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Hamilton–Jacobi method for solving ordinary differential equations*

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The Hamilton–Jacobi method for solving ordinary differential equations is presented in this paper. A system of ordinary differential equations of first order or second order can be expressed as a Hamilton system under certain conditions. Then the Hamilton–Jacobi method is used in the integration of the Hamilton system and the solution of the original ordinary differential equations can be found. Finally, an example is given to illustrate the application of the result.

Keywords: differential equation, integration, Hamilton–Jacobi method
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1. Introduction

The Hamilton–Jacobi integration method is an important integration method in analytical mechanics. The method plays a significant role in the process of development from classical mechanics to quantum mechanics. Arnold pointed out that 'Jacobi's theorem reduces solving the system of ordinary differential equations to finding a complete integral of the partial differential equation. It may appear surprising that this "reduction" from the simple to the complicated provides an effective method for solving concrete problems. Nevertheless, it turns out that this is the most powerful method known for exact integration, and many problems which were solved by Jacobi cannot be solved by other methods.'[1] The Hamilton equations in analytical mechanics have an evident physical meaning. Though the general ordinary differential equations that can be expressed as the Hamilton equations do not have an evident physical meaning, the Hamilton–Jacobi method is also effective for them. In order to apply the Hamilton–Jacobi method for solving ordinary differential equations, they must be expressed in the Hamilton form. Naturally, some differential equations can be realized by the Hamilton formularization and the others cannot. The Hamilton–Jacobi method can be used in solving the differential equations provided that they can be realized by the Hamilton formularization. This method belongs to the mechanical methodology in mathematics.[2]

2. Hamilton formularization of first-order differential equations

Consider the $2n$ first-order ordinary differential equations

$$\dot{x}_\mu = f_\mu(t, x_\nu) \quad (\mu, \nu = 1, \ldots, 2n). \tag{1}$$

Multiplying Eqs.(1) by

$$(\omega_{\mu\nu}) = \begin{pmatrix} 0_{n\times n} & 1_{n\times n} \\ -1_{n\times n} & 0_{n\times n} \end{pmatrix}, \tag{2}$$

and summing them over ν, we obtain

$$\omega_{\mu\nu}\dot{x}_\nu = F_\mu(t, x_\nu), \tag{3}$$

where

$$F_\mu = \omega_{\mu\nu} f_\nu. \tag{4}$$

If the functions F_μ satisfy the condition

$$\frac{\partial F_\mu}{\partial x_\nu} = \frac{\partial F_\nu}{\partial x_\mu}, \tag{5}$$

then Eqs.(3) are self-adjoint and they can be expressed in the form[3]

$$\omega_{\mu\nu}\dot{x}_\nu = \frac{\partial H}{\partial x_\mu}, \tag{6}$$

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where H is the Hamiltonian

$$H = x_\mu \int_0^1 F_\mu(t, \tau x_\nu)\mathrm{d}\tau. \tag{7}$$

If the condition (5) is not satisfied, but after multiplying Eqs.(3) by the factor functions $h_{\mu\nu}(t, \boldsymbol{x})$ the condition (5) holds, then Eqs.(1) can also be realized by the Hamilton formularization.

3. Hamilton formularization of second-order differential equations

Consider a system of second-order differential equations

$$A_{sk}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})\ddot{q}_k + B_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \quad s, k = 1, \ldots, n. \tag{8}$$

A necessary and sufficient condition for the system (8) to be self-adjoint is that all the following conditions are satisfied:[3]

$$\begin{aligned}
&A_{sk} = A_{ks},\\
&\frac{\partial A_{sk}}{\partial \dot{q}_r} = \frac{\partial A_{rk}}{\partial \dot{q}_s},\\
&\frac{\partial B_s}{\partial \dot{q}_k} + \frac{\partial B_k}{\partial \dot{q}_s} = 2\left(\frac{\partial}{\partial t} + \dot{q}_r\frac{\partial}{\partial q_r}\right)A_{sk},\\
&\frac{\partial B_s}{\partial q_k} - \frac{\partial B_k}{\partial q_s} = \frac{1}{2}\left(\frac{\partial}{\partial t} + \dot{q}_r\frac{\partial}{\partial q_r}\right)\left(\frac{\partial B_s}{\partial \dot{q}_k} - \frac{\partial B_k}{\partial \dot{q}_s}\right),\\
&s, k, r = 1, \ldots, n.
\end{aligned} \tag{9}$$

The Lagrangian is given by

$$L = K(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) + D_k\dot{q}_k + C(t, \boldsymbol{q}), \tag{10}$$

where the $n + 2$ functions K, D_k and C are a solution of the linear over determined system of partial differential equations

$$\begin{aligned}
&\frac{\partial^2 K}{\partial \dot{q}_s \partial \dot{q}_k} = A_{sk}(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),\\
&\frac{\partial D_k}{\partial q_s} - \frac{\partial D_s}{\partial q_k} = \frac{1}{2}\left(\frac{\partial B_k}{\partial \dot{q}_s} - \frac{\partial B_s}{\partial \dot{q}_k}\right) + \frac{\partial^2 K}{\partial q_k \partial \dot{q}_s} - \frac{\partial^2 K}{\partial \dot{q}_k \partial q_s} \equiv Z_{ks}(t, \boldsymbol{q}),\\
&\frac{\partial C}{\partial q_k} = \frac{\partial D_k}{\partial t} - B_k - \frac{\partial K}{\partial q_k} + \frac{\partial^2 K}{\partial \dot{q}_k \partial t} + \left[\frac{\partial^2 K}{\partial q_k \partial \dot{q}_s} + \frac{1}{2}\left(\frac{\partial B_k}{\partial \dot{q}_s} - \frac{\partial B_s}{\partial \dot{q}_k}\right)\right]\dot{q}_s \equiv W_k(t, \boldsymbol{q}),
\end{aligned} \tag{11}$$

