Vol 15 No 8, August 2006
1009-1963/2006/15(08)/1705-04

Chinese Physics

© 2006 Chin. Phys. Soc.
and IOP Publishing Ltd

Chaos feature of test particle in the gravitational field with a quadrupole*

Chen Ju-Hua(陈菊华)† and Wang Yong-Jiu(王永久)

College of Physics and Information Science, Hunan Normal University, Changsha 410081, *China*

(Received 24 June 2005; revised manuscript received 8 December 2005)

In this paper we investigate the dynamics of a test particle in the gravitational field with a quadrupole. By constructing Poincaré sections for different values of the parameters and initial conditions, we find a chaotic evolution. From these Poincaré sections, we further confirm that the chaotic evolution of the test particle originates from the quadrupole.

Keywords: chaos, Poincaré section, quadrupole
PACC: 0420, 0547

1. Introduction

Chaotic behaviours were considered as an interesting phenomenon by many physicists since Lorenz[1] found the deterministic non-periodic flow in 1963. Since then, chaos has become one of the most important ideas used to explain various nonlinear phenomena in nature. Following Poincaré's research of the three-body problem, many studies on chaos in celestial mechanics and astrophysics have been performed.[2,3] There are two main lines of the research in general relativity, some deal with the chaos associated with inhomogeneous cosmological models. For example, Oliveira and co-workers[4−7] have studied the chaotic behaviour in the Bianchi IX model. The others assume a given metric then look for chaotic behaviours of the geodesic motion in a given background. Although we know many features of chaos in the Newtonian dynamics, we do not know, as yet, much about those in general relativity. Because the gravitational field around a black hole is very strong and nonlinear, we expect to find a new type of chaotic behaviour in such a strong gravitational field which does not appear in Newtonian dynamics.[8−10] Some authors[11−19] found the chaotic behaviour of a test particle in a relativistic system. Letelier *et al*[20−22] investigated the chaos in black holes with halos. Chen and co-workers[23−31] and Wang and Tang[31] investigated the dynamics of an extremely charged black hole. Saa[32,33] extended the investigation on the integrability of oblique orbits of the test particle in several gravitational fields. And there are many examples in the literature of chaotic motions involving black holes, such as the fixed two-centre problem,[12,34,35] a black hole surrounded by gravitational waves,[36,37] and several core-shell models relevant to the description of galaxies.[38] As a Newtonian case, recent works of Chicone *et al*[39] were devoted to the chaotic behaviour of the Hill system. In order to investigate the chaotic behaviour of the dynamic systems, many researchers concentrated on the study of chaotic dynamics in general relativity with the Poincaré–Melnikov method.[40,41] The Melnikov method is an analytical criterion to determine the occurrence of chaos in integrable systems in which homoclinic (or heteroclinic) manifolds biasymptotic to unstable critical points or to periodic orbits (more generally to invariant tori) are subjected to small perturbations.

It is well known that the non-spherical-symmetrical mass distribution is a common phenomenon. So it is interesting to investigate the evolution of the test particle in the space-time of gravitational field with a quadrupole. In this paper, using Poincaré sections for different initial conditions, we find that there are chaotic motions of the test particle in the gravitational field with a quadrupole. In

*Project supported by the State Key Development Program for Basic Research Program of China (Grant No 2003CB716300), the National Natural Science Foundation of China (Grant No 10547114) and the Scientific Research Fund of Hunan Provincial Education Department (Grant No 02C202).

†E-mail: jhchen@hunnu.edu.cn

Section 2 we investigate the Hamiltonian of the gravitational field with a quadrupole. In Section 3 we construct Poincaré sections for different parameters and find the chaotic evolution of the test particle in the gravitational field with a quadrupole, which originates from the non-spherical-symmetrical mass distribution. In the last section, a brief conclusion is given.

2. Hamiltonian of the gravitational field with a quadrupole

Wang[42] gave the metric of the gravitational field with a quadrupole. Because the space-time of our system is of a static axial-symmetry, using cylindrical coordinates (r, θ, z) to describe the space–time, we obtain the potential to describe the gravitational field with a quadrupole in Newtonian limits,

$$V = -\frac{GM}{\sqrt{r^2+z^2}} + \frac{3\sigma z^2}{(r^2+z^2)^{5/2}} - \frac{\sigma}{(r^2+z^2)^{3/2}}, \quad (1)$$

where M and σ are the mass and quadrupole of the gravitational source, respectively, and $r^2 + z^2 = x^2 + y^2 + z^2$, where x, y, z are the usual Cartesian coordinates.

