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Algebraic structure and Poisson's theory of mechanico-electrical systems*

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The algebraic structure and Poisson's integral theory of mechanico-electrical systems are studied. The Hamilton canonical equations and generalized Hamilton canonical equations and their the contravariant algebraic forms for mechanico-electrical systems are obtained. The Lie algebraic structure and the Poisson's integral theory of Lagrange mechanico-electrical systems are derived. The Lie algebraic structure admitted and Poisson's integral theory of the Lagrange–Maxwell mechanico-electrical systems are presented. Two examples are presented to illustrate these results.

Keywords: algebraic structure, Poisson integral method, mechanico-electrical system
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1. Introduction

It is well known that many methods, such as Ermakov technique,[1] Lutzky's approach,[5−7] group transformation method,[8−11] dynamical algebraic method,[12−14] and algebraic structure and Poisson's theory for constrained mechanical systems,[15−20] have been developed to seek invariants of mechanical and physical systems.[1−19] Among these methods, the invariants of the mechanical systems studied by using algebraic structure seem to have an extra advantage of a straightforward extension to the corresponding quantum mechanical systems.[21] In fact the algebraic structure and Poisson's theory have been used to study the relativistic Birkhoffian systems by Fu *et al.*[22−24] Recently, a series of the symmetries and conservative quantities have been presented to the mechanico-electrical systems.[25−27] Algebraic structure and Poisson's theory, while applied successfully to mechanico-electrical systems, have not been explored so far. In this paper, we make an effort in this direction and demonstrate the applications of the algebraic structure and Poisson's theory of dynamical systems to mechanico-electrical systems.

2. Generalized Hamilton canonical equations of mechanico-electrical systems

Mechanico-electrical systems are systems in which a mechanical process and an electromagnetic process are coupled. The mechanical part constituting N particles is described by n generalized coordinates q_l $(l = 1, \ldots, n)$, and the electrodynamical part constituting m circuits is described by m electric quantities $e_k (k = 1, \ldots, m)$. Consider a mechanico-electrical system composed of m circuits consisting of linear conductors and capacitors. Let i_k denote the current, u_k the electronic potential, R_k the resistance, and C_k the capacitance. Then the Lagrangian of the mechanico-electrical system is

$$L = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - V(\boldsymbol{q}) - W_e(\boldsymbol{q}, \boldsymbol{e}) + W_m(\boldsymbol{q}, \dot{\boldsymbol{e}}), \quad (1)$$

where $\boldsymbol{q} = \{q_1, q_2, \ldots, q_n\}$, $\dot{\boldsymbol{q}} = \{\dot{q}_1, \dot{q}_2, \ldots \dot{q}_n\}$, and $\boldsymbol{e} = \{e_1, e_2, \ldots, e_m\}$ denote generalized coordinate, generalized velocity and generalized electric quantity respectively, $T = \frac{1}{2} a_s(\boldsymbol{q}) \dot{q}_s$ and $V = \sum_{s=1}^{n} v_s(\boldsymbol{q})$ are the kinetic energy and the potential energy, and the electric field energy and the magnetic field energy of the

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mth circuit are respectively defined by

$$W_{\rm e}=\sum_{k=1}^{p}\frac{1}{2}\frac{e_k^2}{C_k},$$

and

$$W_m=\sum_{k=1}^{p}\sum_{r=1}^{p}\frac{1}{2}L_{kr}i_k i_r=\sum_{k=1}^{p}\sum_{r=1}^{p}\frac{1}{2}L_{kr}\dot{e}_k\dot{e}_r, \quad (2)$$

where $L_{kr}=L_{kr}(\boldsymbol{q})$ is the mutual-inductance between the kth circuit and the rth circuit, $L_{kk}=L_{kk}(\boldsymbol{q})$ is the self-inductance of the kth return circuit, $C_k=C_k(\boldsymbol{q})$ is the capacitance of kth capacitor. We point out that the coupling characteristics of mechanical and electromagnetic processes are expressed in $W_{\rm e}$ and W_m. For simplicity, the repeated suffixes conventionally denote summation in the paper.

Motion of the system is governed by the Lagrange–Maxwell equation

$$\begin{aligned}&\frac{\rm d}{{\rm d}t}\frac{\partial L}{\partial\dot{q}_l}-\frac{\partial L}{\partial q_l}+\frac{\partial F}{\partial\dot{q}_l}=Q'_l,\\&\frac{\rm d}{{\rm d}t}\frac{\partial L}{\partial\dot{e}_k}-\frac{\partial L}{\partial e_k}+\frac{\partial F}{\partial\dot{e}_k}=u_k,\\&(l=1,\ldots,n;k=1,\ldots,m),\end{aligned} \quad (3)$$

where Q'_l is non-potential generalized force, F is the dissipative function

$$F=F_{\rm e}(\dot{\boldsymbol{e}})+F_m(\boldsymbol{q},\dot{\boldsymbol{q}}), \quad (4)$$

where $\dot{\boldsymbol{e}}=\{\dot{e}_1,\dot{e}_2,\ldots,\dot{e}_m\}=\boldsymbol{i}$ denotes the electric current, the electric dissipative function is given by

$$F_{\rm e}=\frac{1}{2}R_k i_k^2=\frac{1}{2}R_k\dot{e}_k^2,\quad (k=1,\ldots,m), \quad (5)$$

and F_m is the dissipative function of the viscous frictional damping force.

