Vol 15 No 8, August 2006
1009-1963/2006/15(08)/1678-04

Chinese Physics

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A symmetry and a conserved quantity for the Birkhoff system*

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(Received 14 March 2006; revised manuscript received 28 April 2006)

A symmetry and a conserved quantity of the Birkhoff system are studied. The symmetry is called the Birkhoff symmetry. Its definition and criterion are given in this paper. A conserved quantity can be deduced by using the symmetry. An example is given to illustrate the application of the result.

Keywords: Birkhoff system, symmetry, conserved quantity
PACC: 0320

1. Introduction

'Undoubtedly, the studies of symmetries of physical systems constitute one of the most physically significant, mathematically instructive, and aesthetically appealing topics in physics.'[1] Since the 1960s, people have studied the Noether symmetry,[2−4] the Lie symmetry,[2−12] the form invariance[13−16] and the unified symmetry[17,18] for mechanical systems. In 1966, Currie and Saletan studied the problem on the equivalent Lagrangians for a mechanical system with a single degree of freedom and pointed out that there existed a conserved quantity in this case.[19] In 1981, Hojman and Harleston extended the result above to the general mechanical systems with multi-degree-of-freedom.[20] In Ref.[3], the symmetry under which equivalent Lagrangians can lead to the same Lagrange equations is called the Lagrange symmetry.

The Birkhoff system is one of the important constrained mechanical systems, and the study on symmetries of the system can give some significant results.[21] This paper presents a new symmetry under which the different Birkhoffians and Birkhoff functions can lead to the same Birkhoff equations. The symmetry is called a Birkhoff symmetry. A kind of conserved quantity of the Birkhoff system can be obtained by using the Birkhoff symmetry.

2. Definition and criterion of Birkhoff symmetry

The Birkhoff equations are written in the form

$$\Omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial B}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial t} = 0, \quad (\mu, \nu = 1, \ldots, 2n), \tag{1}$$

where $B = B(t, a)$ is called the Birkhoffian, $R_{\mu} = R_{\mu}(t, a)$ are the Birkhoff functions and

$$\Omega_{\mu\nu} = \frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}} \tag{2}$$

is called the Birkhoff tensor. Suppose that

$$\det(\Omega_{\mu\nu}) \neq 0, \tag{3}$$

and let

$$S_{\mu} = \Omega_{\mu\nu}\dot{a}^{\nu} - \frac{\partial B}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial t}, \tag{4}$$

$$\bar{S}_{\mu} = \bar{\Omega}_{\mu\nu}\dot{a}^{\nu} - \frac{\partial \bar{B}}{\partial a^{\mu}} - \frac{\partial \bar{R}_{\mu}}{\partial t}, \tag{5}$$

where $\bar{B}$ and $\bar{R}_{\mu}$ are respectively other Birkhoffian and Birkhoff functions, and

$$\bar{\Omega}_{\mu\nu} = \frac{\partial \bar{R}_{\nu}}{\partial a^{\mu}} - \frac{\partial \bar{R}_{\mu}}{\partial a^{\nu}}. \tag{6}$$

*Project supported by the National Natural Science Foundation of China (Grant No 10572021) and the Special Research Fund for the Doctoral Program of Higher Education of China (Grant No 20040007022).

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Definition If from $S_\mu = 0$ $(\mu = 1, \ldots, 2n)$ one obtains $\bar{S}_\nu = 0$ $(\nu = 1, \ldots, 2n)$ and vice versa, then the symmetry is called a Birkhoff symmetry of the Birkhoff system.

If $\bar{S}_\mu$ is zero in Eq.(5), we can express $\dot{a}^\mu$ as

$$\dot{a}^\mu = \bar{\Omega}^{\mu\nu}\left(\frac{\partial \bar{B}}{\partial a^\nu} + \frac{\partial \bar{R}_\nu}{\partial t}\right), \tag{7}$$

where

$$\bar{\Omega}^{\mu\nu}\bar{\Omega}_{\nu\rho} = \delta^\mu_\rho. \tag{8}$$

According to the definition above, two groups dynamical functions R_μ, B and $\bar{R}_\mu$, $\bar{B}$ will lead to the same equations. Substituting the expression (7) into $S_\nu = 0$, one can easily obtain

$$\begin{aligned}&\Omega_{\mu\nu}\bar{\Omega}^{\nu\rho}\left(\frac{\partial \bar{B}}{\partial a^\rho} + \frac{\partial \bar{R}_\rho}{\partial t}\right) = \frac{\partial B}{\partial a^\mu} + \frac{\partial R_\mu}{\partial t},\\ &(\mu, \nu, \rho, = 1, \ldots, 2n),\end{aligned} \tag{9}$$

and then we have

Criterion If two groups dynamical functions R_μ, B and $\bar{R}_\mu$, $\bar{B}$ satisfy Eq.(9), then the corresponding symmetry is a Birkhoff symmetry of the Birkhoff system.

