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Lie-form invariance of non-holonomic systems with unilateral constraints

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In this paper the Lie-form invariance of the non-holonomic systems with unilateral constraints is studied. The definition and the criterion of the Lie-form invariance of the system are given. The generalized Hojman conserved quantity and a new type of conserved quantity deduced from the Lie-form invariance are obtained. Finally, an example is presented to illustrate the application of the results.

Keywords: unilateral constraint, non-holonomic system, symmetry, conserved quantity
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1. Introduction

The research on symmetries and conserved quantities of a mechanical system has important theoretical and practical significance. As to symmetries, there are mainly the Noether symmetry,[1−3] the Lie symmetry[2−10] and the form invariance.[11−16] And as to conserved quantities, there are mainly the Noether conserved quantity,[1−3] the Hojman conserved quantity,[17,18] and a new type of conserved quantity.[19] Mei *et al* gave a unified symmetry of a Lagrangian system[20] which could lead to the three kinds of conserved quantities mentioned above. Wu studied the Lie-form invariance of the Lagrange system,[21] from which two kinds of conserved quantities are found. Ge and Zhang studied the Lie-form invariance of holonomic mechanical systems.[22] In this paper, the Lie-form invariance of non-holonomic systems with unilateral constraints is studied, from which two kinds of conserved quantities are found.

2. Definition and criterion of the Lie-form invariance

We suppose that the configuration of the mechanical system is determined by n generalized coordinates $q_s(s = 1, \ldots, n)$. The motion of the system is subjected to the unilateral non-holonomic constraints of Chetaev type,

$$f_\beta = f_\beta(t, q, \dot{q}) \geq 0, \quad \beta = 1, \ldots, g. \tag{1}$$

Then, the differential equations of the system can be expressed in the form

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}, \\ &\lambda_\beta \geq 0, f_\beta \geq 0, \lambda_\beta f_\beta = 0, \\ &s = 1, \ldots, n; \beta = 1, \ldots, g, \end{aligned} \tag{2}$$

where $L = T - V$ is the Lagrangian of the system, Q_s are the generalized non-potential forces, and λ_β are the constraint multipliers.

We consider the infinitesimal transformations of time and generalized coordinates as

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{3}$$

where ε is the infinitesimal parameter, ξ_0 and ξ_s are the generators of infinitesimal transformations.

We introduce the infinitesimal generator vector

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q_s}, \quad s = 1, \ldots, n. \tag{4}$$

The first extended infinitesimal generator is

$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s}, \quad s = 1, \ldots, n, \tag{5}$$

and the second is

$$\begin{aligned} &X^{(2)} = X^{(1)} + (\ddot{\xi}_s - 2\ddot{q}_s\dot{\xi}_0 - \dot{q}_s\ddot{\xi}_0)\frac{\partial}{\partial \ddot{q}_s}, \\ &s = 1, \ldots, n. \end{aligned} \tag{6}$$

Then, we study an invariance of Eq.(2) under the infinitesimal transformations (3).

If the system is constrained, i.e., constraints (1) take the sign of equality, we have

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0,\quad \beta=1,\ldots,g, \tag{7}$$

λ_β can be expressed as function of $t,\boldsymbol{q},\dot{\boldsymbol{q}}$. Thus, Eqs.(2) can be written as

$$E_s(L)=Q_s+\Lambda_s,\quad s=1,\ldots,n, \tag{8}$$

where

$$\begin{aligned} E_s&=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial\dot{q}_s}-\frac{\partial}{\partial q_s},\\ \Lambda_{\mathrm{s}}&=\lambda_\beta\frac{\partial f_\beta}{\partial\dot{q}_s},\quad s=1,\ldots,n;\beta=1,\ldots,g. \end{aligned} \tag{9}$$

Expanding Eqs.(8), we can seek all the generalized accelerations as

$$\ddot{q}_s=A_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\quad s=1,\ldots,n. \tag{10}$$

If the system is free from the constraints, that is, the sign of inequality in constraints (1) holds strictly, we have

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})>0,\quad s=1,\ldots,n;\beta=1,\ldots,g. \tag{11}$$

The Eqs.(2) become

$$E_s(L)=Q_s,\quad s=1,\ldots,n. \tag{12}$$

Expanding Eqs.(12), we obtain

$$\ddot{q}_s=B_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\quad s=1,\ldots,n. \tag{13}$$

Definition The symmetry that is both a Lie symmetry and a form invariance is called a Lie-form invariance.

For the non-holonomic systems with unilateral constraints, the constrained Eqs.(1) retain the form invariance

$$X^{(1)}(f_\beta)=0$$

for

$$f_\beta=0. \tag{14}$$

The determining equations of the Lie symmetry are

$$\begin{aligned} &X^{(2)}\{E_s(L)\}=X^{(1)}(Q_s+\Lambda_s),X^{(1)}(f_\beta)=0,f_\beta=0;\\ &X^{(2)}\{E_s(L)\}=X^{(1)}(Q_s),f_\beta>0. \end{aligned} \tag{15}$$

The criterion equations of the form invariance are

$$\begin{aligned} &E_s\{X^{(1)}(L)\}=X^{(1)}(Q_s+\Lambda_s),X^{(1)}(f_\beta)=0,f_\beta=0;\\ &E_s\{X^{(1)}(L)\}=X^{(1)}(Q_s),f_\beta>0. \end{aligned} \tag{16}$$

According to the above definition, the following result can be obtained.

