{55}$$

Hence,

$$\left\{V, \frac{\partial U}{\partial q}\right\} = -\frac{1}{2} v_1\,, \quad \left\{V, \frac{\partial U}{\partial r}\right\} = -\frac{1}{2} v_2\,,$$

and

$$\left\{V, \frac{\partial U}{\partial \lambda}\right\} = -\frac{1}{2\lambda} r * v_2 + \frac{1}{2}\lambda v_3 \,.$$

In a way similar to that in the last section, we have

$$\frac{\delta}{\delta u} H_n = \begin{pmatrix} v_{2,n} \\ -v_{1,n} \end{pmatrix},$$

$$H_n = -\frac{1}{2n+1}(r * v_{2,n} - v_{3,n+1}) \,.$$

Thus, we obtain the Hamiltonian structure of the multi-component (2+1)-dimensional Guo hierarchy, when $n = 0$, and $H_0 = -\frac{\alpha}{6} q * q$, that is

$$\begin{pmatrix} H_{01} \\ H_{02} \\ \vdots \\ H_{0M} \end{pmatrix} = \frac{\alpha}{6} \begin{pmatrix} q_1^2 \\ q_2^2 \\ \vdots \\ q_m^2 \end{pmatrix}.$$

Hamiltonian structures of integrable coupling of other hierarchies such as AKNS hierarchy, KN hierarchy and so on, can be obtained by use of the quadratic-form identity that is presented in this paper. Discussion of problems of this kind will appear in another paper.

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