given by

$$\begin{aligned}
&K(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{q}_k \int_0^1 \mathrm{d}\tau' \left\{\left[\int_0^1 \mathrm{d}\tau A_{ks}(t, \boldsymbol{q}, \tau\dot{\boldsymbol{q}})\right]\dot{q}_s\right\}(t, \boldsymbol{q}, \tau'\dot{\boldsymbol{q}}),\\
&D_k = \left[\int_0^1 \mathrm{d}\tau\tau Z_{ks}(t, \tau\boldsymbol{q})\right] q_s,\\
&C = \left[\int_0^1 \mathrm{d}\tau W_k(t, \tau\boldsymbol{q})\right] q_k.
\end{aligned} \tag{12}$$

The Lagrangian is called the ordered direct analytic representation of the system (8), i.e.

$$A_{ks}\ddot{q}_s + B_k = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0. \tag{13}$$

If the system (8) is non self-adjoint, but after multiplying it by the factor functions $g_{sk}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ it becomes self-adjoint, then we have

$$g_{sk}(A_{kr}\ddot{q}_r + B_k) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = 0. \tag{14}$$

The Lagrangian is called the indirect analytic representation of the system (8).

Using the Legendre transformation, we can transform Eqs.(13) or Eqs.(14) into the Hamilton equations

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s}, \tag{15}$$

where

$$H = p_s\dot{q}_s - L, \quad p_s = \frac{\partial L}{\partial \dot{q}_s}. \tag{16}$$

A higher order differential equation can be expressed as a system of differential equations of first order or second order, and it can be realized by the Hamilton formularization under certain conditions.

4. Hamilton–Jacobi method

The Hamilton–Jacobi equation for Eqs.(15) is

$$\frac{\partial S}{\partial t}+H\left(q_s,\frac{\partial S}{\partial q_s},t\right)=0. \tag{17}$$

If a complete integral of Eq.(17)

$$S=S(t,q_s,\alpha_s) \tag{18}$$

is found, then finding the solution of Eqs.(15) is reduced to solving the algebraic equations

$$\frac{\partial S}{\partial \alpha_s}=\beta_s,\quad \frac{\partial S}{\partial q_s}=p_s, \tag{19}$$

where α_s and β_s are constants. The main difficulty in using the Hamilton–Jacobi method consists in finding the complete integral of the Hamilton–Jacobi equation (17). In general, one can apply the method of separation of variables to find the complete integral of the Hamilton–Jacobi equation.[4]

5. Illustrative example

Use the Hamilton–Jacobi method to find the solution of the fourth-order differential equation

$$\overset{(4)}{x}=\frac{4}{3}\overset{\cdots}{x}{}^2\ddot{x}^{-1}. \tag{20}$$

Let $q=\ddot{x}$. Then Eq.(20) becomes

$$\ddot{q}-\frac{4}{3}\dot{q}^2q^{-1}=0. \tag{21}$$

It is non-self-adjoint. Multiplying Eq.(21) by the factor $q^{-8/3}$, we have

$$\ddot{q}q^{-8/3}-\frac{4}{3}\dot{q}^2q^{-11/3}=0.$$

It is self-adjoint and the Lagrangian is

$$L=\frac{1}{2}\dot{q}^2q^{-8/3}.$$

We have

$$p=\frac{\partial L}{\partial \dot{q}}=\dot{q}q^{-8/3},\quad H=\frac{1}{2}p^2q^{8/3}.$$

The Hamilton–Jacobi equation (17) gives

$$\frac{\partial S}{\partial t}+\frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2q^{8/3}=0.$$

Its complete integral has the form

$$S=-ht-3\sqrt{2h}q^{-1/3}.$$

Then we have

$$t_0=\frac{\partial S}{\partial h}=-t-\frac{3}{2}\sqrt{2}h^{-1/2}q^{-1/3},$$
$$p=\frac{\partial S}{\partial q}=\sqrt{2h}q^{-4/3}$$

and

$$q=-h^{-3/2}\left(\frac{\sqrt{2}}{3}\right)^{-3}(t+t_0)^{-3}=\ddot{x}.$$

Integrating it, we obtain

$$x=\frac{1}{2}h^{-3/2}\left(\frac{\sqrt{2}}{3}\right)^{-3}(t+t_0)^{-1}+C_1t+C_2, \tag{22}$$

where h, t_0, C_1 and C_2 are constants.

6. Summary

The Hamilton–Jacobi method is an important integration method in analytical mechanics. In this paper, it is used for solving the ordinary differential equations. When a system of ordinary differential equations of first order or second order can be expressed as a Hamilton system, one can integrate it by using the Hamilton–Jacobi method and then find the solution of the system. The main difficulty of the method is to find the complete integral of the Hamilton–Jacobi equation.

References

[1] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (New York: Springer)

[2] Wu H B and Mei F X 2005 *Chin. Phys.* **14** 2391

[3] Santilli R M 1978 *Foundations of Theoretical Mechanics* I (New York: Springer)

[4] Mei F X, Liu D and Luo Y 1991 *Advanced Analytical Mechanics* (Beijing: Beijing Institute of Technology Press) (in Chinese)