We now take it that the $q_s(s=1,\ldots,n,n+1,\ldots,n+m)$ expresses generalized coordinate in which $q_s(s=1,\ldots,n)$ denotes the component of space coordinate, $q_s(s=n+1,\ldots,n+m)$ denotes the electric component, then Eq.(3) can be written in the form

$$\begin{aligned}&\frac{\rm d}{{\rm d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}+\frac{\partial F}{\partial\dot{q}_s}=Q_s,\\&(s=1,\ldots,n,n+1,\ldots,n+m),\end{aligned} \quad (6)$$

where $Q_s(s=1,\ldots,n)$ denotes non-potential generalized force, and $Q_s(s=n+1,\ldots,n+m)$ denotes generalized electrokinetic potential.

The system (5) is called the Lagrangian mechanico-electrical system satisfying the condition $Q_s-\partial F/\partial\dot{q}_s=0$. Equation (6) yields

$$\begin{aligned}&\frac{\rm d}{{\rm d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}=0,\\&(s=1,\ldots,n,n+1,\ldots,n+m),\end{aligned} \quad (7)$$

where the mechanical and electromagnetic process are also coupled with each other.

Assume that the generalized momentum of mechanico-electrical system is

$$p_s=\frac{\partial L}{\partial\dot{q}_s}\quad (s=1,\ldots,n,n+1,\ldots,n+m), \quad (8)$$

and introduce the Hamiltonian of the system

$$\begin{aligned}H(t,\boldsymbol{q},\dot{\boldsymbol{q}})=&p_s\dot{q}_s-L\\=&p_s\dot{q}_s(t,\boldsymbol{q},\boldsymbol{p})-L(t,q_s,\dot{q}_s(t,\boldsymbol{q},\boldsymbol{p}))\\=&H(t,\boldsymbol{q},\boldsymbol{p}),\end{aligned} \quad (9)$$

where $\boldsymbol{p}=\{p_1,p_2,\ldots,p_n\}$ denotes generalized momentum, then Eq.(6) can be shown by

$$\begin{aligned}&\dot{q}_s=\frac{\partial H}{\partial p_s},\\&\dot{p}_s=-\frac{\partial H}{\partial q_s}+(Q_s-\frac{\partial F}{\partial\dot{q}_s})\,\big|_{\dot{\boldsymbol{q}}_s=\dot{\boldsymbol{q}}_s(t,\boldsymbol{q},\boldsymbol{p})}\,,\\&(s=1,\ldots,n,n+1,\ldots,n+m),\end{aligned} \quad (10)$$

which is called the generalized Hamilton canonical equation of mechanico-electrical system. Equation (7) can be written as

$$\begin{aligned}&\dot{q}_s=\frac{\partial H}{\partial p_s},\quad \dot{p}_s=-\frac{\partial H}{\partial q_s},\\&(s=1,\ldots,n,n+1,\ldots,n+m),\end{aligned} \quad (11)$$

which are called the Hamilton canonical equation of Lagrangian mechanico-electrical system. In Eqs.(10) and (11), $\boldsymbol{q}$=$\{q_1,\ldots,q_n,q_{n+1},\ldots,q_{n+m}\}$, and $\boldsymbol{p}=\{p_1,\ldots,p_n,p_{n+1},\ldots,p_{n+m}\}$ denote the generalized coordinate and momentum respectively.

3. Contravariant algebraic form of mechanico-electrical systems

Introduce contravariant vectors

$$a^\mu=\begin{cases}q^\mu, & (\mu=1,\ldots,n,n+1,\ldots,n+m),\\ p_{\mu-(n+m)}, & (\mu=n+m+1,\ldots,2(n+m)),\end{cases} \quad (12)$$

then Hamiltonian of the system will be written in the form

$$H(t,\boldsymbol{q},\boldsymbol{p})=H(t,a^\mu). \quad (13)$$

Using the contravariant tensor

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0_{(n+m)\times(n+m)} & I_{(n+m)\times(n+m)} \\ -I_{(n+m)\times(n+m)} & 0_{(n+m)\times(n+m)} \end{pmatrix}, \quad (14)$$

The Hamilton canonical equation (11) can be expressed in the contravariant algebraic form

$$\begin{aligned} &\dot{a}^{\mu} - \omega^{\mu\nu}\frac{\partial H}{\partial a^{\nu}} = 0, \\ &(\mu,\nu = 1,\ldots,n+m,\ldots, \\ &n+m, n+m+1,\ldots,2n+2m). \end{aligned} \quad (15)$$

For generalized Hamilton canonical equation (10) of Lagrange–Maxwell mechanico-electrical system, we let

$$\begin{aligned} &\left(Q_s - \frac{\partial F}{\partial \dot{q}_s}\right)|_{q_s = q_s(t,q,p)} = \tilde{Q}_s = -\Omega_{sk}\frac{\partial H}{\partial p_k}, \\ &(s,k = 1,\ldots,n,n+1,\ldots,n+m), \end{aligned} \quad (16)$$

where

$$(\Omega_{sk}) = \begin{pmatrix} \Omega_{11} & 0 & \cdots & 0 \\ 0 & \Omega_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Omega_{(n+m)(n+m)} \end{pmatrix}, \quad (17)$$

then Eq. (10) can be expressed in the contravariant algebraic form.

