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Stability of tokamak plasmas with internal transport barriers against high n ideal magnetohydrodynamic ballooning mode*

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A ballooning mode equation for tokamak plasma, with the toroidicity and the Shafranov shift effects included, is derived for a shift circular flux tokamak configuration. Using this equation, the stability of the plasma configuration with an internal transport barrier (ITB) against the high n (the toroidal mode number) ideal magnetohydrodynamic (MHD) ballooning mode is analysed. It is shown that both the toroidicity and the Shafranov shift effects are stabilizing. In the ITB region, these effects give rise to a low shear stable channel between the first and the second stability regions. Out of the ITB region towards the plasma edge, the stabilizing effect of the Shafranov shift causes the unstable zone to be significantly narrowed.

Keywords: toroidal effect, ballooning mode, internal transport barrier
PACC: 5235P, 5255G

1. Introduction

High n (the toroidal mode number) ballooning mode analysis is generally thought to be the most fundamental tool for the study of the magnetohydrodynamic (MHD) stability in magnetically confined toroidal fusion devices (Tokamak, Stellarator, Reversed pinch, spherical torus, etc). The theories about it have been simultaneously developed by several groups, such as Connor, Hastie and Taylor (CHT),[1] the Princeton group,[2] Coppi, Ferreira, Mark and Romas (CFMR),[3] Pogutse and Yurchinko (PY).[4] For tokamak plasmas, a basic and well-known result is the $s-\alpha$ diagram for stability criterion, where $s=(r/q_{\rm c}){\rm d}q_{\rm c}/{\rm d}r$ is the magnetic shear, $\alpha=-(2\mu_0 R_0 q_{\rm c}^2/B_0^2){\rm d}P/{\rm d}r$ is a parameter directly related to the local pressure gradient and then related to the plasma beta value (the ratio of plasma pressure to magnetic field pressure). Here $q_{\rm c}$ is the cylindrical safety factor, μ_0 the vacuum magnetic permeability, R_0 the major radius of the tokamak plasma, r the minor radius, B_0 the toroidal magnetic field at the magnetic axis and P the plasma pressure. This diagram is obtained for a simplified tokamak configuration, i.e. by assuming the equilibrium magnetic surfaces to be a set of shift circular flux surfaces. In this model, only two basic parameters (s and α as mentioned above) appear in the ballooning mode equation. A noticeable feature of the $s-\alpha$ diagram is the presence of the second stable region due to the stabilization associated with geodesic curvature effect. So far, a great number of papers have been published for the ballooning mode study. Very recently, new interest[5,6] in ballooning mode study has switched to the study in internal transport barrier (ITB) phenomena, an improved confinement structure characterized by a central negative shear region and a much smaller energy transport region near the minimum of q (we do not distinguish between the real safety factor q and the cylindrical $q_{\rm c}$ here in this paper, because their difference is of the order of $o(\varepsilon^2)$ in magnitude ($\varepsilon=r/R_0$). The important feature of the ITB is as follows:[7–9] in the near magnetic axis region inside the ITB, the shear is negative, according to the ballooning mode theory, this region is stable; outside but near the ITB, the shear is very small. This requires the original ballooning mode theory to be improved.[5,6,10] In Refs.[5] and [6], a complementary approach is developed based on the recurrence relation describing the toroidal coupling of radially localized modes in adjacent rational surfaces. Meanwhile, the original ballooning mode equation based on the shift circular flux surfaces is still applicable. This new study makes a great contribution to the ballooning mode theory; however, there are two defects in the ballooning mode equation used there. This simplified equation is as follows:

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$$\frac{\mathrm{d}}{\mathrm{d}\eta}\{1+[s(\eta-k)-\alpha\sin\eta]^2\frac{\mathrm{d}\zeta}{\mathrm{d}\eta}\}+\alpha\{\cos\eta+[s(\eta-k)-\alpha\sin\eta]\sin\eta\}\zeta=\gamma^2\zeta, \quad (1)$$

where ζ is the eigen function, η is the extended poloidal angle variable, k is the radial wave number, γ is the normalized growth rate of the unstable mode. The first defect is that from this equation we cannot obtain a stability criterion consistent with the Mercier criterion,[10] i.e.

