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An inverse problem in analytical dynamics*

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This paper presents an inverse problem in analytical dynamics. The inverse problem is to construct the Lagrangian when the integrals of a system are given. Firstly, the differential equations are obtained by using the time derivative of the integrals. Secondly, the differential equations can be written in the Lagrange equations under certain conditions and the Lagrangian can be obtained. Finally, two examples are given to illustrate the application of the result.

Keywords: inverse problem, analytical dynamics, integral, Lagrangian
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1. Introduction

Three hundred yeas ago, Newton formulated his law of universal gravitation according to Kepler's laws of the motion of planets. This is the oldest and the most famous inverse problem in dynamics. There are many inverse problems in every branch of mechanics. With the developing of science and technology, the inverse problem is increasingly receiving a higher expansion and development. Yet, it was not until the 1960s and 1970s that a general description of the inverse problem in dynamics was given.[1,2] Nowadays, the inverse problem in dynamics has become an elementary problem in aeronautics, rocket dynamics and motion planning theory.[1,2] Some conceptions and methods of the inverse problem in nonholonomic dynamics were given in Refs.[3, 4] and the inverse problem in Birkhoff systems dynamics was studied in Refs.[5, 6]. Other correlative studies were given in Ref.[7]. But, most of the results in the above literatures are only in the form of differential equations of systems; further studies are not performed. In this paper, we hope to construct the Lagrangian when the integrals of a system are given. We can turn the systems into Hamilton systems after the Lagrangian is obtained, which has significant meaning either in mechanics or physics.

2. Construction of differential equations from the given integrals

Consider a mechanics system with n degrees of freedom. Independent consistent n integrals of the system have the form

$$I_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = h_s, \quad (s = 1, \ldots, n), \tag{1}$$

where h_s are constants, and

$$\Delta = \det\left(\frac{\partial I_s}{\partial \dot{q}_k}\right) \neq 0. \tag{2}$$

To get the differential equations from integral (1), we take the derivative of integral (1) with respect to t, and we have

$$\frac{\partial I_s}{\partial t} + \frac{\partial I_s}{\partial q_k}\dot{q}_k + \frac{\partial I_s}{\partial \dot{q}_k}\ddot{q}_k = 0, \quad (s, k = 1, \ldots, n). \tag{3}$$

Then we can get the solution

$$\ddot{q}_s = \frac{\triangle_{ls}}{\triangle}\left(-\frac{\partial I_l}{\partial t} - \frac{\partial I_l}{\partial q_k}\dot{q}_k\right) \equiv f_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{4}$$

where Δ_{ls} is algebraic cofactor of $\triangle$ whose elements are (l, s)

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3. To get Lagrangian from differential equations

We will find the Lagrangian corresponding to Eq.(4). Equation (4) can be expressed in the following form:

$$A_{sk}(t,\boldsymbol{q},\dot{\boldsymbol{q}})\ddot{q}_k+B_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0,\quad (s,k=1,\ldots,n), \tag{5}$$

where

$$B_s=-A_{sk}f_k. \tag{6}$$

To get Lagrangian from Eqs.(5), certain conditions should be satisfied. If Eqs.(5) are self-adjoint, namely if they satisfy the conditions[8]

$$\begin{aligned}
&A_{sk}=A_{ks},\\
&\frac{\partial A_{sk}}{\partial\dot{q}_r}=\frac{\partial A_{rk}}{\partial\dot{q}_s},\\
&\frac{\partial B_s}{\partial\dot{q}_k}+\frac{\partial B_k}{\partial\dot{q}_s}=2\left(\frac{\partial}{\partial t}+\dot{q}_r\frac{\partial}{\partial q_r}\right)A_{sk},\\
&\frac{\partial B_s}{\partial q_k}-\frac{\partial B_k}{\partial q_s}=\frac{1}{2}\left(\frac{\partial}{\partial t}+\dot{q}_r\frac{\partial}{\partial q_r}\right)\left(\frac{\partial B_s}{\partial\dot{q}_k}-\frac{\partial B_k}{\partial\dot{q}_s}\right),\qquad (s,k=1,\ldots,n),
\end{aligned} \tag{7}$$

then the Lagrangian has the following form:

$$L=K(t,\boldsymbol{q},\dot{\boldsymbol{q}})+D_k(t,\boldsymbol{q})\dot{\boldsymbol{q}}_k+C(t,\boldsymbol{q}). \tag{8}$$

The $n+2$ functions K, D_k, C are solutions of the following system of partial differential equations:

$$\begin{aligned}
&\frac{\partial^2K}{\partial\dot{q}_s\partial\dot{q}_k}=A_{sk},\\
&\frac{\partial D_s}{\partial q_k}-\frac{\partial D_k}{\partial q_s}=\frac{1}{2}\left(\frac{\partial B_s}{\partial\dot{q}_k}-\frac{\partial B_k}{\partial\dot{q}_s}\right)+\left(\frac{\partial^2K}{\partial q_s\partial\dot{q}_k}-\frac{\partial^2K}{\partial\dot{q}_s\partial q_k}\right)\equiv Z_{sk}(t,\boldsymbol{q}),\\
&\frac{\partial C}{\partial q_s}=\frac{\partial D_s}{\partial t}-B_s-\frac{\partial K}{\partial q_s}+\frac{\partial^2K}{\partial\dot{q}_s\partial t}+\left[\frac{\partial^2K}{\partial q_s\partial\dot{q}_k}+\frac{1}{2}\left(\frac{\partial B_s}{\partial\dot{q}_k}-\frac{\partial B_k}{\partial\dot{q}_s}\right)\right]\dot{q}_k\equiv W_s(t,\boldsymbol{q}).
\end{aligned} \tag{9}$$

