Vol 15 No 8, August 2006
1009-1963/2006/15(08)/1672-06

Chinese Physics

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Three-order form invariance and conserved quantity

Yang Xue-Hui(杨学慧) and Ma Shan-Jun(马善钧)†

College of Physics and Communication Electronics, Jiang Xi Normal University, Nanchang 330022, China

(Received 9 March 2006; revised manuscript received 24 March 2006)

In this paper, the definition of three-order form invariance is given. Then the relation between the three-order form invariance and the three-order Lie symmetry is discussed and the sufficient and necessary condition of Lie symmetry, which comes from the three-order form invariance, is obtained. Finally a three-order Hojman conserved quantity is studied and an example is given to illustrate the application of the obtained results.

Keywords: three-order form invariance, Hojman conserved quantity, three-order Lie symmetry
PACC: 0320

1. Introduction

The conserved quantities of mechanical systems are considered to be in a very important position in physics. The discoveries of the conservation of energy and the conservation of momentum have exerted a great influence on the development of physics. The intimate relationships between symmetries and conserved quantities are increasingly recognized. Using the theory of mechanical symmetry, we can find many new conserved quantities. At present there exist mainly three methods to study conserved quantities, i.e. Noether symmetry, Lie symmetry and form invariance. The studies of Noether symmetry, which was given by German scientist Noether,[1] and Lie symmetry have made considerable progress. In recent years, more and more attention is paid to the study on form invariance. The form invariance of a Lagrangian system was first given by Mei.[2] Then Wang and Mei [3] studied the relation between the form invariance and Lie symmetry for a non-holonomic system. Zhang and Mei [4] gave the definition and criterion of form invariance in a generalized classical mechanical system. Luo *et al*[5] obtained the determining equation, constraint restriction equation and additional restriction equation of form invariance for a non-holonomic system under the special infinitesimal transformation of time invariance. Mei and Xu [6] analysed the form invariance for Lagrangian systems and gave Lutzky conserved quantities for the systems. Xia *et al*[7] discussed the problems on Lie-form invariance of non-holonomic mechanical systems.

The Hojman conserved quantity is a new conserved quantity that was given by Hojman[8] in 1992. Because the Hojman conserved quantity depends on neither Lagrangian function nor Hamiltonian function it is obviously different from the traditional Noether conserved quantity. Some researches were carried out. Using the invariance of differential equation under the infinitesimal transformation, Zhang[9] established the Lie symmetry which depends on infinitesimal transformation of canonical variable in a multi-dimensional phase space and obtained the generalized Hojman conserved quantity. Fang *et al*[10] made study on Hojman conserved quantity of a mechanical system with variable mass under the infinitesimal transformation in phase space. Zhang *et al*[11] used the general Lie symmetrical group in which time and generalized coordinates are variable to construct the generalized Hojman conserved quantity.

All theories are based on the two-order differential equation of motion. In this paper, by using the three-order differential equation the three-order form invariance is determined and the sufficient and necessary condition is discussed when the three-order form invariance becomes the Lie symmetry. Finally a three-order generalized Hojman conserved quantity is obtained and an example is given to illustrate the application of our results.

†Corresponding author, E-mail: shanjun@nc.jx.cn

2. Three-order form invariance

For a holonomic mechanical system that includes N particles, the three-order differential equation of motion can be written as follows:

$$E_\alpha^*(S) = Q_\alpha^*(t, q, \dot{q}, \ddot{q}), \quad (\alpha = 1, 2, 3, \cdots, n), \quad (1)$$

where $E_\alpha^* = \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial}{\partial \dddot{q}_\alpha} - \dfrac{1}{2}\dfrac{\partial}{\partial \ddot{q}_\alpha}$, $Q_\alpha^* = \sum\limits_{i=1}^{N} \dot{\boldsymbol{F}}_i \cdot \dfrac{\partial \boldsymbol{r}_i}{\partial q_\alpha}$, $S = \dfrac{1}{2}\sum\limits_{i=1}^{N} m_i \ddot{\boldsymbol{r}}_i \cdot \ddot{\boldsymbol{r}}_i$ is the acceleration energy. If infinitesimal transformations are

$$\bar{t} = t + \Delta t, \quad (2)$$

$$\bar{q}_\alpha = q_\alpha + \Delta q_\alpha, \quad (3)$$

their expansive formulas are

$$\bar{t} = t + \varepsilon\tau(t, q, \dot{q}, \ddot{q}), \quad (4)$$

$$\bar{q}_\alpha = q_\alpha + \varepsilon\xi_\alpha(t, q, \dot{q}, \ddot{q}), \quad (5)$$

