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Singularly perturbed solution of a sea–air oscillator model for the ENSO*

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A time delay equation for the sea–air oscillator model is studied. The aim is to create an asymptotic solving method of nonlinear equation for the El Niño–Southern Oscillation model. And based on a class of oscillators of the model, employing the method of ENSO singular perturbation, the asymptotic solution of corresponding problem is obtained. It is proven from the results that the method of singular perturbation can be used for analysing the sea surface temperature anomaly in the equatorial eastern Pacific of the atmosphere–ocean oscillation for ENSO model.

Keywords: nonlinear, time delay, ENSO
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1. Introduction

During the last decade, many studies focused on interannual climate variability associated with the El Niño–Southern Oscillation (ENSO). Progress has been made in understanding and simulating the coupled tropical ocean–atmosphere system by using models with varying complexity.[1] A mechanism for the oscillatory nature of ENSO was originally proposed by McCreary,[2] based on reflection of subtropical oceanic upwelling Rossby waves at the western boundary. Suarez and Schopf[3] introduced the conceptual delayed oscillator as a candidate mechanism for ENSO, by considering the effects of equatorially trapped oceanic waves propagating in a closed basin through a delay term. The conceptual delayed oscillator model is represented by a single ordinary differential delay equation with both positive and negative feedbacks. The positive feedback is represented by the local ocean–atmosphere coupling in the equatorial eastern Pacific. The delayed negative feedback is represented by free Rossby waves generated in the eastern Pacific coupling region, which propagate to and reflect from the western boundary, returning as Kelvin waves to reverse the anomalies in the eastern Pacific coupling region.

The ENSO is a natural phenomenon involved in the tropical Pacific ocean–atmosphere interactions. Phenomenon of the ENSO is a very attractive subject of study in the international academic circles.[4−13] Mo, Lin *et al* [14−26] considered also a class of nonlinear problems in atmospheric physics, ocean weather, dynamic system and so on.

Recently, many scholars have investigated the approximate theory of the nonlinear problems.[27−35] Approximate methods have been developed and refined, including the method of averaging, boundary layer method, matched asymptotic expansion method and multiple scales method. In this paper we consider a sea–air oscillator model by using a simple and valid method for nonlinear singularly perturbed theory.

2. Model of equatorial Pacific SST

The delayed oscillator model does not consider the effect of the western Pacific on ENSO. It is assumed that winds in the western Pacific do not affect the sea surface temperature (SST) anomalies in the

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eastern Pacific and the atmosphere is in a steady state or the Niño-4 (over 5°S–5°N,160°E–150°W) zonal wind stress anomalies are linearly proportional to the Niño-3 (over 5°S–5°N,150°E–90°W) SST anomalies. We can obtain the following nonlinear time delay model of the equatorial Pacific:[13]

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{ad}{R}T - \frac{bd}{R}T(t-\eta) - \varepsilon T^3, \tag{1}$$

where T is the Niño-3 SST anomaly, η is a time for waves to travel to the western boundary and return to the eastern Pacific, a is a coefficient representing the positive feedback between T's, b is a coefficient representing the negative feedback due to waves reflection at the western boundary, d is a positive coefficient that relates the Niño-3 SST anomalies to the Niño-4 zonal wind stress anomalies, R is a damping coefficient and ε is a cubic damping coefficient. We also assume that $\eta = c\,\varepsilon$, where c is a positive constant and ε is a small positive parameter.

3. Singularly perturbed solution of SST

We now solve the asymptotic solution of the nonlinear delay model Eq.(1). Let

$$T(t) = T_0(t) + T_1(t)\varepsilon + T_2(t)\varepsilon^2 + \cdots. \tag{2}$$

Substituting Eq.(2) into Eq.(1), expanding the nonlinear term in ε, and equating the coefficients of the same powers correspondingly, for ε^0 we obtain

$$\frac{\mathrm{d}T_0}{\mathrm{d}t} = \frac{d(a-b)}{R}T_0.$$

And we have

$$T_0 = C_0\ \exp(\frac{d(a-b)}{R}t). \tag{3}$$

For ε^1 , we obtain

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} = \frac{d(a-b)}{R}T_1 + \frac{bcd}{R}\frac{\mathrm{d}T_0}{\mathrm{d}t} - T_0^3.$$

From Eq.(3), we have

$$\frac{\mathrm{d}T_1}{\mathrm{d}t} = \frac{d(a-b)}{R}T_1 + \frac{bcd(a-b)C_0}{R^2}\ \exp(\frac{d(a-b)}{R}t) - C_0^3\ \exp(\frac{3d(a-b)}{R}t).$$

Thus we obtain

$$T_1 = C_1\ \exp(\frac{d(a-b)}{R}t) - \frac{bcd(a-b)C_0}{R^2}t\ \exp(\frac{d(a-b)}{R}t) - \frac{C_0^3R}{2ad-3bd}\ \exp(\frac{3d(a-b)}{R}t). \tag{4}$$