$$\frac{s^2}{2}+\varepsilon\alpha(1-q^{-2})>0 \quad (2)$$

for stability; the second defect is that the term $\alpha\sin\eta$ in Eq.(1) does not hold true for the shift surface configuration. However, this term has a significant effect on the stability criterion. Both these defects are due to the fact that some important toroidal effects are not included in Eq.(1), which is derived from a general ballooning mode equation for general axisymmetric toroidal configuration.[10] From this general ballooning mode equation (CHT equation) we find that some toroidal effects are also lost. Complete toroidal effects are included in PY's derivation;[4] however, they gave a very complicated version so that it is hard to compare with a simplified model. The CFMR equation can provide the basis for deriving a model equation including complete toroidal effects. However, in their derivation,[3] only the Shafranov shift is included so that the resulting ballooning mode equation is not able to give the Mercier criterion, either. In this paper, from the CFMR theory, we derive a correct ballooning mode equation for configuration with shift circular flux surfaces, with both these two important toroidal effects included. Then the stability of the ITB is studied. In Section 2, the model equilibrium for the configuration with shift circular flux surfaces is discussed. In Section 3, the correct ballooning mode equation for this configuration is derived. In Section 4, the basic characteristics of the tokamak configuration with ITB are clarified. In Section 5, the stability in the whole plasma region of this configuration against the ideal high n MHD ballooning mode is analysed by using this new eigen-mode equation.

2. Model equilibrium and coordinates

In ballooning mode analysis, a simplified tokamak equilibrium model is widely used.[1–6,11] In this model, the magnetic surfaces are assumed to be shift circular flux surfaces. Meanwhile, it is assumed that the inverse aspect ratio $\varepsilon = r/R_0 << 1$. Being consistent with this assumption, the plasma beta value $\beta = 2\mu < P > /B_0^2$ is of the order of $o(\varepsilon^2)$. For this configuration, convenient coordinates are usually defined as follows :

$$\begin{aligned} R &= R_0+\Delta(r)+r\cos\theta, \\ Z &= r\sin\theta, \end{aligned} \quad (3)$$

where (R,ϕ,Z) are the cylindrical coordinates, Δ is the Shafranov shift function. The (r,θ,ϕ) system has a Jacobian

$$J=1/(\nabla r\times\nabla\theta\cdot\nabla\phi)=Rr(1+\Delta'\cos\theta). \quad (4)$$

To the lowest order, the equilibrium equation (the Grad–Shafranov equation) can be cast into two ordinal differential equations about ψ and Δ:[11]

$$\psi''+\frac{1}{r}\psi'=-\mu_0R_0j_{\phi0}=-\mu_0R_0^2\frac{\mathrm{d}P}{\mathrm{d}\psi}-\frac{1}{2}\frac{\mathrm{d}F^2}{\mathrm{d}\psi}, \quad (5)$$

$$\Delta''+(\frac{1}{r}+\frac{2\psi''}{\psi'})\Delta=-\frac{1}{R_0}+\frac{2\mu_0R_0r}{\psi'}\frac{\mathrm{d}P}{\mathrm{d}\psi}, \quad (6)$$

where $\psi(r)$ is the poloidal magnetic flux, $F=RB_\phi$, the poloidal current flux, and prime denotes the derivative with respect r. The safety factor can be written as follows:

$$\begin{aligned} q(r) &= \frac{F}{2\pi}\oint\frac{\mathrm{d}l}{R^2B_\theta} \\ &= q_{\mathrm{c}}(1-\delta-\frac{\varepsilon\Delta'+\varepsilon^2}{2})\approx q_{\mathrm{c}}(1+o(\varepsilon^2)), \end{aligned} \quad (7)$$

where the cylindrical safety factor $q_{\mathrm{c}}=\dfrac{Fr}{R_0\psi'}$. Using the definition of $(s-\alpha)$ (see Section 1), the Shafranov shift equation (6) can be rewritten as

$$r\Delta''+(3-2s)\Delta'=-\varepsilon-\alpha. \quad (8)$$

Because the (r,θ,ϕ) system is not an orthogonal one, a transformation from the orthogonal coordinates (the CHT and CRMR ballooning mode equations are derived in orthogonal coordinate systems) to non-orthogonal ones is needed:

$$\frac{\partial A(\psi,\chi)}{\partial\psi}=\frac{1}{\psi'}[\frac{\partial A(r,\theta)}{\partial r}+\frac{\Delta'}{r}\sin\theta\frac{\partial A(r,\theta)}{\partial\theta}]. \quad (9)$$

The above is the equilibrium information needed for the following derivation of ballooning mode analysis.