where τ, ξ_α are referred to as functions of infinitesimal transformation, ε is an infinitesimal parameter. Under the infinitesimal transformations of Eqs.(2) and (3), S and Q_α^* have respectively the forms of $\bar{S} = S\left(\bar{t}, \bar{q}, \dfrac{\mathrm{d}\bar{q}}{\mathrm{d}\bar{t}}, \dfrac{\mathrm{d}^2\bar{q}}{\mathrm{d}\bar{t}^2}\right)$ and $\bar{Q}_\alpha^* = Q_\alpha^*\left(\bar{t}, \bar{q}, \dfrac{\mathrm{d}\bar{q}}{\mathrm{d}\bar{t}}, \dfrac{\mathrm{d}^2\bar{q}}{\mathrm{d}\bar{t}^2}\right)$.

Definition: Under the infinitesimal transformations of Eqs.(2) and (3), if the form of equation of motion (1) does not change, i.e.

$$E_\alpha^*(\bar{S}) = \bar{Q}_\alpha^*, \quad (6)$$

then the invariance is termed the form invariance of three-order differential equation of holonomic mechanical system.

Now introducing operators

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_\alpha\frac{\partial}{\partial q_\alpha} + \ddot{q}_\alpha\frac{\partial}{\partial \dot{q}_\alpha} + q_\alpha\frac{\partial}{\partial \ddot{q}_\alpha}, \quad (7)$$

$$X^{(0)} = \frac{\partial}{\partial t} + \xi_k\frac{\partial}{\partial q_k}, \quad (8)$$

$$X^{(1)} = X^{(0)} + \left(\dot{\xi}_k - \dot{q}_k\dot{\tau}\right)\frac{\partial}{\partial \dot{q}_k}, \quad (9)$$

$$X^{(2)} = X^{(1)} + \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\xi}_k - \dot{q}_k\dot{\tau}\right) - \ddot{q}_k\dot{\tau}\right]\frac{\partial}{\partial \ddot{q}_k}, \quad (10)$$

$$X^{(3)} = X^{(2)} + \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\ddot{\xi}_k - 2\ddot{q}_k\dot{\tau} - \dot{q}_k\ddot{\tau}\right) - \dddot{q}_k\,\dot{\tau}\right]\frac{\partial}{\partial q_k}, \quad (11)$$

where $k = 1, 2, 3, \cdots, n$, we have

$$\begin{aligned}\bar{S} =& S\left(\bar{t}, \bar{q}, \frac{\mathrm{d}\bar{q}}{\mathrm{d}\bar{t}}, \frac{\mathrm{d}^2\bar{q}}{\mathrm{d}\bar{t}^2}\right)\\ =& S(t, q, \dot{q}, \ddot{q}) + \varepsilon X^{(2)}(S) + O\left(\varepsilon^2\right),\end{aligned} \quad (12)$$

$$\begin{aligned}\bar{Q}_\alpha^* =& Q_\alpha^*\left(\bar{t}, \bar{q}, \frac{\mathrm{d}\bar{q}}{\mathrm{d}\bar{t}}, \frac{\mathrm{d}^2\bar{q}}{\mathrm{d}\bar{t}^2}\right)\\ =& Q_\alpha^*(t, q, \dot{q}, \ddot{q}) + \varepsilon X^{(2)}(Q_\alpha^*) + \mathrm{O}\left(\varepsilon^2\right).\end{aligned} \quad (13)$$

Substituting Eqs.(12) and (13) into Eq.(6) and considering Eq.(1), we can write the form invariance of three-order differential equation as

$$E_\alpha^*\left(X^{(2)}(S)\right) = X^{(2)}(Q_\alpha^*). \quad (14)$$

3. Three-order form invariance and three-order Lie symmetry

Next we discuss the relation between the form invariance of three-order differential equation and Lie symmetry. If the determinant $\det\left|\dfrac{\partial^2 S}{\partial \ddot{q}_\alpha \partial \ddot{q}_k}\right| \neq 0$ is satisfied, Eq.(1) can be written as

$$q_\alpha = A_\alpha(t, q, \dot{q}, \ddot{q}). \quad (15)$$

On the other hand, from Eq.(1), we can obtain

$$f_\alpha\left(t, q, \dot{q}, \ddot{q}, \dddot{q}\right) = E_\alpha^*(S) - Q_\alpha^* = 0. \quad (16)$$

