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A lattice Bhatnagar–Gross–Krook model for a class of the generalized Burgers equations*

Yu Xiao-Mei(余晓美) and Shi Bao-Chang(施保昌)†

Department of Mathematics, Huazhong University of Science and Technology, Wuhan 430074, China

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A new lattice Bhatnagar–Gross–Krook (LBGK) model for a class of the generalized Burgers equations is proposed. It is a general LBGK model for nonlinear Burgers equations with source term in arbitrary dimensional space. The linear stability of the model is also studied. The model is numerically tested for three problems in different dimensional space, and the numerical results are compared with either analytic solutions or numerical results obtained by other methods. Satisfactory results are obtained by the numerical simulations.

Keywords: LBGK model, a class of generalized Burgers equation, stability, simulation
PACC: 0540, 0340, 0270

1. Introduction

In recent years, the solutions of nonlinear equations have attracted much attention in the study of nonlinear problems, which are central to the deeply developed field of nonlinear science. A class of the generalized Burgers equation has been found to describe various kinds of phenomena such as mathematical modelling of turbulence[1] and the approximation theory of flow through a shock wave travelling in a viscous fluid.[2] The Burgers equation and Navier–Stokes equation are similar in the form of their nonlinear terms and in the occurrence of higher order derivatives with small coefficients in both.[3] It is well known that the accuracy of the numerical computations and the computational efficiency are highly dependent on the numerical schemes used to solve the fluid flow equations. In many cases, numerical solutions are likely to produce results which consist of large non-physical oscillations if the size of the elements is not unrealistically small. Several numerical methods of these equations have been proposed, such as algorithms based on the cubic spline function technique,[4] explicit–implicit method,[5] multilevel scheme with ADI[6] and implicit finite-difference scheme.[7] Most of these existing works are concerned with the one-dimensional Burgers equation. It is difficult to solve higher-dimensional Burgers equations simply and simulate accurately.[8]

The lattice Boltzmann method (LBM) is a new technique for simulating fluid flows and modelling complex systems.[9] Unlike traditional numerical methods which solve for macroscopic variables, the LBM is based on the microscopic kinetic equation for the particle distribution function. The macroscopic quantities, such as density and velocity, are then obtained through moment integrations of the distribution function. The kinetic nature of LBM introduces a number of advantages, such as simple coding, intrinsic parallelism and incorporation of complicated boundary conditions. In the last decade, the lattice Bhatnagar–Gross–Krook (LBGK) model, the most popular LBM, has achieved great success in a variety of fields, from simple laminar flows to thermal flows,[10,11] high-speed compressible flows[12] and so on. The LBGK model also shows the ability to simulate the nonlinear systems. Some LBGK models have been proposed in Refs.[13–15] for Burgers equations, but they are all concerned with the one-dimensional Burgers equations. The LBGK model in Ref.[16] is able to solve two-dimensional Burgers equations; however, it is difficult to extend to three-dimensional equations due to its complex expressions of equilibrium distribution function.

In this paper, we propose a uniform LBGK model for a class of Burgers equation with source term. The

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†Corresponding author. E-mail: sbchust@126.com

rest of the paper is organized as follows. In Section 2 the LBGK model is proposed and the class of generalized Burgers equations is derived. The linear stability analysis of the model is given in Section 3. Numerical results are found in Section 4. Finally, some conclusions are made in Section 5.

2. LBGK model for a class of generalized Burgers equations

In this section, we develop a LBGK model based on the DdQq model[17] to compute approximate solutions for the following generalized d-dimensional Burgers equation:

$$\frac{\partial u}{\partial t}+\beta\nabla\cdot\left(\frac{1}{2}u^2\boldsymbol{E}\right)=\nu\nabla^2u+F(\boldsymbol{x},t). \tag{1}$$

Here, $\boldsymbol{E}$= (1,1,...,1) is the constant vector with length d, β is a constant, and ν is the diffusion constant coefficient. For a small value of ν, the Burgers equation behaves merely as a hyperbolic partial differential equation, and the problem becomes very difficult to solve as steep shock-like wave fronts developed, as reported in Ref.[18]. ∇ is the gradient operator in d dimensions and ∇^2 is the corresponding Laplace operator. The key step to develop a lattice Boltzmann model for the macroscopic equation is how to derive the nonlinear term $\beta\nabla\cdot\left(\frac{1}{2}u^2\boldsymbol{E}\right)$ and the source term $F(\boldsymbol{x},t)$ without additional assumption.