3. Derivation of model ballooning mode equation

As indicated above, the CHT ballooning mode formalism[1,12] does not include complete toroidal effects. Fortunately, the CFMR formalism[3] does. In orthogonal coordinates (ψ, χ, ϕ), this equation reads

$$-\frac{\omega^2}{V_{\rm A}^2}[1+\Sigma^2]\zeta = \frac{h_\chi}{R\nu^2 h_\psi}\frac{\partial}{\partial\chi}[\frac{h_\psi}{Rh_\chi}(1+\Sigma^2)\frac{\partial\zeta}{\partial\chi}] - \frac{2\mu_0 P'(\psi)}{h_\psi B^2}(-k_\psi + \Sigma k_\chi)\zeta \tag{10}$$

with

$$\Sigma = \frac{h_\chi}{h_\psi \nu}\frac{\partial}{\partial\psi}\int^{\chi}\nu(\psi,\chi'){\rm d}\chi', \tag{11}$$

where $h_\psi = 1/RB_\chi$, $h_\chi = JB_\chi$, and the local safety factor

$$\nu = \frac{{\rm d}\phi}{{\rm d}\chi} = \frac{FJ}{R^2}. \tag{12}$$

Now we can transfer the above equation into the (r, θ, ϕ) system by using information used in Section 2 for that model equilibrium configuration. We obtain

$$\Sigma = \frac{(1+\varepsilon\cos\theta)}{(1+\Delta'\cos\theta)^2}[s(\theta-\theta_0) - E(\alpha, s, \Delta', \varepsilon)\sin\theta + o(\varepsilon^2)], \tag{13}$$

with

$$E = \alpha + (2+s)\varepsilon + (2-3s)\Delta'. \tag{14}$$

The normal curvature k_ψ and the geodesic curvature k_χ are respectively defined as follows:

$$k_\psi = \frac{1}{B^2}[|\nabla\psi|\frac{\partial}{\partial\psi} + \frac{\nabla\psi\cdot\nabla\theta}{(\nabla\psi)^2}\frac{\partial}{\partial\theta}](\mu_0 P + \frac{B^2}{2}), \tag{15}$$

$$k_\chi = \frac{1}{B^2}|\nabla\theta|\frac{\partial}{\partial\theta}(\frac{B^2}{2}). \tag{16}$$

Neglecting higher order contributions, we obtain

$$k_\psi = \frac{-1}{(1+\Delta'\cos\theta)R}[\cos\theta + \Delta' + \frac{\varepsilon}{q^2}], \tag{17}$$

$$k_\chi = \frac{1}{R}\sin\theta. \tag{18}$$

Substituting the expressions (12)–(16) into the original ballooning mode equation (1), we finally obtain

$$\frac{(1+\Delta'\cos\theta)^3}{(1+\varepsilon\cos\theta)^3}\Gamma^2(1+\Sigma^2)\zeta = \frac{\partial}{\partial\theta}\left\{\frac{1+\Delta'\cos\theta}{1+\varepsilon\cos\theta}[1+\Sigma^2]\frac{\partial\zeta}{\partial\theta}\right\} + (\frac{1+\Delta'\cos\theta}{1+\varepsilon\cos\theta})^2\alpha[\cos\theta + \frac{\varepsilon}{q^2} + \Sigma\sin\theta]\zeta. \tag{19}$$

Besides the basic parameters s and α , this equation includes more parameters such as ε, Δ' and q. This means that the ballooning stability depends on not only local equilibrium quantities,but also some global quantities such as Δ'. We can see that if we let $\varepsilon = \Delta' = 0$, Eq.(19) will go back to the $s-\alpha$ model equation (1); if we only let $\varepsilon = 0$, it will return to the CFMR equation.[3] The CFMR equation though includes the Shafranov shift effect, it does not include the Mercier term,[13] so that it does not include the complete toroidal effects. As can be seen later, for the ITB configuration, the Mercier term is significant in determining the stability boundary.

