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Spontaneous U(1) symmetry breaking and Bose–Einstein condensation*

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Based on Bogoliubov's truncated Hamiltonian $H_{\rm B}$ for a weakly interacting Bose system, and adding a U(1) symmetry breaking term $\sqrt{V}(\lambda a_0 + \lambda^* a_0^+)$ to $H_{\rm B}$, we show by using the coherent state theory and the mean-field approximation rather than the c-number approximations, that the Bose–Einstein condensation(BEC) occurs if and only if the U(1) symmetry of the system is spontaneously broken. The real ground state energy and the justification of the Bogoliubov c-number substitution are given by solving the Schrödinger eigenvalue equation and using the self-consistent condition.

Keywords: BEC, U(1) symmetry breaking, generalized SU(1,1), coherent state
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Bose–Einstein condensations (BEC) of atoms have been extensively studied both experimentally and theoretically. In treating the many-body Hamiltonian of a weakly interacting Bose system, Bogoliubov made two approximations: the first is to expand the Bosonic Hamiltonian to second order in the operators $a_{\boldsymbol{p}}$ and $a_{\boldsymbol{p}}^+$ ($\boldsymbol{p} \neq 0$) of the excited states, and the second is the c-number substitution for the operators a_0 and a_0^+ of the condensate state ($\boldsymbol{p} = 0$).[1,2] Justifications of the c-number substitutions were given by Ginibre in 1968,[3] and recently by Lieb *et al*[4] and Sütö.[5] In these treatments, the Hamiltonian with a U(1) symmetry breaking term $\sqrt{V}(\lambda a_0 + \lambda^* a_0^+)$ was used to show the equivalence of BEC and the U(1) symmetry breaking. We here show the same point by (i) linearizing the symmetry broken Bogoliubov Hamiltonian but without the c-number approximation, and then (ii) solving the Schrödinger eigenvalue equation. From the solution we can give the condition for BEC and several other relations.

Starting from the grand-canonical Hamiltonian

$$H' = H - \mu N = \sum_{\boldsymbol{p}} \Big(\frac{\boldsymbol{p}^2}{2m} - \mu \Big) a_{\boldsymbol{p}}^+ a_{\boldsymbol{p}} + \frac{U_0}{2V} \sum a_{\boldsymbol{p}_1}^+ a_{\boldsymbol{p}_2}^+ a_{\boldsymbol{p}_2'} a_{\boldsymbol{p}_1'} \delta_{\boldsymbol{p}_1 + \boldsymbol{p}_2, \boldsymbol{p}_1' + \boldsymbol{p}_2'} \quad (1)$$

for a weakly interacting Bose system, we obtain the Bogoliubov truncated Hamiltonian[1] with U(1) symmetry breaking

$$\begin{aligned} H'_\lambda = & -\mu a_0^+ a_0 + \frac{U_0}{2V} {a_0^+}^2 {a_0}^2 \\ & + \sum_{\boldsymbol{p} \neq 0} \Big\{ \Big(\frac{1}{2} \varepsilon_{\boldsymbol{p}} + g_0 a_0^+ a_0 \Big) (a_{\boldsymbol{p}}^+ a_{\boldsymbol{p}} + a_{-\boldsymbol{p}}^+ a_{-\boldsymbol{p}}) \\ & + \frac{g_0}{2} (a_0^2 a_{\boldsymbol{p}}^+ a_{-\boldsymbol{p}}^+ + {a_0^+}^2 a_{\boldsymbol{p}} a_{-\boldsymbol{p}}) \Big\} \\ & + \sqrt{V} (\lambda a_0 + \lambda^* a_0^+) \end{aligned} \quad (2)$$

with $\varepsilon_{\boldsymbol{p}} = \boldsymbol{p}^2/2m - \mu, \lambda = |\lambda| {\rm e}^{{\rm i}\beta}$, and $g_0 = U_0/V$. Here μ is the chemical potential of the system, and $N = \sum_{\boldsymbol{p}} a_{\boldsymbol{p}}^+ a_{\boldsymbol{p}}$ is the total number of the particles. Under the U(1) symmetry transformation $U = \exp[-{\rm i}\theta'(a_0^+ a_0 + a_{\boldsymbol{p}}^+ a_{\boldsymbol{p}} + a_{-\boldsymbol{p}}^+ a_{-\boldsymbol{p}})]$, H'_λ becomes

$$U^+ H'_\lambda U = H'_{\lambda=0} + \sqrt{V} (\lambda {\rm e}^{-{\rm i}\theta'} a_0 + \lambda^* {\rm e}^{{\rm i}\theta'} a_0^+) \quad (3)$$