$$\begin{aligned} &\dot{a}^{\mu} - S^{\mu\nu}\frac{\partial H}{\partial a^{\nu}} = 0, \\ &(\mu,\nu = 1,\ldots,n,n+1,\ldots, \\ &n+m, n+m+1,\ldots,2(n+m)), \end{aligned} \quad (18)$$

$$S^{\mu\nu} = \omega^{\mu\nu} + T^{\mu\nu}, \quad (19)$$

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0_{(n+m)\times(n+m)} & I_{(n+m)\times(n+m)} \\ -I_{(n+m)\times(n+m)} & 0_{(n+m)\times(n+m)} \end{pmatrix}, \quad (20)$$

$$(T^{\mu\nu}) = \begin{pmatrix} 0_{(n+m)\times(n+m)} & 0_{(n+m)\times(n+m)} \\ 0_{(n+m)\times(n+m)} & -\Omega_{kk} \end{pmatrix}. \quad (21)$$

It is obvious that the tensor $S^{\mu\nu}$ is composed of anti-symmetrical tensor $\omega^{\mu\nu}$ and symmetrical tensor $T^{\mu\nu}$.

For example, the Hamiltonian of mechanico-electrical system

$$H = \frac{1}{2}\frac{p_1^2}{m} + \frac{1}{2}\frac{p_2^2}{L_1(x)} + \frac{1}{2}kx^2 - mgx,$$

subjected to non-potential generalized forces $Q_1 = c_1 x\dot{x}$, and $Q_2 = c_2\dot{e}$, dissipative functions $F_{\mathrm{e}} = \frac{1}{2}c_3\dot{e}^2 - c_4\dot{e}$, and $F_m = c_5 x\dot{x} + c_6\dot{x}^2$, where $p_1 = m\dot{x}$, and $p_2 = L_1(x)\dot{e}$ denote generalized momentums, m denotes mass, k and c_l $(l = 1,\ldots,5)$ are constants.

Substituting H into Eq.(16) yields

$$\begin{aligned} \tilde{Q}_1 &= c_1\frac{p_1}{m}x - c_5 x - 2c_6\frac{p_1}{m} = -\Omega_{1k}\frac{\partial H}{\partial p_k}, \\ \tilde{Q}_2 &= (c_2 - c_3)\frac{p_2}{L_1(x)} + c_4 = -\Omega_{2k}\frac{\partial H}{\partial p_k}, \end{aligned}$$

then

$$\begin{aligned} -\Omega_{11} &= c_1 x - c_5\frac{m}{p_1}x + 2c_6, \\ -\Omega_{22} &= -(c_3 - c_2) + c_4\frac{L_1(x)}{p_2}, \end{aligned}$$

$$(\omega^{\mu\nu}) = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \end{pmatrix}, \quad T^{\mu\nu} = \begin{pmatrix} 0\ 0 & 0 & 0 \\ 0\ 0 & 0 & 0 \\ 0\ 0\ c_1 x - c_5\frac{m}{p_1}x + 2c_6 & 0 \\ 0\ 0 & 0 & -(c_3 - c_2) + c_4\frac{L_1(x)}{p_2} \end{pmatrix}.$$

Substituting $\omega^{\mu\nu}$ and $T^{\mu\nu}$ into Eq.(18) leads to the contravariant algebraic form of the mechanico-electrical system

$$\begin{aligned} \dot{a}_1 &= \frac{p_1}{m}, \quad \dot{a}_2 = \frac{p_2}{L_1(x)}, \\ \dot{a}_3 &= \frac{1}{2L_1^2}\frac{\partial L_1}{\partial x}p_2^2 - kx + mg + \left(c_1 x - c_5\frac{m}{p_1}x + 2c_6\right)\frac{p_1}{m}, \\ \dot{a}_4 &= \left[-(c_3 - c_2) + c_4\frac{L_1(x)}{p_2}\right]\frac{p_2}{L_1(x)}. \end{aligned}$$

4. Algebraic structure of mechanico-electrical systems

Firstly, we come to study the algebraic structure of Lagrangian mechanico-electrical systems.

Performing full derivative of function $A(\boldsymbol{a})$ along the Eq.(12), one has

$$\dot{A} = \frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} \quad (\mu,\nu = 1,\ldots,n,n+1,\ldots, n+m,n+m+1,\ldots,2(n+m)), \tag{22}$$

the right-hand side of Eq.(22) is defined as a double-linear product $A \circ H$, i.e.

$$\frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} = A \circ H, \tag{23}$$

which satisfies the right-hand assignment law, left-hand assignment law and scalar law, so Eq.(23) possesses the compatible algebraic structure.