4. Characteristics of tokamak configuration with ITBs

A typical ITB configuration has a negative central shear region and a rapid pressure drop occurring at the near minimum position of the q-profile,[7−9] so, we assume a model for pressure profile as follows:

$$\mathrm{d}P/\mathrm{d}r = -\frac{1}{a} f_p(x), \quad x = r/a, \tag{20a}$$

$$f_p(x) = a_1 2cx(1-x^2)^{c-1} + a_2[\exp(-(\frac{x-x_0}{d})^2) - \exp(-\frac{x_0^2}{d^2})], \tag{20b}$$

$$P(x) = \int_x^1 f_p(x')\mathrm{d}x'. \tag{20c}$$

On the right-hand side of Eq.(20b), the first term proportional to a_1 represents a background pressure without the ITB, the second term proportional to a_2 represents the large pressure gradient in the ITB region (its location is $x = x_0$, width is d) caused by the improvement of the confinement in this region. This means that the plasma internal energy storage that is proportional to the pressure also increases (see Fig.1(a) and Fig.1(b)).

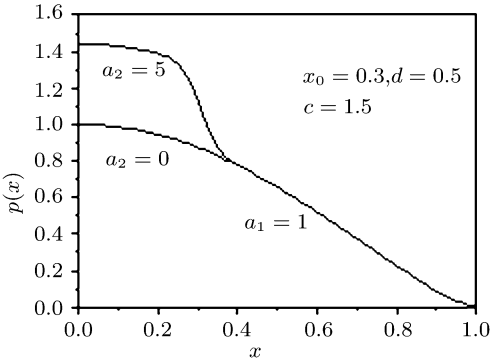


Fig.1(a). Comparison between pressure profiles with and without ITB.

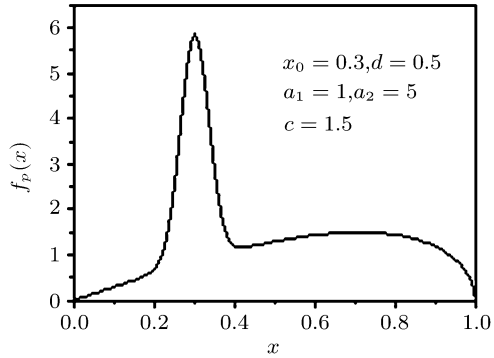


Fig.1(b). Pressure gradient profile $f_p(x)$ for ITB configuration at $x_0 = 0.3$ and $d = 0.05$.

In order to construct a suitable q-profile, we assume a toroidal current profile as follows:

$$j_\phi(r) = j_0[1 + \sum_{i=1} c_i x^i] \tag{21}$$

with $x = r/a$. From Eq.(5), we obtain the first derivative of the poloidal magnetic flux

$$\psi'(r) = \mu_0 R_0 j_0 a^2 \left[\frac{x}{2} + \sum_{i=1} \frac{c_i}{i+2} x^{i+1}\right]. \tag{22}$$

Then the q-profile is given by

$$q(r) = \frac{B_0 r}{\psi'(r)} = q_0/[1 + \sum \frac{2c_i}{i+2} x^i] \tag{23}$$

with $q_0 = \dfrac{2B_0}{\mu_0 R_0 j_{)}}$. We assume that there exists a single minimum $q_{\min}$ of the q-profile at the $x = x_0$ position where the ITB located, and the toroidal current density decreases to zero at the edge, a set of parameters are chosen to satisfy these physical conditions: $c_1 = 2.429$, $c_2 = 3.549$, $c_3 = -42.757$, $c_4 = 62.217$, $c_5 = -26.437$ to fit $q_0 = 2$, $q_{\min} = 1.5$, $x_{\min} = 0.3$ and $q_{\text{edge}} = 4.2$. The corresponding toroidal current profile, q-profile and the shear profile are shown in Figs.2(a)–(c).