3. Conserved quantity deduced by Birkhoff symmetry

Substituting Eq.(9) into the expression (4) yields

$$\begin{aligned}S_\mu =&\Omega_{\mu\nu}\dot{a}^\nu - \Omega_{\mu\nu}\bar{\Omega}^{\nu\rho}\left(\frac{\partial \bar{B}}{\partial a^\rho} + \frac{\partial \bar{R}_\rho}{\partial t}\right)\\ =&\Omega_{\mu\nu}\bar{\Omega}^{\nu\rho}\left(\bar{\Omega}_{\rho\tau}\dot{a}^\tau - \frac{\partial \bar{B}}{\partial a^\rho} - \frac{\partial \bar{R}_\rho}{\partial t}\right)\\ =&\Omega_{\mu\nu}\bar{\Omega}^{\nu\rho}\bar{S}_\rho.\end{aligned} \tag{10}$$

Let

$$\bar{\Omega}_{\mu\nu}\Omega^{\nu\rho} = \Lambda^\rho_\mu, \tag{11}$$

and we will have

$$\bar{S}_\mu = \Lambda^\nu_\mu S_\nu, \tag{12}$$

$$\Lambda^\nu_\mu \Omega_{\nu\rho} = \bar{\Omega}_{\mu\rho}. \tag{13}$$

Taking the partial derivative of Eq.(13) with respect to a^τ, we can have

$$\frac{\partial \Lambda^\nu_\mu}{\partial a^\tau}\Omega_{\nu\rho} + \Lambda^\nu_\mu \frac{\partial \Omega_{\nu\rho}}{\partial a^\tau} = \frac{\partial \bar{\Omega}_{\mu\rho}}{\partial a^\tau},$$

and then

$$\begin{aligned}\frac{\partial \Lambda^\nu_\mu}{\partial a^\tau}\Omega_{\nu\rho} =&\frac{\partial^2 \bar{R}_\rho}{\partial a^\mu \partial a^\tau} - \frac{\partial^2 \bar{R}_\mu}{\partial a^\rho \partial a^\tau}\\ &- \Lambda^\nu_\mu\left(\frac{\partial^2 R_\rho}{\partial a^\nu \partial a^\tau} - \frac{\partial^2 R_\nu}{\partial a^\rho \partial a^\tau}\right).\end{aligned}$$

Similarly we have

$$\begin{aligned}\frac{\partial \Lambda^\nu_\mu}{\partial a^\rho}\Omega_{\nu\tau} =&\frac{\partial^2 \bar{R}_\tau}{\partial a^\mu \partial a^\rho} - \frac{\partial^2 \bar{R}_\mu}{\partial a^\tau \partial a^\rho}\\ &- \Lambda^\nu_\mu\left(\frac{\partial^2 R_\tau}{\partial a^\nu \partial a^\rho} - \frac{\partial^2 R_\nu}{\partial a^\tau \partial a^\rho}\right).\end{aligned}$$

Subtracting the second formula from the first formula above yields

$$\frac{\partial \Lambda^\nu_\mu}{\partial a^\tau}\Omega_{\nu\rho} - \frac{\partial \Lambda^\nu_\mu}{\partial a^\rho}\Omega_{\nu\tau} = \frac{\partial \bar{\Omega}_{\tau\rho}}{\partial a^\mu} - \Lambda^\nu_\mu \frac{\partial \Omega_{\tau\rho}}{\partial a^\nu}. \tag{14}$$

By using $\Lambda^\nu_\rho = \bar{\Omega}_{\rho\tau}\Omega^{\tau\nu}$, we can rewrite Eq.(9) as

$$\frac{\partial \bar{B}}{\partial a^\rho} + \frac{\partial \bar{R}_\rho}{\partial t} = \Lambda^\nu_\rho\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right). \tag{15}$$

Taking the derivative of both sides of Eq.(15) with respect to a^τ leads to

$$\begin{aligned}&\frac{\partial^2 \bar{B}}{\partial a^\rho \partial a^\tau} + \frac{\partial^2 \bar{R}_\rho}{\partial t \partial a^\tau}\\ =&\frac{\partial \Lambda^\nu_\rho}{\partial a^\tau}\left(\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t}\right)\\ &+ \Lambda^\nu_\rho\left(\frac{\partial^2 B}{\partial a^\nu \partial a^\tau} + \frac{\partial^2 R_\nu}{\partial t \partial a^\tau}\right).\end{aligned} \tag{16}$$

By virtue of $S_\nu = 0$, we have

$$\frac{\partial B}{\partial a^\nu} + \frac{\partial R_\nu}{\partial t} = \Omega_{\nu\mu}\dot{a}^\mu.$$

Substituting the above into Eq.(16) gives

$$\begin{aligned}&\frac{\partial^2 \bar{B}}{\partial a^\rho \partial a^\tau} + \frac{\partial^2 \bar{R}_\rho}{\partial t \partial a^\tau}\\ =&\frac{\partial \Lambda^\nu_\rho}{\partial a^\tau}\Omega_{\nu\mu}\dot{a}^\mu + \Lambda^\nu_\rho\left(\frac{\partial^2 B}{\partial a^\nu \partial a^\tau} + \frac{\partial^2 R_\nu}{\partial t \partial a^\tau}\right).\end{aligned} \tag{17}$$