Expanding Eq.(23) yields

$$\frac{\partial A}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} = \frac{\partial A}{\partial q^k}\frac{\partial H}{\partial p_k} - \frac{\partial A}{\partial p_k}\frac{\partial H}{\partial q^k}, \tag{24}$$

which is the classical Poisson's bracket $(A,\ B)$, i.e. $(A,B) = A \circ B$. It is well known that Poisson bracket possesses the anti-symmetrical property

$$A \circ B + B \circ A = 0, \tag{25}$$

and satisfies the Jacobi identical equation

$$A \circ\ (B\ \circ C) + B \circ\ (C \circ\ A) + C\ \circ (A\ \circ B) = 0. \tag{26}$$

Equations (25) and (26) are also called Lie algebra axiom. Then one has the following theorem.

Theorem 1 Equations of motion of Lagrangian mechanico-electrical systems possess compatible algebraic structure as well as Lie algebraic structure.

Secondly, one presents the algebraic structure of Lagrange–Maxwell mechanico-electrical systems.

We perform full derivative of function $A\ (\boldsymbol{a})$ along Eq.(18), and this derivative is defined as a product

$$\dot{A}(\boldsymbol{a}) = \frac{\partial A}{\partial a^\mu}S^{\mu\nu}\frac{\partial H}{\partial a^\nu} \overset{\text{ref}}{=\!=} A \circ\ H, \tag{27}$$

this product also satisfies the right-hand assignment law, left-hand assignment law and scalar law, and then Eq.(18) possesses the compatible algebraic structure.

If Eq.(18) possesses the Lie algebraic structure, then Eq.(27) satisfies Eqs.(25) and (26). When considering Eqs.(19)–(21), Eqs.(25) and (26) lead to conditions with respect to $T^{\mu\nu}$

$$T^{\mu\nu} + T^{\nu\mu} = 0, \tag{28}$$

$$T^{\tau\rho}\frac{\partial T^{\mu\nu}}{\partial a^\rho} + T^{\mu\rho}\frac{\partial T^{\nu\tau}}{\partial a^\rho} + T^{\nu\rho}\frac{\partial T^{\tau\mu}}{\partial a^\rho} = 0, \quad (\mu,\nu,\tau,\rho = 1,\ldots,n,n+1,\ldots, n+m,n+m+1,\ldots,2(n+m)). \tag{29}$$

Further, we have

$$\Omega_{kk} = 0, \quad (k = 1,\ldots,n,n+1,\ldots,n+m), \tag{30}$$

$$Q_s = 0 \quad (s = 1,\ldots,n,n+1,\ldots,n+m). \tag{31}$$

Therefore, Eq.(18) having no Lie algebraic structure, we have

Theorem 2 The equations of motion of Lagrange–Maxwell mechanico-electrical systems do not have Lie algebraic structure.

We can prove that the Lagrange–Maxwell mechanico-electrical systems possess the Lie algebraic structure admitted. For Eq.(27), we define a new product

$$[A,B] \overset{\text{ref}}{=\!=} A\ \circ B - B \circ\ A, \tag{32}$$

which has anti-symmetrical property

$$[A,B] + [B,A] = 0, \tag{33}$$

and satisfies the Jacobi identical equation

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0. \tag{34}$$

Expanding Eq.(34), the $S^{\mu\nu}$ satisfies the following forms:

$$(S^{\mu\rho} - S^{\rho\mu})\frac{\partial}{\partial a^\rho}(S^{\nu\tau} - S^{\tau\nu}) + (S^{\nu\rho} - S^{\rho\nu})\frac{\partial}{\partial a^\rho}(S^{\tau\mu} - S^{\mu\tau}) + (S^{\tau\rho} - S^{\rho\tau})\frac{\partial}{\partial a^\rho}(S^{\mu\nu} - S^{\nu\mu}) = 0, \quad (\mu,\nu,\tau,\rho = 1,\ldots,n,n+1,\ldots, n+m,n+m+1,\ldots,2(n+m)). \tag{35}$$

Using Eqs.(19)–(21), Eq.(35) leads to

$$(T^{\mu\rho} - T^{\rho\mu})\frac{\partial}{\partial a^\rho}(T^{\nu\tau} - T^{\tau\nu}) + (T^{\nu\rho} - T^{\rho\nu})\frac{\partial}{\partial a^\rho}(T^{\tau\mu} - T^{\mu\tau}) + (T^{\tau\rho} - T^{\rho\tau})\frac{\partial}{\partial a^\rho}(T^{\mu\nu} - T^{\nu\mu}) = 0. \tag{36}$$

where $T^{\mu\nu}$ is a symmetrical tensor, then Eq.(36) is explicit. The new product (32) possesses Lie algebraic structure, then Eq.(27) possesses Lie algebraic structure accepted. We have the following theorem.

Theorem 3 Equations of motion of Lagrange–Maxwell mechanico-electrical systems possess the compatible algebraic structure as well as Lie algebraic structure accepted.

5. Poisson's theory of mechanico-electrical systems

It is well known that there are two main contents in Poisson's integral theory of conservative holonomic systems: one is the Poisson conditions on the first integral; the other is the Poisson theorem. Poisson's integral theory also points out that the Poisson bracket of the two first integrals, which are not involutive, is also a first integral. We now give the generalization of the Poisson integral theory in mechanico-electrical systems.

5.1. Poisson's theory of Lagrangian mechanico-electrical systems

We know that the theoretical foundation of Poisson's integral method is equations of motion of systems and possesses Lie algebraic structure.[28] Lagrangian mechanico-electrical system (11) possesses Lie algebraic structure, so the Poisson integral methods of conservative holonomic dynamical systems can be used in the system. Then we have the following proposition.