It is clear that $H'_{\lambda=0}$ has U(1) symmetry; however, this symmetry is broken for $\lambda \neq 0$. Linearizing H'_λ and introducing the following Bose realizations of the generators of the SU(1,1) Lie algebra:[6,7]

$$K_0^+ = \frac{1}{2} {a_0^+}^2, \quad K_0 = \frac{1}{2} a_0^2, \quad K_0^3 = \frac{1}{2} \Big(a_0^+ a_0 + \frac{1}{2} \Big) \quad (4)$$

and

$$\begin{aligned} & K_{\boldsymbol{p}}^+ = a_{\boldsymbol{p}}^+ a_{-\boldsymbol{p}}^+, \quad K_{\boldsymbol{p}}^- = a_{\boldsymbol{p}} a_{-\boldsymbol{p}}, \\ & K_{\boldsymbol{p}}^3 = \frac{1}{2} (a_{\boldsymbol{p}}^+ a_{\boldsymbol{p}} + a_{-\boldsymbol{p}}^+ a_{-\boldsymbol{p}} + 1) \end{aligned} \quad (5)$$

with

$$[K^3, K^\pm] = \pm K^\pm, \qquad [K^+, K^-] = -2K^3, \quad (6)$$

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we have

$$\begin{aligned} H'_\lambda = & -2\mu K_0^3 + 2gK_0^+ + 2g^* K_0^- \\ & + \frac{\mu}{2} - 2g|\langle K_0^+\rangle|^2 \\ & + \sum_{\boldsymbol{p}\neq 0}\Big(\omega_{\boldsymbol{p}} K_{\boldsymbol{p}}^3 + gK_{\boldsymbol{p}}^+ + g^* K_{\boldsymbol{p}}^- - \frac{1}{2}\omega_{\boldsymbol{p}}\Big) \\ & + \sqrt{V}(\lambda a_0 + \lambda^* a_0^+) \end{aligned} \tag{7}$$

with $\omega_{\boldsymbol{p}} = \varepsilon_{\boldsymbol{p}} + 2g_0(2\langle K_0^3\rangle - 1/2)$, and $g = U_0/V\langle K_0^-\rangle = |g|\mathrm{e}^{-\mathrm{i}\varphi'}$. Although the linearization has broken the U(1) symmetry of $H'_{\lambda=0}$, the residual symmetry exists if $\varphi' = 2\theta'$. Thus we can still discuss the effects of U(1) symmetry breaking on the BEC. We should note that the Bogoliubov c-number substitution corresponds to $\langle K_0^3\rangle \approx |\langle K_0^\pm\rangle| \approx \frac{1}{2}N_0 = \frac{1}{2}\langle a_0^+ a_0\rangle$. Since the dynamical algebras of H'_λ are SU(1,1) and harmonic oscillator lie algebras, its eigenstate can be written as displaced generalized SU(1,1) coherent state[7]

$$\begin{aligned} |\psi\rangle = |\psi_0\rangle \otimes |\psi'\rangle = D_0(\alpha_0)T(z_0)|k_0 n_0\rangle \\ \otimes \prod_{\boldsymbol{p}\neq 0} T(z_{\boldsymbol{p}})|k_{\boldsymbol{p}} n_{\boldsymbol{p}}\rangle, \end{aligned} \tag{8}$$

where $D_0(\alpha_0) = \exp(\alpha_0 a_0^+ - \alpha_0^* a_0)$ the displacement operator, $T(z) = \exp(zK^+ - z^*K^-)$ the squeeze operator, $z = -\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\varphi}$, $\alpha_0 = |\alpha_0|\mathrm{e}^{\mathrm{i}\gamma}$, $\theta \in (-\infty, \infty)$, $\varphi \in [0, 2\pi]$, $k_0 = 1/4$, and $k_{\boldsymbol{p}} = 1/2$. If n=0, then $T(z)|k, 0\rangle$ is just the Perelomov SU(1,1) coherent state.[8] Using[6,7]