We know that the angular momentum L in the z direction is conserved and we can also easily reduce the three-dimensional (3D) original problem to a two-dimensional (2D) one in coordinates (r, z). Now we consider the bounded orbit for the test particle under the potential (1), so in Newtonian limits the Hamiltonian is

$$H = \frac{\dot{r}^2+\dot{z}^2}{2} + \frac{L^2}{2r^2} - \frac{GM}{\sqrt{r^2+z^2}} + \frac{3\sigma z^2}{(r^2+z^2)^{5/2}} - \frac{\sigma}{(r^2+z^2)^{3/2}}. \quad (2)$$

Hamiltonian–Jacobi equations corresponding to Hamiltonian (2) can be properly separated in parabolic coordinates,[43,44] leading to the second constant of the motion

$$C = R_z - \alpha\frac{r^2}{2}, \quad (3)$$

where R_z is the z component of the Laplace–Runge–Lenz vector

$$R = \frac{M}{\sqrt{r^2+z^2}}(r\hat{r} + z\hat{z}) + V \times L, \quad (4)$$

where L stands for the total angular momentum. In this case, with two constants of motion H and C, the equations for the trajectories of the test particle can be reduced to quadrature in parabolic coordinates[43,44] and we notice that the equations of the evolution are invariant under the following rescaling:

$$r \to \lambda r, z \to \lambda z, t \to \lambda t, M \to \lambda M, \quad Q \to \lambda Q, P \to \lambda P, H \to \lambda H. \quad (5)$$

3. Numerical computing

In this section, we figure out the evolution of the test particle in the phase space by using the Poincaré section method, and investigate the properties of the dynamics of the test particle in the gravitational field with a quadrupole. In order to investigate the evolutions of the test particle in the gravitational field with a quadrupole and its chaotic behaviour, we use variables $(\dot{r}, \dot{z}, r, z)$ and the package Poincaré[45] to perform the following numerical experiments and to investigate the chaotic behaviour of a dynamical system. Based on the nonlinear dynamical theory, if the evolution is not chaotic, the plotted points form a closed curve in the 2D $(\dot{r}, r)$ plane, because a regular orbit will move on a torus in the phase space and the curve is a cross section of the torus. If the orbit is chaotic, some of those tori will be broken and the Poincaré section does not consist of a set of closed curves, but the points will be distributed randomly in the allowed region to form a chaotic sea. From the distribution of the points in the Poincaré section, we can judge whether the evolution is chaotic or not. Figures 1–4 show the typical Poincaré sections across the plane $z = 0$ for different initial conditions. In Fig.1 and 2 we present the Poincaré section for $GM = 1, L^2 = 0.4, \sigma = 0.01$ and $H = -0.50533901$, and $GM = 1, L^2 = 0.4, \sigma = 0.1$ and $H = -0.48942911$. We can see that some points are distributed randomly in a finite region to form a chaotic sea. If we go on changing the parameters and the initial conditions to draw Figs.3 and 4 for $GM = 1, L^2 = 0.4, \sigma = 1$, $H = -0.33033009$ and $GM = 1, L^2 = 0.4, \sigma = 1.15$, $H = -0.30381358$, a far larger chaotic sea will be obtained in an allowed region. This means that the evolution of the test particle in this gravitational field will change into chaos. By comparing the parameters of Figs.1–4, we can find that the chaotic behaviour mainly originates from the quadrupole.

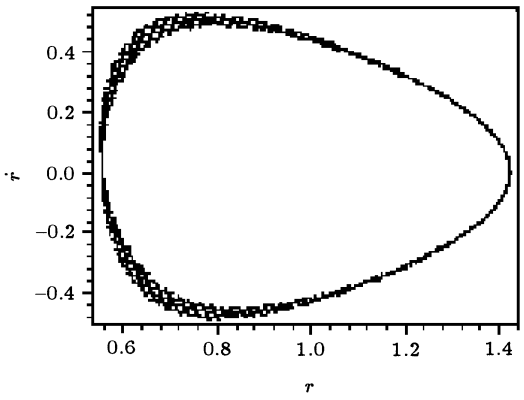


Fig.1. Poincaré section $(\dot{r}, r)$ across the plane $z = 0$, $\dot{z} > 0$ for $GM = 1$, $L^2 = 0.4$, $\sigma = 0.01$ and $H = -0.50533901$.