2.1. LBGK model

The evolution equation of the density distribution function with source in the LBGK model reads

$$\begin{aligned}&f_i(\boldsymbol{x}+\boldsymbol{c}_i\Delta t,t+\Delta t)-f_i(\boldsymbol{x},t)\\=&-\frac{1}{\tau}(f_i(\boldsymbol{x},t)-f_i^{\rm eq}(\boldsymbol{x},t))+\Delta tF_i(\boldsymbol{x},t)\\&+\frac{1}{2}\Delta t^2\frac{\partial F_i(\boldsymbol{x},t)}{\partial t},\end{aligned} \tag{2}$$

where $\boldsymbol{c}_i$ and Δt are the discrete velocity and the time step respectively. τ characterizes the dimensionless relaxation time of the distribution function towards the local equilibrium $f_i^{\rm eq}$, and F_i is the distribution of the source. $\dfrac{\partial F_i(\boldsymbol{x},t)}{\partial t}$ can be computed through forward, backward difference schemes or exact solution. $f_i^{\rm eq}$ and F_i are defined as

$$f_i^{\rm eq}(\boldsymbol{x},t)=\begin{cases}(1-\alpha)u+\omega_i\left[\alpha u+\dfrac{u^2(\boldsymbol{c}_i\cdot\beta\boldsymbol{E})}{2c_{\rm s}^2}+\dfrac{u^3(\boldsymbol{c}_i\cdot\beta\boldsymbol{E})^2}{6c_{\rm s}^4}-\dfrac{u^3\beta^2d}{6c_{\rm s}^2}\right], & i=0,\\ \omega_i\left[\alpha u+\dfrac{u^2(\boldsymbol{c}_i\cdot\beta\boldsymbol{E})}{2c_{\rm s}^2}+\dfrac{u^3(\boldsymbol{c}_i\cdot\beta\boldsymbol{E})^2}{6c_{\rm s}^4}-\dfrac{u^3\beta^2d}{6c_{\rm s}^2}\right], & i=1,2,\cdots,(q-1),\end{cases} \tag{3}$$

$$F_i=\omega_iF\left(1+\frac{\tau-1/2}{\tau}\cdot\frac{(\boldsymbol{c}_i\cdot\beta\boldsymbol{E})u}{c_{\rm s}^2}\right), \tag{4}$$

where ω_i are the weight coefficients which are different for different models. And α is a parameter to adjust different conditions and enhance the ability of the model.

The macroscopic parameters are determined from the moments of the distribution functions as follows:

$$\begin{aligned}\sum_i f_i=&\sum_i f_i^{\rm eq}=u,\\ \sum_i\boldsymbol{c}_if_i^{\rm eq}=&\frac{1}{2}u^2\beta\boldsymbol{E},\\ \sum_i\boldsymbol{c}_i\boldsymbol{c}_if_i^{\rm eq}=&\alpha c_{\rm s}^2u\boldsymbol{I}+\frac{1}{3}u^3\beta^2\boldsymbol{E}\boldsymbol{E},\end{aligned} \tag{5}$$

where $\boldsymbol{I}$ is the unit tensor. And F_i satisfies

$$\sum_iF_i=F,\quad \sum_i\boldsymbol{c}_iF_i=\frac{\tau-1/2}{\tau}Fu\beta\boldsymbol{E}. \tag{6}$$