Substituting Eqs.(3), (4) into Eq.(2), we obtain the asymptotic solution of Eq.(1). But it is not a uniformly valid solution. Therefore, we need to eliminate its secular term. Let

$$\tau = \omega(\varepsilon)t,\ \omega(\varepsilon) = 1 + \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots, \tag{5}$$

where ω_i, i=1,2,$\cdots$, are constants to be undetermined. From Eq.(5), we have

$$t = [\omega(\varepsilon)]^{-1}\tau = (1 - \omega_1\varepsilon + 2(\omega_1+\omega_2)\varepsilon^2 + \cdots)\tau. \tag{6}$$

Keeping in mind Eqs.(3), (4) and substituting Eq.(6) into Eq.(2), we obtain

$$\begin{aligned} T\ =\ & T_0 + T_1\varepsilon + \cdots = C_0\ \exp(\frac{d(a-b)}{R}(1 - \omega_1\varepsilon + 2(\omega_1+\omega_2)\varepsilon^2 + \cdots)\tau) \\ & +[C_1\ \exp(\frac{d(a-b)}{R}(1 - \omega_1\varepsilon + 2(\omega_1+\omega_2)\varepsilon^2 + \cdots)\tau) \\ & -\frac{bcd(a-b)C_0}{R^2}(1 - \omega_1\varepsilon + 2(\omega_1+\omega_2)\varepsilon^2 + \cdots)\tau \\ & \times\exp(\frac{d(a-b)}{R}(1 - \omega_1\varepsilon + 2(\omega_1+\omega_2)\varepsilon^2 + \cdots)\tau) \\ & -\frac{C_0^3R}{2ad-3bd}\ \exp(\frac{3d(a-b)}{R}(1 - \omega_1\varepsilon + 2(\omega_1+\omega_2)\varepsilon^2 + \cdots)\tau)]\varepsilon + \cdots \end{aligned}$$

$$
\begin{aligned}
&= (C_0 + C_1\varepsilon)\ \exp(\frac{d(a-b)}{R}\tau) \\
&-[\frac{d(a-b)C_0}{R}(\omega_1 + \frac{bc}{R})\tau\ \exp(\frac{d(a-b)}{R}\tau)]\varepsilon \\
&-[\frac{C_0^3 R}{(2a-3b)d}\ \exp(\frac{3d(a-b)}{R}\tau)]\varepsilon + O(\varepsilon^2). \qquad (7)
\end{aligned}
$$

Thus we know that if letting $\omega_1 = -\dfrac{bc}{R}$, we can eliminate the secular term in Eq.(7). And from Eq.(5), we obtain

$$\omega = 1 + \frac{bc}{R}\varepsilon + O(\varepsilon^2).$$

Then we have

$$
\begin{aligned}
T(t) =\ & (C_0 + C_1\varepsilon)\exp(\frac{d(a-b)}{R}(1 - \frac{bc}{R}\varepsilon)t) \\
& -[\frac{C_0^3}{(2a-3b)d}\exp(\frac{3d(a-b)}{R}(1 - \frac{bc}{R}\varepsilon)t)]\varepsilon + O(\varepsilon^2), \quad 0 < \varepsilon \ll 1, \qquad (8)
\end{aligned}
$$

where C_0 and C_1 are arbitrary constants. Obviously, Eq.(8) is a first-order uniformly valid asymptotic expansion of the solution to Eq.(1).

We also can obtain even higher-order uniformly valid solution for Eq.(1) using the same method.

4. Precision

In order to evaluate the precision of expansion (8) of the asymptotic solution to Eq.(1), we give a set of parameters as follows:[13] $a = 1.5 \times 10^2\ ^\circ\mathrm{C}\cdot\mathrm{m}^2\cdot\mathrm{N}^{-1}\cdot\mathrm{yr}^{-1}$, $b = 2.5 \times 10^2\ ^\circ\mathrm{C}\cdot\mathrm{m}^2\cdot\mathrm{N}^{-1}\cdot\mathrm{yr}^{-1}$, $d = 3.6 \times 10^{-2}\ ^\circ\mathrm{C}\cdot\mathrm{m}^{-2}\cdot\mathrm{N}\cdot\mathrm{yr}^{-1}$, $r = 2.0\mathrm{yr}^{-1}$, $c = 0$, $C_0 = 1$, $C_1 = 0$.

Now we make a comparison between the values of T_{num} obtained from the numerical calculation and T_{asy} from the asymptotic expansion (8) in the case of different values of ε($^\circ\mathrm{C}^{-2}\cdot\mathrm{yr}^{-1}$) and t(yr) . We obtain the following results(see Table 1).