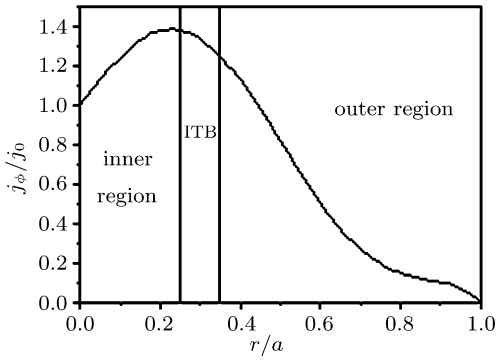


Fig.2(a). Model toroidal current density profile for ITB configuration.

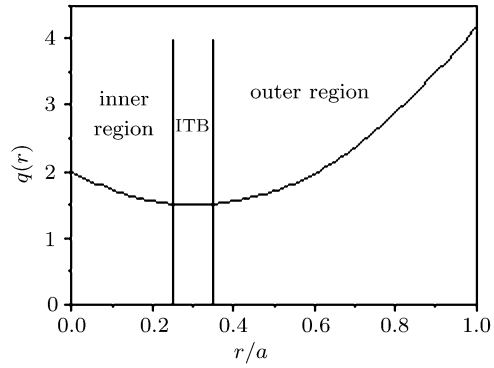


Fig.2(b). The safety factor profile for ITB configuration.

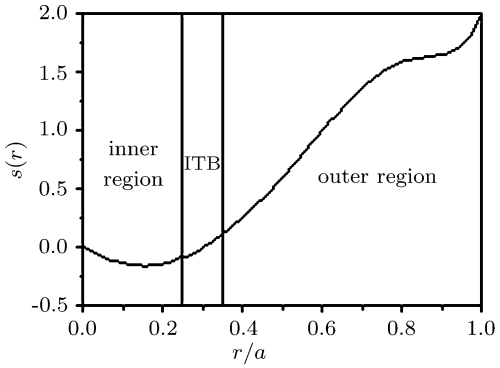


Fig.2(c). The shear profile for the ITB configuration.

The basic parameter α now can be expressed by

$$\alpha(r) = (\beta_0 R_0/a)(\frac{P'}{P_0})q^2 \qquad (24)$$

with $\beta_0 = 2\mu_0 P_0/B_0^2$ and $x = r/a$. The quantity $\alpha a/(\beta_0 R_0)$ as a function of x is shown in Fig.3. To determine the stability of the ITB configuration, we need to obtain the $\Delta'(r)$.

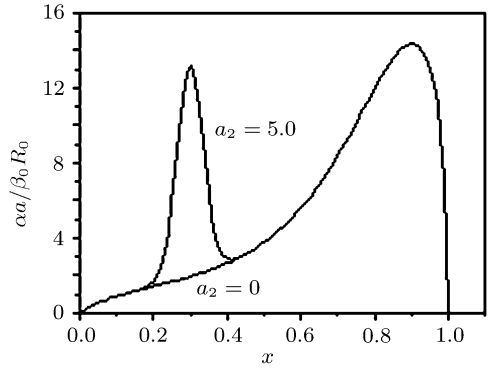


Fig.3. Parameter $\alpha a/(\beta_0 R_0)$ as a function of x, other parameters are the same as those in Figs.1–2.

From Eq.(8) and the definition of α, Eq.(24), we can cast $\Delta'(r)$ into two independent functions as

$$\Delta'(r) = -\frac{a}{R_0} f_{\rm d1}(x) - \beta_0(\frac{R_1}{a}) f_{\rm d2}(x). \qquad (25)$$