$$\begin{aligned} & D_0^+(\alpha_0) a_0 D_0(\alpha_0) = a_0 + \alpha_0, \\ & T^+(z_0) a_0 T(z_0) = a_0 \cosh\frac{\theta_0}{2} - \mathrm{e}^{-\mathrm{i}\varphi_0}\sinh\frac{\theta_0}{2}a_0^+, \\ & T^+(z_0) K_0^\pm T(z_0) = \cosh^2\frac{\theta_0}{2}K_0^\pm + \mathrm{e}^{\pm\mathrm{i}2\varphi_0}\sinh^2\frac{\theta_0}{2}K_0^\mp \\ & \quad - \mathrm{e}^{\pm\mathrm{i}\varphi_0}\sinh\theta_0 K_0^3, \\ & T^+(z_0) K_0^3 T(z_0) = \cosh\theta_0 K_0^3 \\ & \quad - \frac{1}{2}\sinh\theta_0(\mathrm{e}^{-\mathrm{i}\varphi_0}K_0^+ + \mathrm{e}^{\mathrm{i}\varphi_0}K_0^-) \end{aligned} \tag{9}$$

and

$$\begin{aligned} & a_0^+|k_0, n_0\rangle = \sqrt{2n_0+1}\,|k_0, n_0 + \frac{1}{2}\rangle, \\ & a_0|k_0, n_0\rangle = \sqrt{2n_0}\,|k_0, n_0 - \frac{1}{2}\rangle, \\ & K_0^3|k_0, n_0\rangle = (n_0 + k_0)|k_0, n_0\rangle, \\ & K_0^+|k_0, n_0\rangle = [(n_0+1)(n_0+2k_0)]^{1/2}|k_0, n_0+1\rangle, \\ & K_0^-|k_0, n_0\rangle = [n_0(n_0+2k_0-1)]^{1/2}|k_0, n_0-1\rangle. \end{aligned} \tag{10}$$

We can now exactly solve the Schrödinger eigenvalue equation

$$H'_\lambda|\psi\rangle = E'_\lambda|\psi\rangle = (E_\lambda - \mu N)|\psi\rangle \tag{11}$$

and get the eigenvalue E'_λ,

$$\begin{aligned} E'_\lambda = & 2g_0|\langle K_0^+\rangle|^2 - \mu N_0 + 2\sqrt{V}|\lambda\alpha| \\ & + \sum_{\boldsymbol{p}\neq 0}\Big\{(n_{\boldsymbol{p}} + k_{\boldsymbol{p}})[\omega_{\boldsymbol{p}}^2 - 4g_0^2|\langle K_0^+\rangle|^2]^{1/2} - \frac{1}{2}\omega_{\boldsymbol{p}}\Big\} \end{aligned} \tag{12}$$

with $\varphi_0 = \varphi_{\boldsymbol{p}}$, $\varphi_0 + 2\gamma = \pm\pi$, $\gamma + \beta = 0$, $\tanh\theta_{\boldsymbol{p}} = -2g_0|\langle K_0^+\rangle|/\omega_{\boldsymbol{p}}$, $\omega_{\boldsymbol{p}} = \varepsilon_{\boldsymbol{p}} + 2g_0 N_0$, and

$$N_0 = \langle\psi|a_0^+ a_0|\psi\rangle = |\alpha_0|^2 + 2(n_0 + k_0)\cosh\theta_0 - \frac{1}{2}, \tag{13}$$

$$\begin{aligned} \langle K_0^+\rangle = \langle\psi|K_0^\pm|\psi\rangle = \frac{1}{2}[|\alpha_0|^2 + 2(n_0 + k_0) \\ \times \sinh\theta_0]\mathrm{e}^{-\mathrm{i}2\gamma}, \end{aligned} \tag{14}$$

$$\mu = g_0[|\alpha_0|^2 + 2(n_0 + k_0)\sinh\theta_0]\coth\theta_0, \tag{15}$$

$$\begin{aligned} [\sqrt{V}|\lambda| - \mu|\alpha_0|(1 - \tanh\theta_0)] \\ \times\Big(\cosh\frac{\theta_0}{2} + \sinh\frac{\theta_0}{2}\Big) = 0, \end{aligned} \tag{16}$$

$$\langle\psi|a_0|\psi\rangle = \alpha_0. \tag{17}$$

From the above solutions we now discuss the following issues:

(i) Chemical potential μ. The chemical potential can be determined by the condition (See Eq.(12))

$$\frac{\partial}{\partial N_0}E'_\lambda(\boldsymbol{p} = 0) = 0, \tag{18}$$

which gives

$$\mu' = 2g_0\frac{\partial}{\partial N_0}|\langle\psi|K_0^+|\psi\rangle|^2. \tag{19}$$

It is clear that the chemical potential μ' determined by Eq.(19) is not equal to the chemical potential μ determined by Eq.(15). But from the self-consistent condition

$$\lim_{\theta_0\to\infty}\mu' = \lim_{\theta_0\to\infty}\mu = \mu_{\mathrm{c}}, \tag{20}$$

we can obtain the correct value of the chemical potential

$$\mu_{\mathrm{c}} = g_0[|\alpha_0|^2 + (n_0 + k_0)\mathrm{e}^{\theta_0}] = g_0 N_0 \tag{21}$$

and the correct energy of the Bose system

$$\begin{aligned} E^{\mathrm{c}}_{\lambda=0} = \lim_{\theta_0\to\infty}E_{\lambda=0} = \frac{1}{2}g_0 N_0^2 \\ + \sum_{\boldsymbol{p}\neq 0}\Big[(n_{\boldsymbol{p}} + k_{\boldsymbol{p}})\sqrt{\omega_{\boldsymbol{p}}^2 - g_0^2 N_0^2} - \frac{1}{2}\omega_{\boldsymbol{p}}\Big] \end{aligned} \tag{22}$$

with $2|\ \langle\psi|K_0^\pm|\psi\rangle\ | \approx N_0 \approx [|\ \alpha_0\ |^2 + (n_0 + k_0)\mathrm{e}^{\theta_0}]$. It is clear that Eqs.(21–22) are the same as that obtained by Bogoliubov approximations.[1,2] From the above discussions we can give the condition for BEC:

$$\lim_{\theta_0\to\infty}\tanh\theta_0 = 1. \tag{23}$$

(ii) Spontaneous U(1) symmetry breaking.

If $\tanh\theta_0 \not\to 1$, then $\lambda = 0$ leads to $\alpha_0 = 0$ (See Eq.(16)). This means that before the generation of BEC the Bosonic Hamiltonian has U(1) symmetry, and the state of the system is the generalized SU(1,1) coherent state. In this case we have

$$\lim_{\lambda\to 0} \text{angle}\psi|a_0|\psi\rangle = 0. \tag{24}$$

However, if $\tanh\theta_0 \to 1$(BEC), then α_0 can take nonzero values for $\lambda = 0$ (see Eq.(16)), which gives (see Eq.(17))

$$\lim_{\lambda\to 0}\langle\psi|a_0|\psi\rangle = \alpha_0 \tag{25}$$

for any value of λ. These results indicate that BEC implies a spontaneous U(1) symmetry breaking, and causes the displacement of the condensate state. Equations (13), (14) and (17) give the following inequalities:

$$\text{(iii)} \quad |\langle\psi|a_0|\psi\rangle|^2 < \langle\psi|a_0^+ a_0|\psi\rangle(\theta_0 \neq 0), \tag{26}$$

$$\langle\psi|a_0^+ a_0|\psi\rangle\langle\psi|a_0 a_0^+|\psi\rangle \geq \langle\psi|{a_0^+}^2|\psi\rangle\langle\psi|a_0^2|\psi\rangle. \tag{27}$$

The equality in Eq.(27) holds for $\theta_0 \to \infty$ (BEC) (See Eqs.(13–14)).

In conclusion, we have shown the equivalence between the BEC and the spontaneous U(1) symmetry breaking by using the coherent state theory and the mean-field approximation rather than the c-number approximation. Our direct solution to the Schrödinger eigenvalue equation also shows that (i) the Bogoliubov c-number substitutions are correct and (ii) the symmetry breaking causes the displacement of the condensate state. The rigorous solutions to the subject treated in this paper can be found in Refs.[4,5,9]. Our results may be useful in treating the quantum interferences[10] and the quantum statistics[6] of the BEC.

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