Furthermore we can obtain[8]

$$\begin{aligned}
&K(t,\boldsymbol{q},\dot{\boldsymbol{q}})=\dot{q}_s\int_0^1\mathrm{d}\tau'\left\{\left[\int_0^1\mathrm{d}\tau A_{ks}(t,\boldsymbol{q},\tau\dot{\boldsymbol{q}})\right]\dot{q}_k\right\}(t,\boldsymbol{q},\tau'\dot{\boldsymbol{q}}),\\
&D_s=\left[\int_0^1\mathrm{d}\tau\tau Z_{ks}(t,\boldsymbol{\tau}q)\right]q_k,\\
&C=\left[\int_0^1\mathrm{d}\tau W_k(t,\boldsymbol{\tau}q)\right]q_k,
\end{aligned} \tag{10}$$

then Eqs.(5) have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}=0. \tag{11}$$

Thereby, we obtained the Lagrange equation from Eqs.(5).

4. Example

To illustrate the inverse problem in analytical dynamics above, we give the following examples.

Example 1 The integral of a system with single degree of freedom is

$$I=\frac{1}{2}t^3\dot{q}^2+\frac{1}{2}t^2q\dot{q}+\frac{1}{6}t^3q^6=h, \tag{12}$$

where h is a constant. Let us construct the Lagrangian. Firstly, taking the derivative of integral (12) with respect to t, we have

$$\frac{\mathrm{d}I}{\mathrm{d}t}=\left(t^3\dot{q}+\frac{1}{2}t^2q\right)\left(\ddot{q}+\frac{2}{t}\dot{q}+q^5\right)=0.$$

Then the differential equation is obtained

$$\ddot{q} + \frac{2}{t}\dot{q} + q^5 = 0, \tag{13}$$

Eq.(13) is the renowned Emden equation.

Secondly, we want to get the Lagrangian from Eq.(13). Equation (13) does not satisfy condition (7). Multiplying Eq.(13) by t^2, we have

$$t^2\ddot{q} + 2t\dot{q} + t^2q^5 = 0, \tag{14}$$

Eq.(14) is self-adjoint. From the first equation of the Eq.(10), we can obtain

$$K = \frac{1}{2}t^2\dot{q}^2.$$

Furthermore, by Eq.(10) and Eq.(9) we can obtain

$$Z_{11} = 0, \quad W_1 = -t^2q^5,$$
$$D_1 = 0, \quad C = -\frac{1}{6}t^2q^6.$$

Thus we obtain the Lagrangian

$$L = \frac{1}{2}t^2\dot{q}^2 - \frac{1}{6}t^2q^6. \tag{15}$$

Example 2 The integrals of a system with three degrees of freedom are

$$I_1 = \frac{1}{2}(\dot{q_1}^2 + q_1^2\dot{q_2}^2 + q_1^2\dot{q_3}^2) - \frac{1}{2}q_1^2 = h_1,$$
$$I_2 = q_1^2\dot{q_2} = h_2,$$
$$I_3 = q_1^2\dot{q_3} = h_3, \tag{16}$$

where h_1, h_2, h_3 are constants. Find the Lagrangian of Eqs.(16).

Taking the derivative of integrals (16) with respect to t, we have

$$\dot{q_1}\ddot{q_1} + q_1^2\dot{q_2}\ddot{q_2} + q_1^2\dot{q_3}\ddot{q_3} +$$
$$q_1\dot{q_1}\dot{q_2}^2 + q_1\dot{q_1}\dot{q_3}^2 - q_1\dot{q_1} = 0, \tag{17}$$
$$q_1^2\ddot{q_2} + 2q_1\dot{q_1}\dot{q_2} = 0, \tag{18}$$
$$q_1^2\ddot{q_3} + 2q_1\dot{q_1}\dot{q_3} = 0. \tag{19}$$

The three equations are not adjoint of itself. Substituting $\ddot{q_2}$, $\ddot{q_3}$ obtained by Eqs.(18) and Eqs.(19) into Eqs.(17), we have

$$\ddot{q_1} - q_1(\dot{q_2}^2 + \dot{q_3}^2) - q_1 = 0. \tag{20}$$

It is easy to validate that Eqs.(18), (19) and (20) form a group of equations which are adjoint of themselves. Furthermore, we have

$$A_{11} = 1, \quad A_{22} = q_1^2, \quad A_{33} = q_1^2,$$

$$A_{12} = A_{21}, \quad A_{13} = A_{31} = 0, \quad A_{23} = A_{32} = 0,$$

$$B_1 = -q_1(\dot{q_2}^2 + \dot{q_3}^2) - q_1,$$
$$B_2 = 2q_1\dot{q_1}\dot{q_2}, \quad B_3 = 2q_1\dot{q_1}\dot{q_3}.$$

By Eqs.(10) and (9) we can obtain

$$K(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}(\dot{q_1}^2 + q_1^2\dot{q_2}^2 + q_1^2\dot{q_3}^2),$$

$$D_1 = D_2 = D_3 = 0, \quad W_1 = \frac{1}{2}q_1^2, \quad W_2 = W_3 = 0.$$

Then we obtain the Lagrangian

$$L = \frac{1}{2}(\dot{q_1}^2 + q_1^2\dot{q_2}^2 + q_1^2\dot{q_3}^2) + \frac{1}{2}q_1^2. \tag{21}$$

5. Conclusions

In this paper, we have brought forward an inverse problem in analytical dynamics: namely, to construct the Lagrangian from the given integrals of a system. The corresponding theory for solving it has been established. Of course, not all systems have such a Lagrangian.

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