Making Taylor expansion on the left of Eq.(2) we get

$$\begin{aligned}&f_i(\boldsymbol{x}+\boldsymbol{c}_i\Delta t,t+\Delta t)-f_i(\boldsymbol{x},t)\\=&\Delta t\left(\frac{\partial}{\partial t}+\boldsymbol{c}_i\cdot\nabla\right)f_i+\frac{\Delta t^2}{2}\left(\frac{\partial}{\partial t}+\boldsymbol{c}_i\cdot\nabla\right)^2f_i.\end{aligned} \tag{7}$$

Next, the Chapman–Enskog expansion is applied to f_i under the assumption that the mean free path is of the same order as ε.

$$f_i=f_i^0+\varepsilon f_i^{(1)}+\varepsilon^2f_i^{(2)}+.... \tag{8}$$

And we discuss changes in different time scales

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t_0}+\varepsilon\frac{\partial}{\partial t_1}+\varepsilon^2\frac{\partial}{\partial t_2}. \tag{9}$$

Here we take $\varepsilon=\Delta t$. Applying Eqs.(7), (8) and (9) to Eq.(2), the series lattice Boltzmann equations

are derived as follows:

$$f_i^{\rm eq} = f_i^0, \tag{10}$$

$$\left(\frac{\partial}{\partial t_0} + \boldsymbol{c}_i \cdot \nabla\right) f_i^{(0)} = -\frac{1}{\tau} f_i^{(1)} + F_i, \tag{11}$$

$$\begin{aligned}&\frac{\partial}{\partial t_1} f_i^{(0)} + \left(1 - \frac{1}{2\tau}\right)\left(\frac{\partial}{\partial t_0} + \boldsymbol{c}_i \cdot \nabla\right) f_i^{(1)} \\ &+ \frac{1}{2}\left(\frac{\partial}{\partial t_0} + \boldsymbol{c}_i \cdot \nabla\right) F_i \\ =& -\frac{1}{\tau} f_i^{(2)} + \frac{1}{2}\frac{\partial F_i}{\partial t_0}.\end{aligned} \tag{12}$$

From Eqs.(5), (8) and (10), we know that

$$\sum_i f_i^{(k)} = 0, \quad k \geq 1. \tag{13}$$

Summing Eq.(11) over i and using Eqs.(5), (6) and (13), we obtain

$$\frac{\partial u}{\partial t_0} + \nabla \cdot \left(\frac{1}{2} u^2 \beta E\right) = F. \tag{14}$$

Then taking the summation in Eqs.(11) and (12) about i, computing (11)+(12)$\times\varepsilon$, and using Eqs.(5), (9), (10) and (10), we get

$$\begin{aligned}&\frac{\partial u}{\partial t} + \nabla \cdot \left(\frac{1}{2} u^2 \beta E\right) + \varepsilon\left(1 - \frac{1}{2\tau}\right)\nabla \\ &\cdot \left(\sum_i \boldsymbol{c}_i f_i^{(1)}\right) + \frac{\varepsilon}{2}\nabla \cdot \left(\sum_i \boldsymbol{c}_i F_i\right) = F;\end{aligned}$$

Using Eqs.(11), (14), (5) and (6), we obtain

$$\begin{aligned}-\frac{1}{\tau}\sum_i \boldsymbol{c}_i f_i^{(1)} =& \frac{\partial}{\partial t_0}\left(\sum_i \boldsymbol{c}_i f_i^{(0)}\right) + \nabla \cdot (\boldsymbol{c}_i \boldsymbol{c}_i f_i^{(0)}) - \sum_i \boldsymbol{c}_i F_i \\ =& \frac{\partial}{\partial t_0}\left(\frac{1}{2} u^2 \beta \boldsymbol{E}\right) + \nabla \cdot \left(\alpha c_{\rm s}^2 u I + \frac{1}{3} u^3 \beta^2 \boldsymbol{E}\boldsymbol{E}\right) - \frac{\tau - 1/2}{\tau} F u \beta \boldsymbol{E} \\ =& u\beta \boldsymbol{E}\left(\frac{\partial u}{\partial t_0} + \nabla \cdot \left(\frac{1}{2} u^2 \beta \boldsymbol{E}\right) - F\right) + \nabla \cdot (\alpha c_{\rm s}^2 u \boldsymbol{I}) + \frac{1}{2\tau} F u \beta \boldsymbol{E} \\ =& \nabla \cdot (\alpha c_{\rm s}^2 u \boldsymbol{I}) + \frac{1}{2\tau} F u \beta \boldsymbol{E}.\end{aligned}$$

