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Relative entropy as a measure of entanglement for Gaussian states*

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In this paper, we derive an explicit analytic expression of the relative entropy between two general Gaussian states. In the restriction of the set for Gaussian states and with the help of relative entropy formula and Peres–Simon separability criterion, one can conveniently obtain the relative entropy entanglement for Gaussian states. As an example, the relative entanglement for a two-mode squeezed thermal state has been obtained.

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Quantum entanglement has proved to play an essential role in all branches of the emerging field of quantum information theory.[1] One of the most basic problems in quantum information theory is the measure of entangled states. Now, Gaussian states have attracted much attention[2−12] because of immense progresses both in quantum information processing protocols[13] and in experiments[14] for continuous-variable (CV) quantum states. As is well known, as the distances of the best distinguishability for the quantum mixed states, the relative entropy[3] and the Bures distance[15] are considered as good candidates of quantum entanglement.[10] Theoretically, based on the relative entropy or Bures distance, one can calculate the distance between two Gaussian states. As mentioned in Ref.[16], however, due to the lack of a parametrization for the whole CV separable states, one can hardly calculate the distance of a Gaussian state to the set of all separable states. Hence, one has to restrict the set of all CV separable states to separable Gaussian states so as to compute the distance of Gaussian states.[10,16] It has been proved that the upper bounds on the entanglement for the Gaussian states can be obtained under this restricted approximation.[10,16] Under this restriction, the main problem concerning the measure of entanglement for Gaussian state can be taken as the calculation of the relative entropy or Bures fidelity[17] for Gaussian states. Therefore, the explicit expressions of the relative entropy and Bures fidelity for Gaussian states become desirable in the measure of entanglement for Gaussian states.

The explicit expression of Bures fidelity for Gaussian state has been proposed in Refs.[18, 19]. Recently, Marian *et al*[10] have also derived the explicit expression of Bures fidelity for a two-mode squeezed thermal state (STS), and obtained its entanglement in the approximation mentioned above. However, from the Bures fidelity given in Ref.[10], when using the Peres–Simon separability criterion, one has to work through rather lengthy algebra so as to achieve the entanglement for two-mode STS. Here, we approach the same problem from a different perspective, i.e. we use the relative entropy, instead of Bures fidelity, as a measure of entanglement for Gaussian states. We will show that one can conveniently obtain the entanglement for the two-mode STS by using the relative entropy formula proposed in this paper.

To our knowledge, few explicit analytic expressions of the relative entropy for general Gaussian

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states have been obtained so far. In this paper, we will derive a general formula of the relative entropy between two general Gaussian states. As we shall see below, this general formula can provide an efficient approach to measuring the entanglement of Gaussian states under the restricted approximation. The convenience of the formula can be seen from the example of the two-mode STS.

Now let us consider the relative entropy entanglement for a general multi-mode Gaussian quantum state ρ. The relative entropy entanglement is defined as[3]

$$E(\rho) = \min_{\sigma\in D} S(\rho\,|\sigma\,). \tag{1}$$

Here D is the set of all separable bipartite states of the system.[3] As pointed out in Refs.[10, 16], in order to calculate the entanglement for a general Gaussian state, we will restrict ourselves to separable Gaussian states. So, under this approximation condition, Eq.(1) can be written as

$$E_{\rm G}(\rho) = \min_{\sigma\in G} S(\rho\,|\sigma\,), \tag{2}$$

where G is the set of all separable bipartite Gaussian states, and $S(\rho\,|\sigma\,) = {\rm tr}\,\{\rho(\ln\rho - \ln\sigma)\}$ is the relative entropy between two Gaussian mixed states ρ and σ.

It is easy to see from Eq.(2) that one of the main problems concerning the entanglement for Gaussian state ρ is to calculate the relative entropy between two Gaussian states. Hence, the following derivation will be concentrated on the calculation of the relative entropy. To do this, for convenience, we denote the relative entropy between two Gaussian states ρ_1 and ρ_2 by

$$S(\rho_1\,|\rho_2\,) = {\rm tr}\,\{\rho_1(\ln\rho_1 - \ln\rho_2)\}\,. \tag{3}$$

Consider a general Gaussian state whose density operator reads

$$\rho_i = A_i {\rm e}^{H(N_i)} = A_i {\rm exp}\left[\frac{1}{2}\Lambda N_i \Sigma \Lambda^{\rm T}\right]. \tag{4}$$