Proposition 1 The necessary and sufficient condition on which $I(a^\mu, t)$ $(\mu = 1, \ldots, n+m)$ is first integral of the Lagrangian mechanico-electrical system (11) is that the $I(a^\mu, t)$ satisfies

$$\frac{\partial I}{\partial t} + (I, H) = 0, \tag{37}$$

where the Poisson bracket (I, H) is expressed as

$$(I, H) = \frac{\partial I}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial I}{\partial p_k}\frac{\partial H}{\partial q_k}, \quad (k = 1, \ldots, n, n+1, \ldots, n+m). \tag{38}$$

Proof In fact, the I is a first integral if and only if $\dot{I} = 0$. Notice that

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial a^\mu}\Omega^{\mu\nu}\frac{\partial H}{\partial a^\nu} = \frac{\partial I}{\partial t} + (I, H) = 0.$$

Equation (38) is called the Poisson condition of the first integral for Lagrangian mechanico-electrical system.

Proposition 2 $H = c$ is a first integral of Lagrangian mechanico-electrical system, if the Hamiltonian of the system does not depend explicitly on time t.

Proposition 3 For the Lagrangian mechanico-electrical system (11), which possesses two first integrals $I_1(a^\mu, t)$ and $I_2(a^\mu, t)$ having not involution, their Poisson bracket (I_1, I_2) is also the first integral of the system.

Proof Supposing that the Lagrangian mechanico-electrical system (11) possesses the following two first integrals not having involution

$$I_1(a^\mu, t) = c_1, \quad I_2(a^\mu, t) = c_2, \tag{39}$$

which satisfying Poisson conditions

$$\frac{\partial I_1}{\partial t} + (I_1, H) = 0, \quad \frac{\partial I_2}{\partial t} + (I_2, H) = 0. \tag{40}$$

Performing the following operation:

$$\frac{\partial}{\partial t}(I_1, I_2) = \left(\frac{\partial I_1}{\partial t}, I_2\right) + \left(I_1, \frac{\partial I_2}{\partial t}\right) + \frac{\partial I_1}{\partial a^\mu}\frac{\partial \omega^{\mu\nu}}{\partial t}\frac{\partial I_2}{\partial t}, \tag{41}$$

in respect that

$$\frac{\partial \omega^{\mu\nu}}{\partial t} = 0,$$

then Eq.(41) becomes

$$\frac{\partial}{\partial t}(I_1, I_2) = \left(\frac{\partial I_1}{\partial t}, I_2\right) + \left(\frac{\partial I_2}{\partial t}, I_1\right). \tag{42}$$

Using Lie algebraic axioms (25), (26) leads to

$$\begin{aligned}((I_1, I_2), H) =& -((I_2, H), I_1) - ((H, I_1), I_2)\\ =& (I_1, (I_2, H)) + ((I_1, H), I_2).\end{aligned} \tag{43}$$

Combining Eq.(42) with Eq.(43) and considering Eq.(40) yields

$$\begin{aligned}&\frac{\partial}{\partial t}(I_1, I_2) + ((I_1, I_2), H)\\ =&\left(\left(\frac{\partial I_1}{\partial t}\right) + (I_1, H), I_2\right)\\ &+ \left(I_1, \frac{\partial I_2}{\partial t} + (I_2, H)\right) = 0,\end{aligned} \tag{44}$$

then (I_1, I_2) is also a first integral of the system and obtained in Eq.(44).

Proposition 4 For the Lagrangian mechanico-electrical system (11) that possesses a first integral $I(a^\mu, t)$ containing t, and Hamiltonian that does not depend explicitly on t, $\dfrac{\partial I}{\partial t}$, $\dfrac{\partial^2 I}{\partial t^2}, \ldots$, are also first integrals of the system.

Proof By partially differentiating Eq.(37) with respect to t, we obtain

$$\frac{\partial}{\partial t}\frac{\partial I}{\partial t} + \left(\frac{\partial I}{\partial t}, H\right) + \left(I, \frac{\partial H}{\partial t}\right) = 0,$$

where H does not depend explicitly on t, and

$$\frac{\partial}{\partial t}\frac{\partial I}{\partial t} + \left(\frac{\partial I}{\partial t}, H\right) = 0, \tag{45}$$

where $\dfrac{\partial I}{\partial t}$ is a first integral of Eq.(11). Similarly, one can prove that $\dfrac{\partial^2 I}{\partial t^2}$, $\dfrac{\partial^3 I}{\partial t^3}$, ..., are also the first integrals of the system.

Proposition 5 For the Lagrangian mechanico-electrical system (11) that possesses a first integral $I(a^\mu, t)$ containing a^ρ, and Hamiltonian H that does not depend explicitly on a^ρ, $\dfrac{\partial I}{\partial a^\rho}$, $\dfrac{\partial^2 I}{\partial a^{\rho^2}}$, ..., are also first integrals of the system.