Criterion If the generators of infinitesimal transformations ξ_0 and ξ_s satisfy

$$\begin{aligned} &\{X^{(2)}[E_s(L)]-X^{(1)}(Q_s+\Lambda_s)\}^2+\{E_s[X^{(1)}(L)]-X^{(1)}(Q_s+\Lambda_s)\}^2=0,X^{(1)}(f_\beta)=0,f_\beta=0;\\ &\{X^{(2)}[E_s(L)]-X^{(1)}(Q_s)\}^2+\{E_s[X^{(1)}(L)]-X^{(1)}(Q_s)\}^2=0,f_\beta>0, \end{aligned} \tag{17}$$

then the corresponding symmetry is a Lie-form invariance.

3. The conserved quantity deduced from the Lie-form invariance

Under certain conditions, by using the Lie-form invariance, both the generalized Hojman conserved quantity and a new type of conserved quantity can be obtained.

Proposition 1 For the non-holonomic systems with unilateral constraints, if there is a function $\mu=\mu(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ such that

$$\begin{aligned} &\frac{\partial A_s}{\partial\dot{q}_s}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu=0,f_\beta=0;\\ &\frac{\partial B_s}{\partial\dot{q}_s}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu=0,f_\beta>0, \end{aligned} \tag{18}$$

where

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t}&=\frac{\partial}{\partial t}+\dot{q}_s\frac{\partial}{\partial q_s}+A_s\frac{\partial}{\partial\dot{q}_s},f_\beta=0;\\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}&=\frac{\partial}{\partial t}+\dot{q}_s\frac{\partial}{\partial q_s}+B_s\frac{\partial}{\partial\dot{q}_s},f_\beta>0, \end{aligned} \tag{19}$$

then the Lie-form invariance will lead to a generalized Hojman conserved quantity

$$I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial(\mu\xi_0)}{\partial t} + \frac{1}{\mu}\frac{\partial(\mu\xi_s)}{\partial q_s} + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}\left[\mu\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\right] - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = \text{const.} \tag{20}$$

Proof According to the definition, a Lie-form invariance must be a Lie symmetry, by using generalized Hojman's proof method, Proposition 1 can be proved.

Proposition 2 If there is a function $G_{\mathrm{F}} = G_{\mathrm{F}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ such that

$$\begin{aligned}&\tilde{X}^{(1)}\{\tilde{X}^{(1)}(L)\} + \tilde{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \tilde{X}^{(1)}(Q_{\mathrm{s}} + \Lambda_s)(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{F}} = 0, f_\beta = 0;\\ &\tilde{X}^{(1)}\{\tilde{X}^{(1)}(L)\} + \tilde{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \tilde{X}^{(1)}(Q_s)(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{F}} = 0, f_\beta > 0,\end{aligned} \tag{21}$$

where

$$\tilde{X}^{(1)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial \dot{q}_s}, \tag{22}$$

then the Lie-form invariance will lead to a new type of conserved quantity

$$I_{\mathrm{F}} = \tilde{X}^{(1)}(L)\xi_0 + \frac{\partial\tilde{X}^{(1)}(L)}{\partial\dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\mathrm{F}} = \text{const.} \tag{23}$$

Proof According to the definition, a Lie-form invariance must be a form invariance, when a form invariance satisfies condition (21), there is a new type of conserved quantity (23).

4. Illustrative example

If the Lagrangian of a system is

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2), \tag{24}$$

the non-potential generalized forces are

$$Q_1 = Q_2 = 0, \tag{25}$$

and the motion of the material point is subjected to the non-holonomic unilateral constraint

$$f = \dot{q}_1 + bt\dot{q}_2 - bq_2 + t \geq 0, \tag{26}$$

we study the Lie-form invariance and conserved quantities of the system.

The differential equations of motion of the system can be expressed as

$$\begin{aligned}&\ddot{q}_1 = \lambda, \ddot{q}_2 = bt\lambda,\\ &\lambda \geq 0, f \geq 0, \lambda f = 0.\end{aligned} \tag{27}$$

If the system is under the constraint, from Eq.(26) and (27) we have

$$\lambda = -\frac{1}{1+b^2t^2}, \quad \ddot{q}_1 = -\frac{1}{1+b^2t^2}, \quad \ddot{q}_2 = -\frac{bt}{1+b^2t^2}. \tag{28}$$

If the system is free from the constraint, from Eq.(27) we obtain

$$\ddot{q}_1 = \ddot{q}_2 = 0. \tag{29}$$

Take generators as

$$\xi_0 = 0, \xi_1 = 1, \xi_2 = 0, \tag{30}$$

it is easy to prove that the corresponding symmetry is Lie-form invariant.

Expression (18) gives

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{31}$$

form which we obtain

$$\mu = 1 \tag{32}$$

and

$$\mu = \dot{q}_2 - bt\dot{q}_1 + bq_1. \tag{33}$$

From expressions (30) and (33), by using Proposition 1, the conserved quantity expression (20) yields

$$I_{\mathrm{H}} = b(\dot{q}_2 - bt\dot{q}_1 + bq_1)^{-1} = \text{const.} \tag{34}$$

This is a generalized Hojman conserved quantity of the system deduced from the Lie-form invariance.

Expression (21) gives

$$G_{\mathrm{F}} = \dot{q}_2 - bt\dot{q}_1 + bq_1, \tag{35}$$

and expression (23) gives

$$I_{\mathrm{F}} = \dot{q}_2 - bt\dot{q}_1 + bq_1 = \text{const.} \tag{36}$$

This is a new type of conserved quantity of the system deduced from the Lie-form invariance.

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