Through the transformation of Lie symmetry, Eq.(16) becomes

$$\bar{f}_\alpha = f_\alpha\left(\bar{t}, \bar{q}, \frac{\mathrm{d}\bar{q}}{\mathrm{d}\bar{t}}, \frac{\mathrm{d}^2\bar{q}}{\mathrm{d}\bar{t}^2}, \frac{\mathrm{d}^3\bar{q}}{\mathrm{d}\bar{t}^3}\right) = 0. \quad (17)$$

Expanding the above equation, one has

$$\begin{aligned}& f_\alpha\left(\bar{t}, \bar{q}, \frac{\mathrm{d}\bar{q}}{\mathrm{d}\bar{t}}, \frac{\mathrm{d}^2\bar{q}}{\mathrm{d}\bar{t}^2}, \frac{\mathrm{d}^3\bar{q}}{\mathrm{d}\bar{t}^3}\right)\\ =& f_\alpha(t, q, \dot{q}, \ddot{q}, q)\\ &+ \varepsilon X^{(3)}\left(f_\alpha\left(t, q, \dot{q}, \ddot{q}, \dddot{q}\right)\right) + O\left(\varepsilon^2\right).\end{aligned} \quad (18)$$

Substituting Eqs.(16) and (18) into Eq.(17), one can have

$$X^{(3)}(E_\alpha^*(S)) = X^{(2)}(Q_\alpha^*), \quad (19)$$

or[12]

$$\dddot{\xi}_\alpha - \ddot{q}_\alpha\,\dddot{\tau} - 3\ddot{q}_\alpha\ddot{\tau} - 3A_\alpha\dot{\tau} = X^{(2)}(A_\alpha). \quad (20)$$

Equation (19) is the determining equation of three-order Lie symmetry. Comparing Eq.(14) with Eq.(19), we find that the terms on the right-hand sides of the two equations are identical. Expanding the term on the left-hand side of Eq.(14), we have

$$
\begin{aligned}
&E_\alpha^*\left(X^{(2)}\left(S\right)\right)\\
=&\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\ddot{q}_\alpha}-\frac{1}{2}\frac{\partial}{\partial\dot{q}_\alpha}\right)\left\{\tau\frac{\partial S}{\partial t}+\xi_k\frac{\partial S}{\partial q_k}+\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)\frac{\partial S}{\partial\dot{q}_k}+\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)-\ddot{q}_k\dot{\tau}\right]\frac{\partial S}{\partial\ddot{q}_k}\right\}\\
=&E_\alpha^*\left(t\right)\frac{\partial S}{\partial t}+\tau E_\alpha^*\left(\frac{\partial S}{\partial t}\right)+\dot{\tau}\frac{\partial^2 S}{\partial\ddot{q}_\alpha\partial t}+\frac{\partial\tau}{\partial\ddot{q}_\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial t}\right)+E_\alpha^*\left(\xi_k\right)\frac{\partial S}{\partial q_k}+\xi_k E_\alpha^*\left(\frac{\partial S}{\partial q_k}\right)+\dot{\xi}_k\frac{\partial^2 S}{\partial\ddot{q}_\alpha\partial q_k}\\
&+\frac{\partial\xi_k}{\partial\ddot{q}_\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial q_k}\right)+E_\alpha^*\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)\frac{\partial S}{\partial\dot{q}_k}+\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)E_\alpha^*\left(\frac{\partial S}{\partial\dot{q}_k}\right)+\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)\frac{\partial^2 S}{\partial\ddot{q}_\alpha\partial\dot{q}_k}\\
&+\frac{\partial}{\partial\ddot{q}_\alpha}\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial\dot{q}_k}\right)+E_\alpha^*\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)-\ddot{q}_k\dot{\tau}\right]\frac{\partial S}{\partial\ddot{q}_k}+\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)-\ddot{q}_k\dot{\tau}\right]E_\alpha^*\left(\frac{\partial S}{\partial\ddot{q}_k}\right)\\
&+\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)-\ddot{q}_k\dot{\tau}\right]\frac{\partial^2 S}{\partial\ddot{q}_\alpha\partial\ddot{q}_k}+\frac{\partial}{\partial\ddot{q}_\alpha}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)-\ddot{q}_k\dot{\tau}\right]\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial\ddot{q}_k}\right)\\
=&X^{(3)}\left(E_\alpha^*\left(S\right)\right)+E_\alpha^*\left(\tau\right)\frac{\partial S}{\partial t}+E_\alpha^*\left(\xi_k\right)\frac{\partial S}{\partial q_k}+E_\alpha^*\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)\frac{\partial S}{\partial\dot{q}_k}+E_\alpha^*\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)-\ddot{q}_k\dot{\tau}\right]\frac{\partial S}{\partial\ddot{q}_k}\\
&+\frac{\partial\tau}{\partial\ddot{q}_\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial t}\right)+\frac{\partial\xi_k}{\partial\ddot{q}_\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial q_k}\right)+\frac{\partial}{\partial\ddot{q}_\alpha}\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial\dot{q}_k}\right)\\
&+\left\{\frac{\partial}{\partial\ddot{q}_\alpha}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)\right]-\ddot{q}_k\frac{\partial\dot{\tau}}{\partial\ddot{q}_\alpha}\right\}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial\ddot{q}_k}\right).
\end{aligned}
\tag{21}
$$