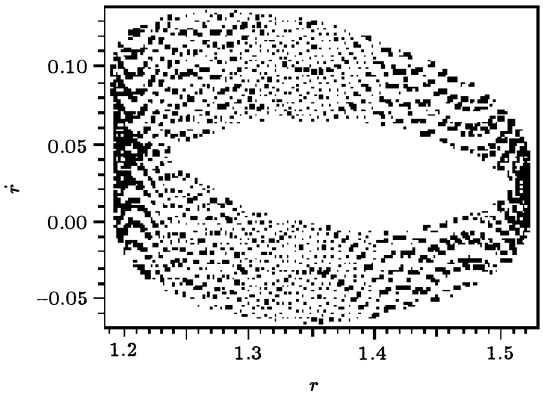


Fig.3. Poincaré section $(\dot{r}, r)$ across the plane $z = 0$, $\dot{z} > 0$ for $GM = 1$, $L^2 = 0.4$, $\sigma = 1$ and $H = -0.33033009$.

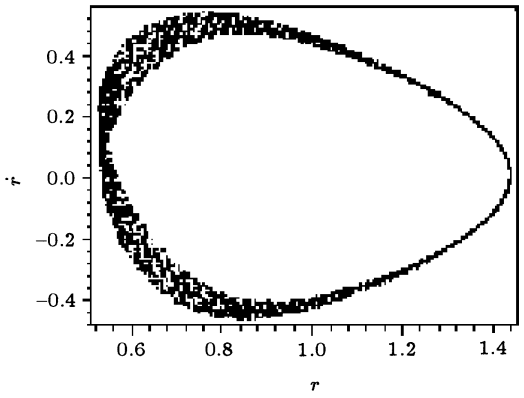


Fig.2. Poincaré section $(\dot{r}, r)$ across the plane $z = 0$, $\dot{z} > 0$ for $GM = 1$, $L^2 = 0.4$, $\sigma = 0.1$ and $H = -0.48942911$.

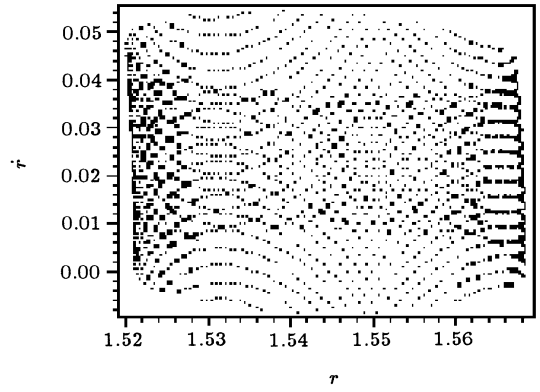


Fig.4. Poincaré section $(\dot{r}, r)$ across the plane $z = 0$, $\dot{z} > 0$ for $GM = 1$, $L^2 = 0.4$, $\sigma = 1.15$ and $H = -0.30381358$.

4. Conclusions

In this paper we have investigated the evolution of a test particle in the gravitational field with a quadrupole. We have constructed Poincaré sections for different values of parameters and initial conditions. From these Poincaré sections we have found the chaotic evolution. By comparing these Poincaré sections, we confirm that the chaotic evolution of the test particle mainly originates from the quadrupole.

Acknowledgments

The authors are very grateful to Professor Alberto Saa (Departamento de Matemática Aplicada, IMECC-UNICANP Campinas. S.P. Brazil), Professor H. P. de Oliveira (NASA/Fermilab Astrophysics Centre, Fermi National Accelerator Batavia, Illinois) and Professor Wenhua Hai, Jiliang Jing.