Therefore, we obtain

$$\frac{\partial}{\partial t} u + \nabla \cdot \left(\frac{1}{2} u^2 \beta \boldsymbol{E}\right) = \alpha c_{\rm s}^2 \varepsilon\left(\tau - \frac{1}{2}\right)\nabla^2 u + F. \tag{15}$$

From the above analysis, we can see that the diffusion coefficient can be determined by $\nu = \alpha c_{\rm s}^2 \Delta t\left(\tau - \frac{1}{2}\right)$. And the source term is without any assumption as shown in Ref.[19].

We can find that a class of important equations is derived at the same time. Just take the one-dimensional condition for example:

Burgers equation: without source term, F=0, $u_t + \beta u u_x = D u_{xx}$;

Fisher equation: $\beta = 0$, $u_t = D u_{xx} + \lambda u(1 - u)$;

Pure diffusion equation: $\beta = 0$, F=0,$u_t = D u_{xx}$

Diffusionless Burgers equation: ν=0, F=0, $u_t + a u u_x = 0$;

Ordinary difference equation: $\beta = 0$, α=0, $u_t = F$.

That is to say we can use the LBGK model to solve the class of the important equation. Moreover, just through the change of the dimensions of the vector $\boldsymbol{E}$, we can solve the equations in any higher dimension, while the model in Ref.[16] is difficult to apply even in two-dimensional equations due to their complex expressions of the equilibrium function. We use some DdQq models to solve several problems for example. It is easy to develop any other DdQq model by the same method.

2.2. The general LBGK models for the class of the generalized Burgers equations

In this paper, three models D1Q3, D2Q9 and D3Q15 are adapted to simulate different problems respectively. From Eqs.(3) and (4), we know that only the discrete velocities sets and the weight coefficients are different in these models.

The particles velocity vectors for D1Q3 are defined by $(\boldsymbol{c}_0, \boldsymbol{c}_1, \boldsymbol{c}_2)$= (0, $-c$, c) with the weight coefficients ω_0=2/3, $\omega_1 = \omega_2$=1/6. $c_{\rm s} = c/\sqrt{3}$ is the sound speed of the model, and $c = \Delta x/\Delta t$ is the particle speed.

The D2Q9 LBGK model for the two-dimensional Burgers equation has the particles velocity vectors defined by

$$\boldsymbol{c}_i = \begin{cases} (0,0), & i=0, \\ c(\cos\theta_i, \sin\theta_i), & (\theta_i=(i-1)\pi/2, i=1,2,3,4), \\ \sqrt{2}c(\cos\theta_i, \sin\theta_i), & (\theta_i=(i-5)\pi/2+\pi/4, i=5,6,7,8). \end{cases}$$

The weight coefficients are chosen asω_0=4/9, $\omega_1 = \omega_2 = \omega_3 = \omega_4$=1/9, $\omega_5 = \omega_6 = \omega_7 = \omega_8$=1/36.