Table 1. A comparison of the values of between T_{num} and T_{asy}

ε	T	$t=0.0$	$t=0.5$	$t=1.0$	$t=1.5$	$t=2.0$	$t=2.5$	$t=3.0$
1	T_{num}	1.5000	0.4621	0.1735	0.0685	0.0275	0.0111	0.0045
	T_{asy}	1.0062	0.4070	0.1653	0.0672	0.0273	0.0111	0.0045
0.1	T_{num}	1.0290	0.4112	0.1661	0.0673	0.0273	0.0111	0.0045
	T_{asy}	1.0006	0.4066	0.1653	0.0672	0.0273	0.0111	0.0045
0.01	T_{num}	1.0028	0.4070	0.1654	0.0672	0.0273	0.0111	0.0045
	T_{asy}	1.0001	0.4066	0.1653	0.0672	0.0273	0.0111	0.0045
0.001	T_{num}	1,0003	0.4066	0.1653	0.0672	0.0273	0.0111	0.0045
	T_{asy}	1.0000	0.4066	0.1653	0.0672	0.0273	0.0111	0.0045

5. Conclusions

(1) From the comparison of precisions obtained as above, we can conclude that for small parameter ε, the asymptotic expansion (8) is a good approximate expansion of the solution for the nonlinear time delay model (1).

(2) The ENSO is a complicated natural phenomenon. Hence we need to reduce the basic models for the sea–air oscillator and solve them by using the approximate method. The singular perturbation approach is a simple and valid method.

(3) The singular perturbation is an asymptotic method, which differs from general numerical method. The expansion of a solution by means of the singular perturbation method can be made in the analytic op-

eration. Thus, from Eq.(8), we can further study the quality and quantitative behaviours of the temperature anomaly and the zonal wind stress anomaly in the equatorial eastern Pacific. But in this paper we do not discuss this any further.

References

[1] Neelin J D, Battisti D S, Hirst A C, Jin F F, Wakata Y, Yamagata T and Zebiak S E 1998 *J. Geophys. Res.* **103** 14

[2] McCreary J P 1983 *Mon. Wea. Rev.* **111** 370

[3] Suarez M J and Schopf P S 1988 *J. Atmos. Sci.* **45** 3283

[4] McPhaden M J and Zhang D 2002 *Nature* **415** 603

[5] Gu D F and Philander S G H 1997 *Science* **275** 805

[6] An S I and Wang B 2000 *J. Climate* **13** 2044

[7] Nonaka M, Xie S P and McCreary J P 2001 *Geophys. Res. Lett.* **20** 1116

[8] Han H L 2004 *Acta Phys. Sin.* **53** 4061(in Chinese)

[9] Feng G L, Dong W J, Jia X J and Cao H X 2002 *Acta Phys. Sin.* **51** 1181

[10] Liu S K, Fu Z T, Liu S D and Zhao Q 2002 *Chin. Phys. Sin.* **51** 10

[11] Wang C Z 2001 *Advances in Atmospheric Sciences* **18** 674

[12] Biondi F, Gershunov A and Cayan D R 2001 *J. Climate* **14** 5

[13] Wang C Z 2001 *J. Climate* **14** 98

[14] Mo J Q, Zhu J and Wang H 2003 *Progress in Natural Sci.* **13** 768

[15] Mo J Q, Lin W T and Zhu J 2004 *Progress in Natural Sci.* **14** 550

[16] Mo J Q, Lin W T and Zhu J 2004 *Progress in Natural Sci.* **14** 1126

[17] Mo J Q and Lin W T 2004 *Acta Phys. Sin.* **53** 996 (in Chinese)

[18] Mo J Q, Lin W T and Zhu J 2004 *Acta Phys. Sin.* **53** 3245 (in Chinese)

[19] Mo J Q and Lin W T 2005 *Acta Phys. Sin.* **54** 993 (in Chinese)

[20] Mo J Q and Lin W T 2005 *Acta Phys. Sin.* **54** 1081 (in Chinese)

[21] Mo J Q and Lin W T 2005 *Chin. Phys.* **14** 875

[22] Lin Y H and Ceng Q C 1999 *Progress in Natural Sci.* **9** 211

[23] Lin Y H, Ji Z Z and Zeng Q C 1999 *Progress in Natural Sci.* **9** 532

[24] Lin Y H, Xue F and Lian S M 2001 *Chin. J. Atmospheric Sci.* **25** 111

[25] Lin Y H and Fu Y F 2001 *Acta Meteorologica Sin.* **59** 1

[26] Lin W T and Mo J Q 2004 *Chin. Sci. Bull.* **48** II 5

[27] de Jager E M and Jiang F R 1996 *Theory of Singular Perturbation* (Amsterdam: North-Holland)

[28] Hwangm S 2004 *J. Diff. Eqns.* **200** 191

[29] Zhang F 2004 *J. Diff. Eqns.* **205** 77

[30] Ammari H, Kang H and Touibi K 2005 *Asymptotic Anal.* **41** 119

[31] Khasminskii R Z and Yin G 2005 *J. Diff. Eqns.* **212** 85

[32] Marques I 2005 *Nonlinear Anal.* **61** 21

[33] Ouyang C 2004 *Acta Phys. Sin.* **53** 1900 (in Chinese)

[34] Han H L 2005 *Acta Phys. Sin.* **54** 2590 (in Chinese)

[35] Wu Q K 2005 *Acta Phys. Sin.* **54** 2510 (in Chinese)