Comparing Eq.(21) with Eq.(19), we can obtain the sufficient and necessary condition as follows:

$$
\begin{aligned}
&E_\alpha^*\left(\tau\right)\frac{\partial S}{\partial t}+E_\alpha^*\left(\xi_k\right)\frac{\partial S}{\partial q_k}+E_\alpha^*\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)\frac{\partial S}{\partial\dot{q}_k}+E_\alpha^*\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)-\ddot{q}_k\dot{\tau}\right]\frac{\partial S}{\partial\ddot{q}_k}\\
&+\frac{\partial\tau}{\partial\ddot{q}_\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial t}\right)+\frac{\partial\xi_k}{\partial\ddot{q}_\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial q_k}\right)+\frac{\partial}{\partial\ddot{q}_\alpha}\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial\dot{q}_k}\right)\\
&+\left\{\frac{\partial}{\partial\ddot{q}_\alpha}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\xi}_k-\dot{q}_k\dot{\tau}\right)\right]-\ddot{q}_k\frac{\partial\dot{\tau}}{\partial\ddot{q}_\alpha}\right\}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial\ddot{q}_k}\right)=0.
\end{aligned}
\tag{22}
$$

If the system is considered to have three-order form invariance and this condition is satisfied, one can conclude that the system has a three-order Lie symmetry.

4. Three-order generalized Hojman conserved quantity

Considering the above conclusion and using the property of Lie symmetry, we can study the conserved quantity of the system when the system has the three-order form invariance. Substituting Eq.(15) into Eqs.(7)–(11) yields

$$
\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}=\frac{\partial}{\partial t}+\dot{q}_\alpha\frac{\partial}{\partial q_\alpha}+\ddot{q}_\alpha\frac{\partial}{\partial\dot{q}_\alpha}+A_\alpha\frac{\partial}{\partial\ddot{q}_\alpha},
\tag{23}
$$

$$
\tilde{X}^{(0)}=\tau\frac{\partial}{\partial t}+\xi_k\frac{\partial}{\partial q_k},
\tag{24}
$$

$$
\tilde{X}^{(1)}=\tilde{X}^{(0)}+\left(\frac{\tilde{\mathrm{d}}\xi_k}{\mathrm{d}t}-\dot{q}_k\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}\right)\frac{\partial}{\partial\dot{q}_k},
\tag{25}
$$

$$
\tilde{X}^{(2)}=\tilde{X}^{(1)}+\left(\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_k-2\ddot{q}_k\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}-\dot{q}_k\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\right)\frac{\partial}{\partial\ddot{q}_k}.
\tag{26}
$$

Equation (20) can be written as follows:

$$
\begin{aligned}
&\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_\alpha-\dot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\\
&-3\ddot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau-3A_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}\\
=&\tilde{X}^{(2)}\left(A_\alpha\right).
\end{aligned}
\tag{27}
$$