Proof By partially differentiating (37) with respect to a^ρ, we have

$$\frac{\partial}{\partial a^\rho}\frac{\partial I}{\partial t} + \left(\frac{\partial I}{\partial a^\rho}, H\right) + \left(I, \frac{\partial H}{\partial a^\rho}\right) = 0,$$

where H does not depend explicitly on a^ρ, and

$$\frac{\partial}{\partial t}\frac{\partial I}{\partial a^\rho} + \left(\frac{\partial I}{\partial a^\rho}, H\right) = 0, \tag{46}$$

therefore $\dfrac{\partial I}{\partial a^\rho}$ is a first integral of Eq.(11). Similarly we can prove that $\dfrac{\partial^2 I}{\partial a^{\rho^2}}$, $\dfrac{\partial^3 I}{\partial a^{\rho^3}}$, ..., are also first integrals of the system.

5.2. Poisson's theory of Lagrange–Maxwell mechanico-electrical systems

Poisson's theory also points out that[28] part of Poisson's integral method may be used for such systems that their equations of motion possess Lie algebraic structure admitted, and the Poisson's theorem does not hold true any more in the systems.

Proposition 6 The necessary and sufficient condition on which $I(a^\mu, t)$ ($\mu = 1, \ldots, n+m$)is a first integral of the Lagrange–Maxwell mechanico-electrical system (18) is that the $I(a^\mu, t)$ satisfies

$$\frac{\partial I}{\partial t} + [I, H] = 0, \tag{47}$$

where

$$\begin{aligned}[I, H] =& I \circ H = \frac{\partial I}{\partial a^\mu} S^{\mu\nu} \frac{\partial I}{\partial a^\nu} \\ & (\mu, \nu = 1, \ldots, n, n+1, \ldots, \\ & n+m, n+m+1, \ldots, 2(n+m)),\end{aligned}$$

is Poisson bracket, and $S^{\mu\nu}$ is given by Eqs.(19)–(21).

Proposition 7 $H = C$ is a first integral of the conservative and non-dissipative Lagrange–Maxwell mechanico-electrical system (18), if the Hamiltonian H of the system does not depend explicitly on time t.

Proof Substituting $I = H$ into Eq.(47) leads to

$$\begin{aligned}\frac{\partial H}{\partial t} + [H, H] =& \frac{\partial H}{\partial t} + \frac{\partial H}{\partial a^\mu} S^{\mu\nu} \frac{\partial H}{\partial a^\nu} \\ =& \frac{\partial H}{\partial t} + \frac{\partial H}{\partial a^\mu} T^{\mu\nu} \frac{\partial H}{\partial a^\nu} \\ =& \frac{\partial H}{\partial t} - \Omega_{kk}\left(\frac{\partial H}{\partial a^{n+m+s}}\right)^2.\end{aligned} \tag{48}$$

Using Eq.(15), we obtain

$$\begin{aligned}&\frac{\partial H}{\partial t} + [H, H] = \frac{\partial H}{\partial t} + \tilde{Q}_s\left(\frac{\partial H}{\partial a^{n+m+s}}\right) \\ &(s = 1, \ldots, n, n+1, \ldots, n+m).\end{aligned} \tag{49}$$

Then $H = C$ is a first integral of Lagrange–Maxwell mechanico-electrical system (18) for the case of $\tilde{Q}_s = 0$ and no dissipation, when H does not depend explicitly on t.

Proposition 8 For the Lagrange–Maxwell mechanico-electrical system (18), which possesses a first integral $I(a^\mu, t)$ containing t, when Hamiltonian H, non-potential generalized force Q_s and the dissipative function F do not depend explicitly on t, then $\dfrac{\partial I}{\partial t}$, $\dfrac{\partial^2 I}{\partial t^2}$, ..., are also first integrals of the system.

Proof By partially differentiating (47) with respect to t, we can obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial t}\right) + \left[\frac{\partial I}{\partial t}, H\right] + \frac{\partial \tilde{Q}_s}{\partial t}\frac{\partial I}{\partial a^\mu} \\ &+ \frac{\partial I}{\partial a^\mu} S^{\mu\nu} \frac{\partial}{\partial a^\nu}\left(\frac{\partial H}{\partial t}\right) = 0.\end{aligned} \tag{50}$$

When Hamiltonian H, non-potential generalized force Q_s and the dissipative function F do not depend explicitly on t, i.e. $\dfrac{\partial}{\partial t}\tilde{Q}_s$=0 and $\dfrac{\partial H}{\partial t} = 0$, one has

$$\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial t}\right) + \left[\frac{\partial I}{\partial t}, H\right] = 0. \tag{51}$$

Namely $\dfrac{\partial I}{\partial t}$ is a first integral of the system (18). Similarly, we can prove that $\dfrac{\partial^2 I}{\partial t^2}$, ..., are also first integrals of the system.

Proposition 9 For the Lagrange–Maxwell mechanico-electrical system (18), which possesses a first integral $I(a^\mu, t)$ containing a^ρ, when Hamiltonian H, non-potential generalized force Q_s and the dissipative function Fdo not depend explicitly on a^ρ, then $\dfrac{\partial I}{\partial a^\rho}$, $\dfrac{\partial^2 I}{\partial a^{\rho^2}}$, ..., are also first integrals of the system.