It is more difficult for higher dimensions. A simple way to extend the square lattice, with vectors to the sides and corners of the square, to three dimensions is to use vectors to the sides and corners of a cube.[20] This defines a body-centred-cubic lattice with $\boldsymbol{c}_i^{\mathrm{I}} \in \{(\pm c,0,0),(0,\pm c,0),(0,0,\pm c)\}$, $\boldsymbol{c}_i^{\mathrm{II}} \in \{(\pm c,\pm c,\pm c)\}$, $\boldsymbol{c}_0$=(0,0,0). The equilibrium distribution can be written as

$$f_i^{\mathrm{eq}}(\boldsymbol{x},t) = \begin{cases} (1-\alpha)u + \omega_0\left[au + \dfrac{u^2(\boldsymbol{c}_0\cdot\beta\boldsymbol{E})}{2c_{\mathrm{s}}^2} + \dfrac{u^3(\boldsymbol{c}_0\cdot\beta\boldsymbol{E})^2}{6c_{\mathrm{s}}^4} - \dfrac{u^3\beta^2 d}{6c_{\mathrm{s}}^2}\right], & i=0, \\ \omega_i^{\mathrm{I}}\left[\alpha u + \dfrac{u^2(\boldsymbol{c}_i^{\mathrm{I}}\cdot\beta\boldsymbol{E})}{2c_{\mathrm{s}}^2} + \dfrac{u^3(\boldsymbol{c}_i^{\mathrm{I}}\cdot\beta\boldsymbol{E})^2}{6c_{\mathrm{s}}^4} - \dfrac{u^3\beta^2 d}{6c_{\mathrm{s}}^2}\right], & i=1,2,\cdots,7, \\ \omega_i^{\mathrm{II}}\left[\alpha u + \dfrac{u^2(\boldsymbol{c}_i^{\mathrm{II}}\cdot\beta\boldsymbol{E})}{2c_{\mathrm{s}}^2} + \dfrac{u^3(\boldsymbol{c}_i^{\mathrm{II}}\cdot\beta\boldsymbol{E})^2}{6c_{\mathrm{s}}^4} - \dfrac{u^3\beta^2 d}{6c_{\mathrm{s}}^2}\right], & i=8,9,\cdots,15, \end{cases} \tag{16}$$

with ω_0=2/9, ω_i^{I}=1/9, ω_i^{II}=1/72 and $c_{\mathrm{s}} = c/\sqrt{3}$.

3. Stability analysis

The analysis presented in what follows is similar to that presented in Ref.[21], where the goal of the work is to determine the stability conditions for the new LBGK model.

We consider the condition of the evolution equation (2) without source term

$$\begin{aligned} &f_i(\boldsymbol{x}+\boldsymbol{c}_i\Delta t, t+\Delta t) - f_i(\boldsymbol{x},t) \\ =&-\frac{1}{\tau}(f_i(\boldsymbol{x},t) - f_i^{\mathrm{eq}}(\boldsymbol{x},t)). \end{aligned} \tag{17}$$

The update operator is defined as

$$g_i(f_j) = f_i(\boldsymbol{x},t) - \frac{1}{\tau}(f_i(\boldsymbol{x},t) - f_i^{\mathrm{eq}}(\boldsymbol{x},t)). \tag{18}$$

Just as the analysis in the much of the previous work showed in Ref.[22], the resulting linearized system is

$$f_i'(\boldsymbol{x}+\boldsymbol{c}_i\Delta t, t+\Delta t) = G_{ij} f_j'(\boldsymbol{x},t). \tag{19}$$

And the spatial dependence of the stability is invested by taking the Fourier transform of Eq.(19) to obtain

$$F_i(\boldsymbol{k}, t+\Delta t) = L_{ij} F_j(\boldsymbol{k},t), \tag{20}$$

$$L_{ij} = \mathrm{diag}\{\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{c}_i\Delta t)\}G_{ij}. \tag{21}$$

Where G_{ij} is the Jacobian matrix corresponding to the coefficient of the linear term of $g_i(f_j)$ in Eq.(18) at the $\overline{f_j^{\mathrm{eq}}}$ which does not depend on location or time.