Here

$$A_i = Z_i^{-1}(\beta)\,|_{\beta=-1} = \left|{\rm det}({\rm e}^{-N_i} - 1)\right|^{\frac{1}{2}} \quad (i = 1, 2)$$

is the normalization factor; $\Lambda = (a_1^\dagger,\ a_2^\dagger \ldots a_n^\dagger,\ a_1,\ a_2 \ldots a_n) \equiv (a^\dagger, \tilde{a})$, $a_j^\dagger$ and a_j $(j = 1, 2 \ldots, n)$ are n-mode boson creation and annihilation operators, respectively, and satisfy the commutation rule $[a_j, a_{j'}^\dagger] = \delta_{jj'}$, while N is the $2n \times 2n$ negative Hermitian matrix,[20] $\Sigma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ and I is the $n \times n$ unity matrix.

From Eq.(3), we have

$$\begin{aligned} S(\rho_1\,|\rho_2\,) =& {\rm tr}\,\{\rho_1(\ln\rho_1 - \ln\rho_2)\} \\ =& {\rm tr}(\rho_1 \ln\rho_1) - tr(\rho_1 \ln\rho_2). \end{aligned} \tag{5}$$

For a given Gaussian state ρ_i in Eq.(4), by using the partition function formula presented in Ref.[21]

$$Z(\beta) = {\rm tr}({\rm e}^{-\beta_i H(N_i)}) = |{\rm det}[\exp(\beta_i N_i) - 1]|^{-1/2}\,, \tag{6}$$

we can readily evaluate the first term in Eq.(5) as follows:

$$\begin{aligned} {\rm tr}(\rho_1 \ln\rho_1) =& {\rm tr}\left\{A_1 {\rm e}^{H(N_1)} \ln[A_1 {\rm e}^{H(N_1)}]\right\} \\ =& \ln A_1 - A_1 \frac{{\rm d}Z_1(\beta)}{{\rm d}\beta}\,|_{\beta=-1} \\ =& \frac{1}{2}\left[\ln\left|{\rm det}({\rm e}^{-N_1} - 1)\right| + {\rm tr}\Big(\frac{N_1}{1 - {\rm e}^{N_1}}\Big)\right]. \end{aligned} \tag{7}$$

The second term in Eq.(5) can be written as

$${\rm tr}(\rho_1 \ln\rho_2) = \ln A_2 + A_1 {\rm tr}[{\rm e}^{H(N_1)} H(N_2)]. \tag{8}$$

Similarly, the second term in Eq.(8) can also be calculated as follows:

$$\begin{aligned} {\rm tr}[{\rm e}^{H(N_1)} H(N_2)] =& \frac{\rm d}{{\rm d}\xi} {\rm tr}({\rm e}^{[H(N_1) + (\xi+1)H(N_2)]}\,|_{\xi=-1}\,) \\ =& \frac{1}{2} Z_1(-1) {\rm tr}\left(\frac{N_2}{1 - {\rm e}^{N_1}}\right). \end{aligned} \tag{9}$$

Substituting Eqs.(6)–(9) into Eq.(5), we finally derive a general formula of the relative entropy between two general Gaussian states ρ_1 and ρ_2, i.e.

$$\begin{aligned} S(\rho_1\,|\rho_2\,) =& {\rm tr}\,\{\rho_1(\ln\rho_1 - \ln\rho_2)\} \\ =& \frac{1}{2}\left[\ln\left|{\rm det}({\rm e}^{-N_1} - 1)\,{\rm det}({\rm e}^{-N_2} - 1)^{-1}\right|\right] \\ &+ \frac{1}{2}{\rm tr}\left(\frac{N_1 - N_2}{1 - {\rm e}^{N_1}}\right). \end{aligned} \tag{10}$$

Based on the Peres–Simon separability criterion,[5] by using Eq.(10) one can easily calculate the entanglement for a general Gaussian state under the restriction Eq.(2). The following example will illuminate the convenience of the formula in Eq.(10).

Before proceeding, here we review briefly the Peres–Simon separability criterion. As is well known, in order to use the Peres–Simon separability criterion to calculate the entanglement of the Gaussian states, one needs to know the corresponding characteristic function that is determined by its covariance matrix. A two-mode Gaussian state with the density operator ρ has the following Wigner characteristic function:[22]

$$\begin{aligned} \chi^{(w)}(\lambda_1, \lambda_2) =& {\rm tr}[\rho \exp(\lambda_1 a_1 - \lambda_1^* a_1^\dagger + \lambda_2 a_2 - \lambda_2^* a_2^\dagger)] \\ =& \exp\Big(-\frac{1}{2}\lambda V \lambda^{\rm T}\Big). \end{aligned} \tag{11}$$