Proof By partially differentiating (47) with respect to a^ρ, we can have

$$\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial a^\rho}\right)+\frac{\partial}{\partial a^\mu}\left(\frac{\partial I}{\partial a^\rho}\right)S^{\mu\nu}\frac{\partial H}{\partial a^\nu}+\frac{\partial \tilde{Q}_s}{\partial a^\rho}\frac{\partial I}{\partial a^{n+m+s}}+\frac{\partial I}{\partial a^\mu}S^{\mu\nu}\frac{\partial}{\partial a^\nu}\left(\frac{\partial H}{\partial a^\rho}\right)=0. \tag{52}$$

When Hamiltonian H, non-potential general force Q_s and the dissipative function F do not depend explicitly on a^ρ, i.e. $\frac{\partial}{\partial a^\rho}\tilde{Q}_s$=0 and $\frac{\partial H}{\partial a^\rho}=0$, we have

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial a^\rho}\right)+\frac{\partial}{\partial a^\mu}\left(\frac{\partial I}{\partial a^\rho}\right)S^{\mu\nu}\frac{\partial H}{\partial a^\nu}\\ =&\frac{\partial}{\partial t}\left(\frac{\partial I}{\partial a^\rho}\right)+\left[\frac{\partial I}{\partial a^\rho},H\right]=0.\end{aligned} \tag{53}$$

Namely $\frac{\partial I}{\partial a^\rho}$ is a first integral of the system (18). Similarly, we can prove that $\frac{\partial^2 I}{\partial a^{\rho^2}},\ldots$, are also first integrals of the system.

6. Example

In Fig.1 we consider a circuitry of dynamo-electric transducer, which memorizes mechanical vibration. Let m denote the mass of armature, k the stiffness factor, $L_1=L_1(x)=bx$ (b is constant) the self-inductance of winding. The circuitry constitutes winding, battery and resistance, where x denotes the displacement of plumb line, which is calculated from original length of spring. E denotes the electromotive force of battery, and R denotes the resistance. We take it that the $q_s(s=1,2)$ expresses generalized coordinates in which $q_1=x$ denotes the component of

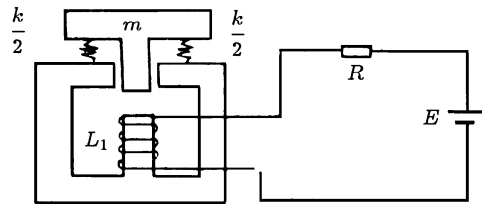


Fig.1. Circuit of dynamo-electric transducer.

space coordinate, $q_2=q$ denotes the electric component, and, further, suppose that the armature is subjected to nonconservative force $E\dot{q}/\dot{x}+R\dot{q}^2/\dot{x}$, then we obtain the kinetic energy and magnetic energy of the system

$$T=\frac{1}{2}m\dot{x}+\frac{1}{2}bx\dot{q}^2, \tag{54}$$

potential energy

$$V=\frac{1}{2}kx^2-mgx, \tag{55}$$

and dissipative function

$$F=-\frac{1}{2}R\dot{q}^2. \tag{56}$$

The Lagrange function of the system is written in the form

$$L=\frac{1}{2}m\dot{x}^2+\frac{1}{2}bx\dot{q}^2-\frac{1}{2}kx^2+mgx. \tag{57}$$

Taking the generalized momenta and velocities of the system

$$\begin{aligned}p_1&=\frac{\partial L}{\partial \dot{x}}=m\dot{x},\quad \dot{x}=\frac{p_1}{m},\\ p_2&=\frac{\partial L}{\partial \dot{q}}=bx\dot{q},\quad \dot{q}=\frac{p_2}{bx},\end{aligned} \tag{58}$$

Hamiltonian of the system can be expressed as

$$H=\frac{1}{2}\frac{p_1^2}{m}+\frac{1}{2}\frac{p_2^2}{bx}+\frac{1}{2}kx^2-mgx. \tag{59}$$

Equations of motion of the system are described in the partial canonical form

$$\begin{aligned}\dot{x}=&\frac{p_1}{m},\quad \dot{q}=\frac{p_2}{bx},\\ \dot{p}_1=&-\frac{\partial H}{\partial x}-E\frac{\dot{q}}{\dot{x}}+R\frac{\dot{q}^2}{\dot{x}}\\ =&-kx+mg+\frac{1}{2b}\frac{p_2^2}{x^2}-E\frac{\dot{q}}{\dot{x}}+R\frac{\dot{q}^2}{\dot{x}},\\ \dot{p}_2=&-\frac{\partial H}{\partial q}+E-R\dot{q}=E-R\dot{q},\end{aligned} \tag{60}$$

in which

$$\begin{aligned}\tilde{Q}_1&=-E\frac{\dot{q}}{\dot{x}}+R\frac{\dot{q}^2}{\dot{x}}=-E\frac{mp_2}{bxp_1}+R\frac{mp_2^2}{b^2x^2},\\ \tilde{Q}_2&=E-R\frac{p_2}{bx}.\end{aligned} \tag{61}$$