According to the expression (3), we obtain that

$$\frac{\partial f_i^{\mathrm{eq}}}{\partial f_j} = \begin{cases} (1-\alpha) + \omega_i\left[\alpha + \dfrac{u(\boldsymbol{c}_i\cdot\beta\boldsymbol{E})}{c_{\mathrm{s}}^2} + \dfrac{u^2(\boldsymbol{c}_i\cdot\beta\boldsymbol{E})^2}{2c_{\mathrm{s}}^4} - \dfrac{u^2\beta^2 d}{2c_{\mathrm{s}}^2}\right], & i=0, \\ \omega_i\left[\alpha + \dfrac{u(\boldsymbol{c}_i\cdot\beta\boldsymbol{E})}{c_{\mathrm{s}}^2} + \dfrac{u^2(\boldsymbol{c}_i\cdot\beta\boldsymbol{E})^2}{2c_{\mathrm{s}}^4} - \dfrac{u^2\beta^2 d}{2c_{\mathrm{s}}^2}\right], & i\neq 0. \end{cases} \tag{22}$$

Obviously, the expression of G_{ij} is

$$G_{ij} = (1-\frac{1}{\tau})\boldsymbol{I} + \frac{1}{\tau} f_{if_j}^{\mathrm{eq}}(\overline{u}), \tag{23}$$

where $\boldsymbol{I}$ is the unit diagonal matrix, and $\overline{u}$ is a constant which does not change with node and time. If

we denote $J_i(f) = -\dfrac{(f_i - f_i^{\rm eq})}{\tau}$, the Jacobian matrix is $J_f^*(f) := ([f_{if_j}^{\rm eq}(\overline{u})] - \boldsymbol{I})/\tau$. Then Eq.(23) can be written as

$$G_{ij} = \boldsymbol{I} + J_f^*(f). \tag{24}$$

Firstly, we discuss how to choose some parameters in the new LBGK model by requiring that the model fulfills the stability conditions. Just take the D2Q9 model for example, and unit particle speed is adapted for simplicity without loss of generality.[21] Thus, Eq.(22) can be written as

$$\frac{\partial f_i^{\rm eq}}{\partial f_j} = \begin{cases} (1-\alpha) + A\left[\alpha + 3u(\boldsymbol{e}_i \cdot \beta\boldsymbol{E}) + \dfrac{9u^2(\boldsymbol{e}_i \cdot \beta\boldsymbol{E})^2}{2} - 3u^2\beta^2\right], & i = 0, \\ B\left[\alpha + 3u(\boldsymbol{e}_i \cdot \beta\boldsymbol{E}) + \dfrac{9u^2(\boldsymbol{e}_i \cdot \beta\boldsymbol{E})^2}{2} - 3u^2\beta^2\right], & i = 1,2,3,4, \\ C\left[\alpha + 3u(\boldsymbol{e}_i \cdot \beta\boldsymbol{E}) + \dfrac{9u^2(\boldsymbol{e}_i \cdot \beta\boldsymbol{E})^2}{2} - 3u^2\beta^2\right], & i = 5,6,\cdots,9, \end{cases} \tag{25}$$

where $\boldsymbol{e}_i = \boldsymbol{c}_i/\boldsymbol{c}$.

If $\beta = 1$, Eq.(1) is the standard Burgers equation. The eigenvalues of Jacobian matrix $J_f^*(f)$ with $\overline{u} = 0$ is $a(-1 + A + 4B + 4C)/\tau$ and $-1/\tau$. Since A, B, C represent the weight coefficients in Eq.(25) and they satisfy $A + 4B + 4C = 1$, hence the eigenvalues of $J_f^*(f)$ are 0 and $-1/\tau$, which have no relation with α. That is to say, the imported parameter α is not only to make the model for flexible and stronger ability to simulate different conditions, but also does not limit the stability. Moreover, Eq.(1) is the pure diffusion equation with $\beta = 0$. From the above analysis, we may know that the eigenvalues have no relation with u. That is to say, it is easier to keep stability than the Burgers equation.