Proposition: if a holonomic mechanical system satisfies Eq.(27) with infinitesimal quantities $\tau\left(t,q,\dot{q},\ddot{q}\right)$ and $\xi_\alpha\left(t,q,\dot{q},\ddot{q}\right)$, and there is a function $\mu=\mu\left(t,q,\dot{q},\ddot{q}\right)$ that satisfies the following equations

$$
\frac{\partial A_\alpha}{\partial\ddot{q}_\alpha}+\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\ln\mu=0,
\tag{28}
$$

then the system has the following conserved quantity:

$$\begin{aligned} I^* =&\frac{\partial\tau}{\partial t}+\frac{\partial\xi_\alpha}{\partial q_\alpha}+\frac{\partial}{\partial\dot{q}_\alpha}\left(\frac{\tilde{\mathrm{d}}\xi_\alpha}{\mathrm{d}t}-\dot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\right)\\ &+\frac{\partial}{\partial\ddot{q}_\alpha}\left(\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_\alpha-2\ddot{q}_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}-\dot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\right)\\ &+\tilde{X}^{(2)}\left(\ln\mu\right)-\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}. \end{aligned} \tag{29}$$

Equation (29) is called the three-order generalized Hojman conserved quantity and it can be proved as follows:

$$\begin{aligned} \frac{\tilde{\mathrm{d}}}{\mathrm{d}t}I^* =&\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial\tau}{\partial t}+\frac{\partial\xi_\alpha}{\partial q_\alpha}+\frac{\partial}{\partial\dot{q}_\alpha}\left(\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_\alpha-\dot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\right)\right.\\ &\left.+\frac{\partial}{\partial\ddot{q}_\alpha}\left(\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_\alpha-2\ddot{q}_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}-\dot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\right)\right]\\ &+\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\left(\tilde{X}^{(2)}\left(\ln\mu\right)\right)-\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau. \end{aligned} \tag{30}$$

For a general function $\varphi\left(t,q,\dot{q},\ddot{q}\right)$ we can have

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\varphi}{\partial t}=\frac{\partial}{\partial t}\frac{\tilde{\mathrm{d}}\varphi}{\mathrm{d}t}-\frac{\partial A_k}{\partial t}\frac{\partial\varphi}{\partial\ddot{q}_k}, \tag{31}$$

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\varphi}{\partial q_\alpha}=\frac{\partial}{\partial q_\alpha}\frac{\tilde{\mathrm{d}}\varphi}{\mathrm{d}t}-\frac{\partial A_k}{\partial q_\alpha}\frac{\partial\varphi}{\partial\ddot{q}_k}, \tag{32}$$

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\varphi}{\partial\dot{q}_\alpha}=\frac{\partial}{\partial\dot{q}_\alpha}\frac{\tilde{\mathrm{d}}\varphi}{\mathrm{d}t}-\frac{\partial\varphi}{\partial q_\alpha}-\frac{\partial A_k}{\partial\dot{q}_\alpha}\frac{\partial\varphi}{\partial\ddot{q}_k}, \tag{33}$$

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\partial\varphi}{\partial\ddot{q}_\alpha}=\frac{\partial}{\partial\ddot{q}_\alpha}\frac{\tilde{\mathrm{d}}\varphi}{\mathrm{d}t}-\frac{\partial\varphi}{\partial\dot{q}_\alpha}-\frac{\partial A_k}{\partial\ddot{q}_\alpha}\frac{\partial\varphi}{\partial\ddot{q}_k}, \tag{34}$$

$$\begin{aligned} &\tilde{X}^{(2)}\left(\frac{\partial\varphi}{\partial\ddot{q}_\alpha}\right)\\ =&\frac{\partial}{\partial\ddot{q}_\alpha}\left(\tilde{X}^{(2)}\left(\varphi\right)\right)-\frac{\partial\tau}{\partial\ddot{q}_\alpha}\frac{\partial\varphi}{\partial t}-\frac{\partial\xi_k}{\partial\ddot{q}_\alpha}\frac{\partial\varphi}{\partial q_k}\\ &-\frac{\partial}{\partial\ddot{q}_\alpha}\left(\frac{\tilde{\mathrm{d}}\xi_k}{\mathrm{d}t}-\dot{q}_k\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}\right)\frac{\partial\varphi}{\partial\dot{q}_k}\\ &-\frac{\partial}{\partial\ddot{q}_\alpha}\left(\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_k-2\ddot{q}_k\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}-\dot{q}_k\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\right)\frac{\partial\varphi}{\partial\ddot{q}_k}. \end{aligned} \tag{35}$$