For given functions of $P(r)$, $q(r)$, $s(r)$, and $q(r)$ above, these two functions are shown in Fig.4(a). As seen from this figure, the part of the shift due to the vacuum toroidal effect $f_{\rm d1}$ is much smaller than the part of the pressure induced shift $f_{\rm d2}$. Meanwhile, we see that in the ITB region, the Δ' rises rapidly. To see more clearly, we define

$$\delta_{\rm c} = -\Delta'/\alpha. \qquad (26)$$

This quantity as a function of x is shown in Fig.4(b). The Shafranov shift effect in the eigen-mode equation (19) makes it different from the original CHT's $s-\alpha$ model equation in several aspects : at first it appears in all three terms (the first stabilizing term and the second destabilizing term on the right-hand side and the initial term on the left-hand side of this equation). Besides, the parameter E in Eq.(14) is also different from α greatly. As a result, the original CHT ballooning equation is somewhat inadequate for describing the ballooning stability of an ITB-type configuration.

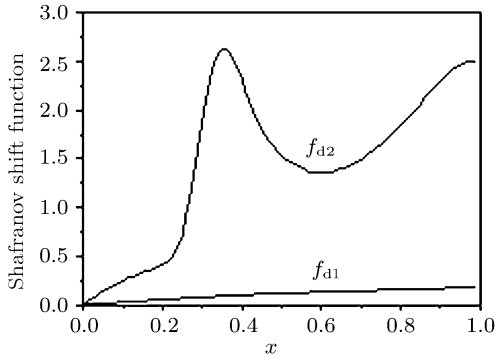


Fig.4(a). Relative Shafranov shift functions $f_{\rm d1}$ and $f_{\rm d2}$.

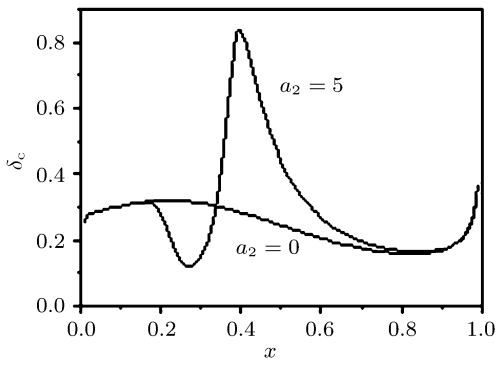


Fig.4(b). The ratio $\delta_{\rm c} = -\Delta'/\alpha$ as a function of x for cases with ITB ($a_2 = 5.0$) and without the ITB ($a_2 = 0$).

5. Stability analysis

5.1. Stability of plasma in ITB region

Both CHT's equation and our new equation indicate that the plasma in the negative shear region (in our case, $x \leq 0.3$) is stable against the high n ideal MHD ballooning mode so that we are interested only in the positive shear region. In and near the ITB region ($0.3 \leq x < 0.4$ somehow), the shear is small, α is very large, and $\delta_{\rm c} > 0.2$. A salient feature in this region is that both the toroidicity effect and the Shafranov shift can play a role in stabilizing plasma so well that the first and second stability regions in the $s-\alpha$ space,[1–6] which are originally separated, now are connected by a low shear stable channel. A typical example is shown in Fig.5. This means that the plasma is always stable in the region no matter

how large the plasma pressure gradient is.

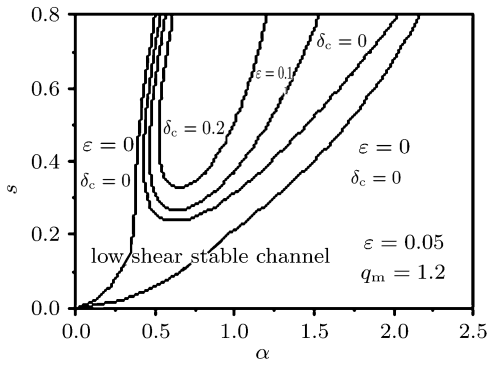


Fig.5. In the ITB region, the toroidicity effect ($\varepsilon \neq 0$) and the Shafranov shift effect ($\delta_c > 0$) produce a low shear stable channel in the $s - \alpha$ space, which makes the plasma stable.

5.2. Stability in an intermediate region

Now we consider the stability in an intermediate region defined as $0.4 \leq x \leq 0.8$. In our example, we have selected such a pressure profile model that outside the ITB region it is almost the same as that in the case without the ITB. The only difference between these two cases is the different Shafranov shift, or the quantity δ_c as seen from Fig.4(b). We envisage that due to the larger Shafranov shift induced by the ITB, stability of the plasma in this region is improved compared with the counterpart case without the ITB (see Fig.6).