In Eq.(11), $\lambda = (\lambda_1^{\rm I}, \lambda_1^{\rm R}, \lambda_2^{\rm I}, \lambda_2^{\rm R})$ is a real row vector, where $\lambda_j = \lambda_j^{\rm I} + {\rm i}\lambda_j^{\rm R}$. The real, symmetric, and positive 4×4 covariance matrix V has the following block structure:

$$V = \begin{pmatrix} V_1 & \mathcal{C} \\ \mathcal{C}^{\rm T} & V_2 \end{pmatrix}. \tag{12}$$

Here V_1, V_2 and $\mathcal{C}$ are 2×2 matrices. Their entries are covariances of the quadrature amplitudes $\hat{x}_j = \dfrac{1}{\sqrt{2}}(a_j + a_j^\dagger)$ and $\hat{p}_j = \dfrac{1}{\sqrt{2}}(a_j - a_j^\dagger)$ with the annihilation (creation) operators a_j $(a_j^\dagger)$ satisfying the commutators $[a_{j'}, a_j^\dagger] = \delta_{j'j}$ $(j, j' = 1, 2)$. According to the covariance matrix corresponding to the density operator ρ, Simon[5] derived the following inequality that any separable state has to obey:

$$\begin{aligned} S(\rho) =& \det V - \frac{1}{4}(\det V_1 + \det V_2 + 2\,|\det \mathcal{C}|) \\ &+ \frac{1}{16} \geq 0. \end{aligned} \tag{13}$$

Using Eq.(13), one can easily check whether a two-mode Gaussian state is separable or not. This equation is known as Peres–Simon separability criterion.

Now as an example, we consider a two-mode STS whose density operator reads

$$\rho = AS(r,\theta)T(\beta)S^\dagger(r,\theta). \tag{14}$$

Here

$$S(r,\theta) = \exp[r(\mathrm{e}^{{\rm i}\theta}a_1^\dagger a_2^\dagger - \mathrm{e}^{-{\rm i}\theta}a_1 a_2)] \tag{15}$$

is a two-mode squeezing operator[23] with the squeezing factor $r > 0$ and squeezing angle $\theta \in (-\pi, \pi]$,

$$T(\beta) = \exp\Big[-\frac{\beta}{2}\sum_{j=1}^{2}(a_j^\dagger a_j + a_j a_j^\dagger)\Big] \tag{16}$$

is a two-mode thermal state density operator, and A is a normalization factor.

It is easy to write density operator ρ in Eq.(14) in the following quadratic form:

$$\rho = A\exp\left[\frac{1}{2}\Lambda N \Sigma \Lambda^{\rm T}\right], \tag{17}$$

where

$$N = \begin{pmatrix} -\beta\cosh(2rI) & -\beta\mathrm{e}^{{\rm i}\theta}\sinh(2r\sigma_1) \\ \beta\mathrm{e}^{-{\rm i}\theta}\sinh(2r\sigma_1) & \beta\cosh(2rI) \end{pmatrix}. \tag{18}$$

In Eq.(18), $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is Pauli matrix, and I is the 2×2 unity matrix. From Eq.(18), one can obtain the following results:

$$\det(\mathrm{e}^{-N} - 1) = (2\cosh\beta - 2)^2, \tag{19}$$

$$\mathrm{e}^N = \begin{pmatrix} D & -B \\ B^* & C \end{pmatrix}, \tag{20}$$

$$D = (\cosh\beta - \sinh\beta\cosh(2r))I,$$

$$B = \mathrm{e}^{{\rm i}\theta}\sinh\beta\sinh(2r)\sigma_1,$$

$$C = (\cosh\beta + \sinh\beta\cosh(2r))I.$$

If we consider another two-mode STS whose density operator is given by

$$\begin{aligned} \rho' =& A'S(r',\theta')T(\beta')S^\dagger(r',\theta') \\ =& A'\exp\left[\frac{1}{2}\Lambda N' \Sigma \Lambda^{\rm T}\right], \end{aligned} \tag{21}$$

then we obtain

$$\begin{aligned} {\rm tr}\left(\frac{N - N'}{1 - \mathrm{e}^N}\right) =& \frac{2\sinh\beta}{1-\cosh\beta} \\ &\times [\beta'\sinh(2r)\sinh(2r')\cos(\theta - \theta') \\ &-\beta'\cosh 2r\cosh(2r') + \beta]. \end{aligned} \tag{22}$$