Considering Eqs.(31)–(34) and (27), one has

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tilde{X}^{(2)}\left(\varphi\right)=\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}\varphi}{\mathrm{d}t}+\tilde{X}^{(2)}\left(\frac{\tilde{\mathrm{d}}\varphi}{\mathrm{d}t}\right), \tag{36}$$

$$\begin{aligned} &\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial\tau}{\partial t}+\frac{\partial\xi_\alpha}{\partial q_\alpha}+\frac{\partial}{\partial\dot{q}_\alpha}\left(\frac{\tilde{\mathrm{d}}\xi_\alpha}{\mathrm{d}t}-\dot{q}_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}\right)+\frac{\partial}{\partial\ddot{q}_\alpha}\left(\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_\alpha-2\ddot{q}_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}-\dot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\right)\right]\\ =&\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau+\frac{\partial A_\alpha}{\partial\ddot{q}_\alpha}\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}+\frac{\partial}{\partial\ddot{q}_\alpha}\left(\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_\alpha-3\ddot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau-\dot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau-3A_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}\right)-\frac{\partial A_k}{\partial t}\frac{\partial\tau}{\partial\ddot{q}_k}\\ &-\frac{\partial A_k}{\partial q_\alpha}\frac{\partial\xi_\alpha}{\partial\ddot{q}_k}-\frac{\partial A_k}{\partial\dot{q}_\alpha}\frac{\partial}{\partial\ddot{q}_k}\left(\frac{\tilde{\mathrm{d}}\xi_\alpha}{\mathrm{d}t}-\dot{q}_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}\right)-\frac{\partial A_k}{\partial\ddot{q}_\alpha}\frac{\partial}{\partial\ddot{q}_k}\left(\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_\alpha-2\ddot{q}_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}-\dot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\right). \end{aligned} \tag{37}$$

Combining Eqs.(35) and (37), we have

$$\begin{aligned} \tilde{X}^{(2)}\left(\frac{\partial A_\alpha}{\partial\ddot{q}_\alpha}\right)=&\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial\tau}{\partial t}+\frac{\partial\xi_\alpha}{\partial q_\alpha}+\frac{\partial}{\partial\dot{q}_\alpha}\left(\frac{\tilde{\mathrm{d}}\xi_\alpha}{\mathrm{d}t}-\dot{q}_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}\right)+\frac{\partial}{\partial\ddot{q}_\alpha}\left(\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_\alpha-2\ddot{q}_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}\right.\right.\\ &\left.\left.-\dot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\ \frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\right)\right]-\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau-\frac{\partial A_\alpha}{\partial\ddot{q}_\alpha}\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}. \end{aligned} \tag{38}$$

Finally we can obtain the following equation from Eqs.(38), (28) and (36):

$$\begin{aligned} \frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\left(\tilde{X}^{(2)}\left(\ln\mu\right)\right)=&\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau-\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\left[\frac{\partial\tau}{\partial t}+\frac{\partial\xi_\alpha}{\partial q_\alpha}+\frac{\partial}{\partial\dot{q}_\alpha}\left(\frac{\tilde{\mathrm{d}}\xi_\alpha}{\mathrm{d}t}-\dot{q}_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}\right)\right.\\ &\left.+\frac{\partial}{\partial\ddot{q}_\alpha}\left(\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_\alpha-2\ddot{q}_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}-\dot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\ \frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\right)\right]. \end{aligned} \tag{39}$$

Substituting Eq.(39) into Eq.(30) gives

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}I^* = 0. \tag{40}$$

5. Example

An example of one-dimensional oscillator is given below to show the application of the above result. For a one-dimensional oscillator, we have

$$S = \frac{1}{2}m\dot{q}^2, \quad Q^* = -k\dot{q}. \tag{41}$$

Substituting Eq.(41) into Eq.(1) yields

$$\dddot{q} = -\frac{k}{m}\dot{q}. \tag{42}$$

If the oscillator has the three-order form invariance and satisfies Eq.(22), then the following equation is satisfied

$$\begin{aligned}&\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi - \dot{q}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\\ &- 3\ddot{q}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau + 3\frac{k}{m}\dot{q}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\\ =&-\frac{k}{m}\left(\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi - \dot{q}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\right).\end{aligned} \tag{43}$$