References

[1] Lorenz E N 1963 *J. Atoms. Sci.* **20** 215

[2] Moser T 1978 *Math. Intelligencer* **1** 65

[3] Wisdom J 1987 *Proc. R. Soc. London* **A413** 109

[4] de Oliveira H P, Soares I Damiao and Stuchi T J 1997 *Phys. Rev.* D **56** 730

[5] Monerat G A, de Oliveira H P and Soares I Damiao 1998 *Phys. Rev.* D **58** 063504

[6] Barguine R, de Oliveira H P, Soares I D and Tonini E V 2001 *Phys. Rev.* D **63** 063502

[7] de Oliveira H P, Almeida A M, Ozorio de, Soares I Damiao and Tonini E V 2002 gr-qc/0202047

[8] Misner C W 1969 *Phys. Rev. Lett.* **22** 1071

[9] Belinskii V A, Khalatnikov I M and Lifshitz E M 1970 *Adv. Phys.* **19** 525
[10] Barrow J D 1982 *Phys. Rep.* **85** 1
[11] Soota Y, Suzuki S and Maeda K 1996 *Class. Quantum Grav.* **13** 1241
[12] Contopoulos G 1991 *Proc. R. Soc. London* **A435** 551
[13] Dettmann C P, Frankel N E and Cornish N J 1994 *Phys. Rev.* D **50** R618
[14] Yustsever U 1995 *Phys. Rev.* D **52** 3176
[15] Karas V and Vokrouhlický D 1992 *Gen. Relativ. Gravit.* **24** 729
[16] Varvoglis H and Papadopoulos D 1992 *Astron. Astrophys.***261** 664
[17] Bombelli L and Calzetta E 1992 *Class. Quantum Grav.* **9** 2573
[18] Moeckel R 1992 *Commun. Math. Phys.* **150** 415
[19] Letelier P S and Vieira W M 1997 *Class. Quantum Grav.* **14** 1249
[20] Guéron E and Letelier P S 2001 astro-ph/0101140
[21] Vieira W M and Letelier P S 1997 *Phys. Lett.* A **228** 22
[22] Letelier P S and Vieira W M 1997 *Phys. Rev.* D **56** 8095
[23] Chen J H and Wang Y J 2001 *Acta Phys. Sin.* **50** 1833 (in Chinese)
[24] Wang Y J and Tang Z M 2001 *Acta Phys. Sin.* **50** 1829 (in Chinese)
[25] Wang Y J and Tang Z M 2001 *Acta Phys. Sin.* **50** 2284 (in Chinese)
[26] Wang Y J and Cao D 2004 *Chin. Phys.* **13** 579
[27] Chen J H and Wang Y J 2003 *Class. Quantum Grav.* **20** 3897
[28] Chen J H and Wang Y J 2003 *Chin. Phys.* **12** 836
[29] Chen J H and Wang Y J 2004 *Chin. Phys.* **13** 583
[30] Chen J H and Wang Y J 2005 *Chin. Phys.* **14** 1282
[31] Wang Y J and Tang Z M 2001 *Chin.Phys.* **10** 679
[32] Saa A 1999 *Phys. Lett.* A **259** 201
[33] Saa A 2000 *Phys. Lett.* A **269** 204
[34] Contopoulos G 1990 *Proc. R. Soc. London* A **431** 183
[35] Cornish N J and Gibbons G W 1997 *Class. Quantum Grav.* **14** 1865
[36] Bombelli L and Calzetra E 1992 *Class. Quantum Grav.* **9** 2573
[37] Letelier P S and Vieira W M 1997 *Class. Quantum Grav.* **14** 1249
[38] Vieira W M and Letelier P S 1999 *Astrophys. J.* **513** 383
[39] Chicone C, Mashhoon B and Helv D G Retzloff 1999 *Phys. Acta.* **72** 123
[40] Melnikov V K 1963 *Trans. Moscow. Math. Soc.* **12** 1
[41] Wiggins S 1988 *Global Bifurcations and Chaos* (New York: Springer)
[42] Wang Y J 2000 *General Relativity and Cosmology* (Hunan: Hunan Science and Technology Press)
[43] Dorizzi B, Grammaticos B and Ramani A 1984 *J. Math. Phys.* **25** 481
[44] Grammaticos B, Dorizzi B, Ramani A and Hietarinta J 1985 *Phys. Lett.* **109**A 81
[45] Ceb-Terrab E S and de Oliveira H P 1996 *Comput. Phys. Commun.* **95** 171