By substituting the expressions Eqs.(19)–(22) into Eq.(10), we arrive at the relative entropy of the two-mode STS as follows:

$$\begin{aligned} S(\rho\,|\rho') =& \ln\left(\frac{\cosh\beta - 1}{\cosh\beta' - 1}\right) + \frac{\beta\sinh\beta}{1-\cosh\beta} \\ &+ \frac{\beta'\sinh\beta}{1-\cosh\beta} \\ &\times [\sinh(2r)\sinh(2r')\cos(\theta - \theta') \\ &- \cosh(2r)\cosh(2r')]. \end{aligned} \tag{23}$$

Thus, based on the Peres–Simon separability criterion[5] and the results above, one can easily obtain the relative entropy entanglement of the two-mode STS.

It follows from the definition of the characteristic function (11) that the corresponding covariance matrix V for the two-mode STS in Eq.(14) is given by[10]

$$V_1 = V_2 = \frac{1}{2}\cosh(2r)\cosh(2r)I, \tag{24}$$

$$\mathcal{C} = \frac{1}{2}\coth\frac{\beta}{2}\sinh(2r)\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}. \tag{25}$$

Substituting Eq.(24) and Eq.(25) into Eq.(13) gives the Peres–Simon separability condition for the two-mode STS in Eq.(14) as follows:

$$\cosh(2r) \leq \cosh(2r_{\rm s}) = \coth\beta, \tag{26}$$

where $r_{\rm s}$ is separable threshold that is determined by $\cosh(2r_{\rm s}) = \coth\beta$. Furthermore, the two-mode STS becomes entangled under the following condition:

$$S(\rho) < 0 \Longleftrightarrow r > r_{\rm s}. \tag{27}$$

Let $\tilde{\rho}$ be the separable Gaussian mixed state. We denote its parameters by $\tilde{\beta}, \tilde{\theta}, \tilde{r}$. Under the threshold condition

$$S(\tilde{\rho}) = 0 \Longleftrightarrow \tilde{r} = \tilde{r}_{\rm s}, \tag{28}$$

from Eq.(23), direct calculation gives:

$$\cosh\tilde{\beta} = \frac{{\rm e}^{2r_{\rm s}}\sinh(2r)+1}{{\rm e}^{2r_{\rm s}}\cosh(2r)-1}, \tag{29}$$

$$\tilde{\beta} = \ln\left(\frac{{\rm e}^{2r_{\rm s}}\sinh(2r)+1+{\rm e}^{r+r_{\rm s}}\sqrt{2-{\rm e}^{-2(r-r_{\rm s})}}}{{\rm e}^{2r_{\rm s}}\cosh(2r)-1}\right), \quad (r > r_{\rm s}),$$

$$\cosh(2\tilde{r}) = \coth\tilde{\beta}, \tag{30}$$

$$\tilde{\theta} = \theta. \tag{31}$$

Substituting Eqs.(29)–(31) into Eq.(23), we obtain the entanglement for the two-mode STS in Eq.(14) as follows:

$$E_{\rm G}(\rho) = -{\rm e}^{2r_{\rm s}}\ln(\coth r_{\rm s}) + \ln\left(\frac{\cosh(2r)-{\rm e}^{-2r_{\rm s}}}{\sinh(2r_{\rm s})[2-{\rm e}^{-2(r-r_{\rm s})}]}\right)$$
$$+\frac{{\rm e}^{r+r_{\rm s}}}{\sqrt{2-{\rm e}^{-2(r-r_{\rm s})}}}\ln\left(\frac{{\rm e}^{2r_{\rm s}}\sinh(2r)+1+{\rm e}^{r+r_{\rm s}}\sqrt{2-{\rm e}^{-2(r-r_{\rm s})}}}{{\rm e}^{2r_{\rm s}}\cosh(2r)-1}\right), \quad (r > r_{\rm s}), \tag{32}$$

$$E_{\rm G}(\rho) = 0, \quad (r \leq r_{\rm s}). \tag{33}$$

In conclusion, based on the approximation in Refs.[10, 16], we have obtained the entanglement for two-mode STS in terms of its relative entropy to the set of all separable states of the same kind. Through the example, it is obvious that, by using the relative entropy formula presented in this paper, one can easily calculate the entanglement for the two-mode STS. It is worth pointing out that this approach can also be applied to the multi-mode Gaussian states. For a given Gaussian state, based on the Peres–Simon separability criterion, one can conveniently calculate its entanglement by using the formula in Eq.(10). Therefore, our method also provides an approach to the measure of the relative entanglement for a general Gaussian state.

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