According to Eq.(14), we have

$$\begin{aligned}\frac{\partial \Lambda^\nu_\rho}{\partial a^\tau}\Omega_{\nu\mu}\dot{a}^\mu =&\left(\frac{\partial \Lambda^\nu_\rho}{\partial a^\mu}\Omega_{\nu\tau} + \frac{\partial \bar{\Omega}_{\tau\mu}}{\partial a^\rho} - \Lambda^\nu_\rho \frac{\partial \Omega_{\tau\mu}}{\partial a^\nu}\right)\dot{a}^\mu\\ =&\left(\frac{d}{dt}\Lambda^\nu_\rho - \frac{\partial \Lambda^\nu_\rho}{\partial t}\right)\Omega_{\nu\tau}\\ &+ \frac{\partial}{\partial a^\rho}\left(\frac{\partial \bar{B}}{\partial a^\tau} + \frac{\partial \bar{R}_\tau}{\partial t}\right)\\ &- \Lambda^\nu_\rho \frac{\partial}{\partial a^\nu}\left(\frac{\partial B}{\partial a^\tau} + \frac{\partial R_\tau}{\partial t}\right).\end{aligned} \tag{18}$$

Substituting Eq.(18) into Eq.(17) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\Lambda^\nu_\rho\right)\Omega_{\nu\tau} = 0, \tag{19}$$

which can be written in the matrix form as follows:

$$\dot{\Lambda}\Omega = 0. \tag{20}$$

Hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{tr}\Lambda^m\right)=0, \tag{21}$$

where m is a positive integer. We see that Eq.(21) is equivalent to

$$\mathrm{tr}\Lambda^m=\mathrm{const.} \tag{22}$$

Proposition The Birkhoff symmetry of the Birkhoff system will lead to a conserved quantity as shown in expression (22).

4. Example

A Birkhoff system of the fourth order is

$$R_1=a^3,\ \ R_2=a^4,\ \ R_3=R_4=0,$$

$$\begin{aligned}B=&\frac{1}{2}\left(a^3-\frac{1}{b}\arctan bt\right)^2\\&+\frac{1}{2}\left(a^4-\frac{1}{2b}\ln\left(1+b^2t^2\right)\right)^2.\end{aligned} \tag{23}$$

Using this example we study the Birkhoff symmetry and the conserved quantity in the way proposed above.

Take another group dynamical functions as

$$\bar{R}_1=\frac{1}{2}\left(a^3\right)^2,\ \ \bar{R}_2=a^4,\ \ \bar{R}_3=\bar{R}_4=0,$$

$$\begin{aligned}\bar{B}=&\frac{1}{3}\left(a^3\right)^3-\frac{1}{2}\left(a^3\right)^2\frac{1}{b}\arctan bt\\&+\frac{1}{2}\left(a^4-\frac{1}{2b}\ln\left(1+b^2t^2\right)\right)^2,\end{aligned} \tag{24}$$

and we will have

$$\left(\bar{\Omega}_{\mu\nu}\right)=\begin{pmatrix}0&0&-a^3&0\\0&0&0&-1\\a^3&0&0&0\\0&1&0&0\end{pmatrix},$$

and

$$\left(\Omega^{\nu\rho}\right)=\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}.$$

In this case, the formula (9) is verified. Formula (11) gives

$$\left(\Lambda^{\rho}_{\mu}\right)=\begin{pmatrix}0&0&-a^3&0\\0&0&0&-1\\a^3&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}=\begin{pmatrix}a^3&0&0&0\\0&1&0&0\\0&0&a^3&0\\0&0&0&1\end{pmatrix}.$$

By using the proposition above, the conserved quantity is

$$I=a^3=\mathrm{const.} \tag{25}$$

Similarly, by taking the dynamical functions as

$$\bar{R}_1=a^3,\ \ \bar{R}_2=\frac{1}{2}\left(a^4\right)^2,\ \ \bar{R}_3=\bar{R}_4=0,$$

$$\begin{aligned}\bar{B}=&\frac{1}{3}\left(a^4\right)^3-\frac{1}{2}\left(a^4\right)^2\frac{1}{2b}\ln\left(1+b^2t^2\right)\\&+\frac{1}{2}\left(a^3-\frac{1}{b}\arctan bt\right)^2,\end{aligned} \tag{26}$$

the conserved quantity can be written as

$$I=a^4=\mathrm{const.} \tag{27}$$

5. Conclusion

This paper has presented a new symmetry and a new conserved quantity for the Birkhoff system. This new symmetry is different from the other known symmetries, i.e. the Noether symmetry, the Lie symmetry, the form invariance. Under the symmetry proposed in this paper, two groups of dynamical functions lead to the same differential equations, and the conserved quantity deduced from the above symmetry depends only on the two Birkhoff tensors.

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