Assuming that ξ and τ are the functions of t, q, $\dot{q}$, then Eq.(43) can be expanded as follows:

$$\begin{aligned}&\frac{\partial^3}{\partial t^3}\xi + 3\dot{q}\frac{\partial^3}{\partial q\partial t\partial t}\xi + 3\ddot{q}\frac{\partial^3}{\partial t\partial\dot{q}\partial t}\xi + 6\ddot{q}\dot{q}\frac{\partial^3}{\partial t\partial q\partial\dot{q}}\xi + 3\ddot{q}\frac{\partial^2}{\partial t\partial q}\xi + 3\dot{q}^2\frac{\partial^3}{\partial t\partial q\partial q}\xi\\ &+ 3\ddot{q}^2\frac{\partial^3}{\partial t\partial\dot{q}\partial\dot{q}}\xi + 3\ddot{q}\dot{q}^2\frac{\partial^3}{\partial q\partial q\partial\dot{q}}\xi + 3\ddot{q}\dot{q}\frac{\partial^2}{\partial q^2}\xi + 3\ddot{q}^2\dot{q}\frac{\partial^3}{\partial q\partial\dot{q}\partial\dot{q}}\xi + \ddot{q}^3\frac{\partial^3}{\partial\dot{q}^3} + 3\ddot{q}^2\frac{\partial^2}{\partial q\partial\dot{q}}\xi\\ &+ \dot{q}^3\frac{\partial^3}{\partial q^3}\xi - \dot{q}\frac{\partial^3}{\partial t^3}\tau - 3\dot{q}^2\frac{\partial^3}{\partial q\partial t\partial t}\tau - 3\ddot{q}\dot{q}\frac{\partial^3}{\partial t\partial\dot{q}\partial t}\tau - 6\ddot{q}\dot{q}^2\frac{\partial^3}{\partial t\partial\dot{q}\partial q}\tau - 3\ddot{q}\dot{q}\frac{\partial^2}{\partial t\partial q}\tau\\ &- 3\dot{q}^3\frac{\partial^3}{\partial t\partial q\partial q}\tau - 3\ddot{q}^2\dot{q}\frac{\partial^3}{\partial t\partial\dot{q}\partial\dot{q}}\tau - 3\ddot{q}\dot{q}^3\frac{\partial^3}{\partial q\partial q\partial\dot{q}}\tau - 3\ddot{q}\dot{q}^2\frac{\partial^2}{\partial q^2}\tau - 3\ddot{q}^2\dot{q}^2\frac{\partial^3}{\partial q\partial\dot{q}\partial\dot{q}}\tau\\ &- \ddot{q}^3\dot{q}\frac{\partial^3}{\partial\dot{q}^3}\tau - 3\ddot{q}^2\dot{q}\frac{\partial^2}{\partial q\partial\dot{q}}\tau - \dot{q}^4\frac{\partial^3}{\partial q^3}\tau - 3\ddot{q}\frac{\partial^2}{\partial t^2}\tau - 6\dot{q}\ddot{q}\frac{\partial^2}{\partial q\partial t}\tau - 6\ddot{q}^2\frac{\partial^2}{\partial\dot{q}\partial t}\tau - 6\ddot{q}^2\dot{q}\frac{\partial^2}{\partial q\partial\dot{q}}\tau\\ &- 3\ddot{q}^2\frac{\partial\tau}{\partial q} - 3\dot{q}^2\ddot{q}\frac{\partial^2}{\partial q^2}\tau - 3\ddot{q}^3\frac{\partial^2}{\partial\dot{q}^2}\tau + 3\frac{k}{m}\dot{q}\frac{\partial\tau}{\partial t} + 3\frac{k}{m}\dot{q}^2\frac{\partial\tau}{\partial q} + 3\frac{k}{m}\dot{q}\ddot{q}\frac{\partial\tau}{\partial\dot{q}}\\ =&-\frac{k}{m}\left(\frac{\partial\xi}{\partial t} + \dot{q}\frac{\partial\xi}{\partial q} + \ddot{q}\frac{\partial\xi}{\partial\dot{q}} - \dot{q}\frac{\partial\tau}{\partial t} - \dot{q}^2\frac{\partial\tau}{\partial q} - \dot{q}\ddot{q}\frac{\partial\tau}{\partial\dot{q}}\right),\end{aligned} \tag{44}$$

where $\beta = \sqrt{\dfrac{k}{m}}$. Equation (44) has the following solution:

$$\begin{aligned}\xi =&-\frac{3}{4}\left(c_1\cos\beta t - c_2\sin\beta t\right)\dot{q}\\ &-\frac{3}{2}\beta\left(c_1\sin\beta t + c_2\cos\beta t\right)q\\ &+\left(c_3\sin\beta t + c_4\cos\beta t\right),\end{aligned} \tag{45}$$

$$\tau = c_1\cos\beta t - c_2\sin\beta t. \tag{46}$$

Substituting Eq.(42) into Eq.(28) gives

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\left(\ln\mu\right) = 0. \tag{47}$$