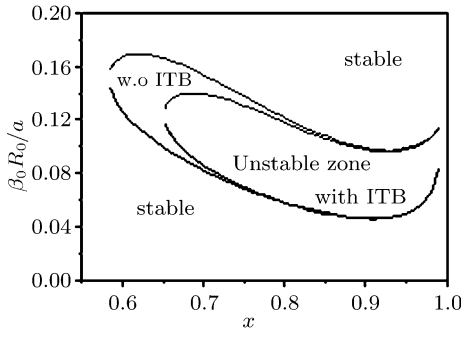


Fig.6. Unstable zone in the $(\beta_0 R_0/a) - x$ space for the configuration without the ITB.

5.3. Unstable zone in the edge region

In the outer region of the ITB, with the increase of the distance from the location of ITB, the effect of the ITB becomes less important so that the stability margin will only be slightly extended compared with the case without the ITB (also see above Fig.6). If there exists no other stabilizing mechanism (such as the parallel flow or flow shear, the poloidal flow or flow shear, the electric field and its shear, etc.), usually, there will appear an unstable zone in this region. With the parameters cited above, this unstable zone is shown in Fig.7.

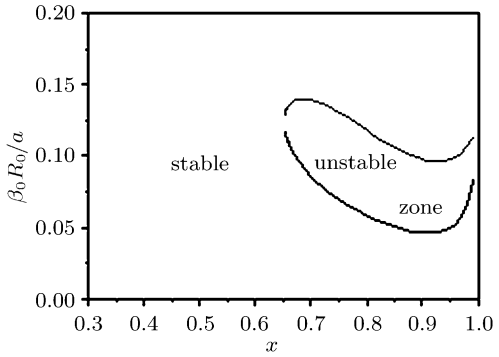


Fig.7. Unstable region in the $(\beta_0 R_0/a) - x$ space, showing the allowable beta for an ITB configuration. Only in the near edge region there exists an unstable zone for this configuration with parameters cited above.

The toroidicity effect ($\varepsilon \neq 0$) on the ballooning mode, which basically is the averaged favourable curvature stabilizing effect (the Mercier term), has been indicated recently.[10] However, if we take only this effect into account, the corresponding unstable zone for the ITB configuration is much wider than that in the case with the Shafranov shift as shown in Fig.8. This is one of our major new results.

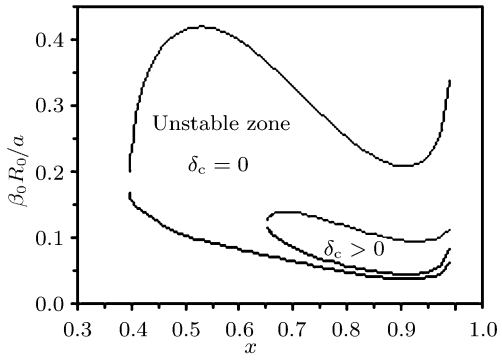


Fig.8. Unstable zone in the $(\beta_0 R_0/a) - x$ space for the configuration if only the toroidicity effect is considered without the Shafranov shift ($\delta_c = 0$).

6. Summary

In this paper we derive a new eigen-mode equation of the ideal MHD ballooning mode for configuration with shift circular flux surfaces. This equation is used to study the whole region stability of tokamak plasma with ITB. The plasma is completely stabilized inside the ITB region. Outside the ITB region, it is much stabler than in the case without the ITB. It is indicated that the stabilizing effect due to the Shafranov shift is rather strong. With this effect taken

into account, the unstable zone in the $(\beta_0 R_0/a) - x$ space is much narrower than with only the toroidicity effect (the Mercier term) considered as indicated in Ref.[10]. Physically, the complete stabilizing of the configuration, which means a free transition of the plasma from the first stable region to the second stable region, needs other stabilizing mechanisms like some plasma flow or flow shear in the boundary region to be supported.

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