Let

$$a_1=x,\quad a_2=q,\quad a_3=p_1,\quad a_4=p_2, \tag{62}$$

Eq.(58) can be written in the contravariant algebraic form, therefore one has

$$\dot{a}^u-S^{\mu\nu}\frac{\partial H}{\partial a^\nu}=0,\quad (\mu\quad,\nu=1,\ldots,4), \tag{63}$$

where

$$S^{\mu\nu}=\omega^{\mu\nu}+T^{\mu\nu},$$

$$(\omega^{\mu\nu})=\begin{bmatrix}0_{2\times2} & I_{2\times2}\\ -I_{2\times2} & 0_{2\times2}\end{bmatrix},$$

$$(T^{\mu\nu})=\begin{bmatrix}0_{2\times2} & 0_{2\times2}\\ 0_{2\times2} & -\Omega_{kk}\end{bmatrix}.$$

Using Eqs.(15) and (61), we obtain

$$\begin{aligned} -\Omega_{11} =& -E\frac{m^2a_4}{ba_1a_3^2}+R\frac{m^2a_4^2}{b^2a_1^2a_3^2},\\ -\Omega_{22} =& E\frac{ba_1}{a^4}-R. \end{aligned} \tag{64}$$

Therefore, we give the contravariant algebraic form as

$$\begin{aligned} \dot{a}_1 =&\frac{a_3}{m},\\ \dot{a}_2 =&\frac{a_4}{ba_1},\\ \dot{a}_3 =&ka_1-mg+\frac{1}{2b}\frac{a_4^2}{a_1^2}-E\frac{ma_4}{ba_1a_3}+R\frac{ma_4^2}{b^2a_1^2a_3},\\ \dot{a}_4 =&E-R\frac{a_4}{ba_1}, \end{aligned} \tag{65}$$

when

$$\begin{aligned} \frac{\partial H}{\partial t} =&0,\\ \tilde{Q}_s\dot{q}_s =&\tilde{Q}_1\dot{x}+\tilde{Q}_2\dot{q}\\ =&-E\frac{a_4}{ba_1}+R\frac{a_4^2}{b^2a_1^2}+E\frac{a_4}{ba_1}\\ &-R\frac{a_4^2}{b^2a_1^2}=0. \end{aligned} \tag{66}$$

We know, from Proposition (7), that Hamiltonian of the system

$$H=\frac{1}{2}\frac{a_3^2}{m}+\frac{1}{2}\frac{a_4^2}{ba_1}+\frac{1}{2}ka_1^2-mga_1=C_1 \tag{67}$$

is the first integral. Using Proposition 6, we can obtain

$$\begin{aligned} \frac{\partial I}{\partial t}+I\ H =&\frac{\partial I}{\partial t}+\frac{\partial I}{\partial a^\mu}\omega^{\mu\nu}\frac{\partial H}{\partial a^\nu}+\frac{\partial I}{\partial a^\mu}T^{\mu\nu}\frac{\partial H}{\partial a^\nu}=\frac{\partial I}{\partial t}+\frac{\partial I}{\partial a^1}\frac{a_3}{m}+\frac{\partial I}{\partial a_2}\frac{a_4}{ba^1}\\ &+\frac{\partial I}{\partial a^3}\left(-E\frac{ma_4}{ba_1a_3}+R\frac{ma_4^2}{b^2a_1^2a_3}+\frac{1}{2}\frac{a_4^2}{ba_1^2}-ka_1+mg\right)+\frac{\partial I}{\partial a_4}\left(E-R\frac{a_4}{ba_1}\right)=0, \end{aligned} \tag{68}$$

which is a one-order linear homogeneous partial differential equation with the characteristic equation being

$$\frac{\mathrm{d}t}{1}=\frac{m\mathrm{d}a_1}{a_3}=\frac{ba_1\mathrm{d}a_2}{a_4}=\frac{a_3\mathrm{d}a_3}{\left(-E\frac{ma_4}{ba_1}+\frac{Rma_4^2}{b^2a_1^2}+\frac{a_3a_4^2}{2ba_1^2}-ka_1a_3+mga_3\right)}=\frac{\mathrm{d}a_4}{E-\frac{Ra_4}{ba_1}}. \tag{69}$$

We, from Eq.(69), can obtain the following integrals:

$$I_1 =ma_1-\int a_3\mathrm{d}t=c_1, \tag{70}$$

$$I_2 =a_2-\int\frac{a_4}{ba_1}\mathrm{d}t=c_2, \tag{71}$$

$$\begin{aligned} I_3 =&\frac{1}{2}a_3^2+E\frac{ma_2}{ba_1}+\frac{mRa_2a_4}{b^2a_1^2}+\frac{a_2a_3a_4}{2ba_1^2}\\ &-\frac{1}{2}ka_1^2+mga_1=c_3, \end{aligned} \tag{72}$$

$$I_4 =a_2+\frac{a_4}{R}+\frac{Eba_1}{R^2}\ln\left(E-\frac{Ra_4}{ba_1}\right)=C_4. \tag{73}$$

The first integral I_3 includes a_2. Using Proposition 9, from I_3 we obtain a new integral

$$I_5=E\frac{m}{ba_1}+\frac{mRa_4}{b^2a_1^2}+\frac{a_3a_4}{2ba_1^2}=c_5. \tag{74}$$

7. Conclusion

The algebraic structure and the Poisson's theory are extended to mechanico-electrical systems in this paper. The results here present significant approaches to seeking for conserved quantities in mechanico-electrical systems.

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