The solution of above equation is

$$\mu = c_5\left(\beta^2\dot{q}^2 + \ddot{q}^2\right). \tag{48}$$

Substituting Eqs.(45), (46) and (48) into Eq.(32), the oscillator with one dimension has a conserved quantity

$$\begin{aligned}I^* =&\frac{3}{4}\beta\left(c_1\sin\beta t + c_2\cos\beta t\right) + \left[-\frac{3}{4}\left(c_1\cos\beta t - c_2\sin\beta t\right)\ddot{q} + \frac{1}{4}\beta\left(c_1\sin\beta t + c_2\cos\beta t\right)\dot{q}\right.\\ &\left.-\frac{3}{2}\beta^2\left(c_1\cos\beta t - c_2\sin\beta t\right)q + \beta(c_3\cos\beta t - c_4\sin\beta t)\right]\frac{2\beta^2\dot{q}}{\beta^2\dot{q}^2 + \ddot{q}^2}\\ &+\left[2\beta(c_1\sin\beta t + c_2\cos\beta t)\ddot{q} - \frac{5}{4}\beta^2\left(c_1\cos\beta t - c_2\sin\beta t\right)\dot{q} + \frac{3}{2}\beta^3\left(c_1\sin\beta t + c_2\cos\beta t\right)q\right.\\ &\left.-\beta^2\left(c_1\sin\beta t + c_2\cos\beta t\right)\right]\frac{2\ddot{q}}{\beta^2\dot{q}^2 + \ddot{q}^2},\end{aligned} \tag{49}$$

where c_1, c_2, c_3, c_4 and c_5 are constant.

6. Discussion

First, if $\tau\left(t,q,\dot{q},\ddot{q}\right)=0$ and $\xi_\alpha\left(t,q,\dot{q},\ddot{q}\right)\neq 0$, Eq.(27) can be simplified as follows:

$$\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_\alpha-\xi_k\frac{\partial A_\alpha}{\partial q_k}-\frac{\tilde{\mathrm{d}}\xi_k}{\mathrm{d}t}\frac{\partial A_\alpha}{\partial\dot{q}_k}-\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_k\frac{\partial A_\alpha}{\partial\ddot{q}_k}=0. \tag{50}$$

And if $\mu\left(t,q,\dot{q},\ddot{q}\right)$ satisfies Eq.(28) then the three-order conserved quantity is

$$\begin{aligned}I^*=&\frac{1}{\mu}\frac{\partial}{\partial q_\alpha}\left(\mu\xi_\alpha\right)+\frac{1}{\mu}\frac{\partial}{\partial\dot{q}_\alpha}\left(\mu\frac{\tilde{\mathrm{d}}\xi_\alpha}{\mathrm{d}t}\right)\\&+\frac{1}{\mu}\frac{\partial}{\partial\ddot{q}_\alpha}\left(\mu\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\xi_\alpha\right).\end{aligned} \tag{51}$$

Second, if $\xi_\alpha\left(t,q,\dot{q},\ddot{q}\right)=0$ and $\tau\left(t,q,\dot{q},\ddot{q}\right)\neq 0$, Eq.(27) can be simplified as follows:

$$\begin{aligned}&\dot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau+3\ddot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\\&+3A_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}+\tau\frac{\partial A_\alpha}{\partial t}-q_k\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}\frac{\partial A_\alpha}{\partial\dot{q}_k}\\&-\left(2\ddot{q}_k\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}+\dot{q}_k\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\right)\frac{\partial A_\alpha}{\partial\ddot{q}_k}=0.\end{aligned} \tag{52}$$

And if $\mu\left(t,q,\dot{q},\ddot{q}\right)$ satisfies Eq.(28) then the three-order conserved quantity is

$$\begin{aligned}I^*=&\frac{1}{\mu}\frac{\partial}{\partial t}\left(\mu\tau\right)-\frac{1}{\mu}\frac{\partial}{\partial\dot{q}_\alpha}\left(\mu\dot{q}_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}\right)\\&-\frac{1}{\mu}\frac{\partial}{\partial\ddot{q}_\alpha}\left[\mu\left(2\ddot{q}_\alpha\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}+\dot{q}_\alpha\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\frac{\tilde{\mathrm{d}}}{\mathrm{d}t}\tau\right)\right]-\frac{\tilde{\mathrm{d}}\tau}{\mathrm{d}t}.\end{aligned} \tag{53}$$

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