Furthermore, when $\boldsymbol{k}$=0, the eigenvalues of the 9×9 matrix G_{ij} are $1 - \dfrac{1}{2}u^2$ and $1 - \dfrac{1}{\tau}$. Thus, the stability of the new LBGK model is guaranteed. And then in the whole $\boldsymbol{k}$ space, iterate the mean flow density and the relaxation time to determine the neutral stability boundary. The results are presented in Fig.1 as the solid curve. We can see in Fig.1 that as the value of τ increases, the maximum stable velocity is seen to increase from about 1.4. Since there is no limiting stable velocity comparing with the other D2Q9 models shown in Ref.[21], the new LBGK model is more stable at large velocity and high Reynolds, that is large density u and low diffusive coefficient D in our model respectively. This is different from other lattice Boltzmann model to simulate the fluid flow. We also find that the choice of the relaxation time values τ in Ref.[15] satisfies stability conditions, the same as the choice in Ref.[16].

Some idiographic tests near the stability boundary are performed. The relative global errors varying as time with different τ are validated at ν=1, 0.01 respectively. The results are shown in Fig.2 and Fig.3. It can be observed that the relative error will increase as time if computation does not satisfy the stability conditions. But the error will become lower and lower as time if it satisfies the stability conditions. The results show that the new LBGK model is easy to achieve stability conditions and it is consistent. This is very important for numerical methods.

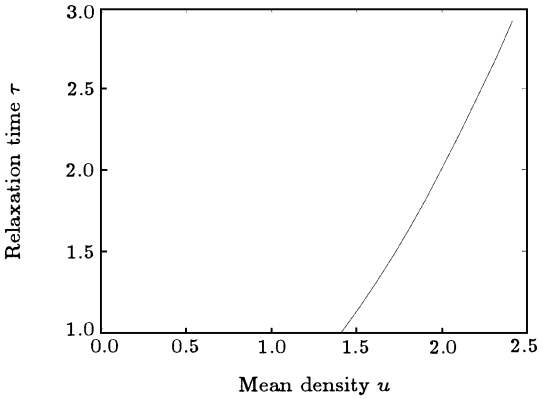


Fig.1. Stability boundary as a function of u and τ for the most unstable wavenumbers. It is the neutral stability curve for the proposed D2Q9 model for each α.

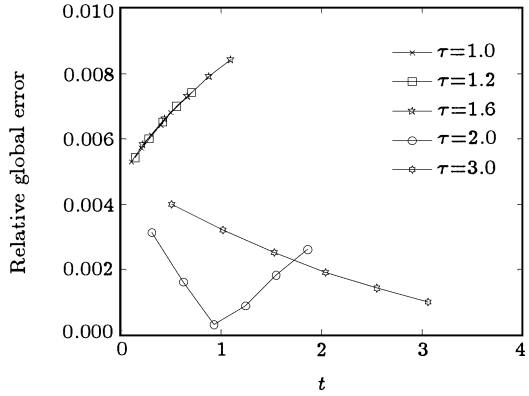


Fig.2. The relative global error as function of time varied by τ. ν=1.0.

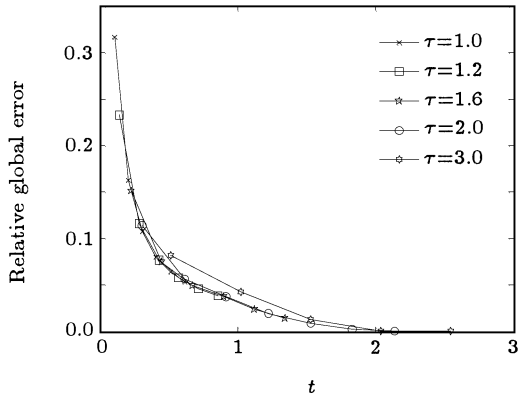


Fig.3. The relative global error as function of time varied by τ. ν=0.01.

4. Numerical results

To evaluate the accuracy and the suitable ranges of the LBGK model for the class of the generalized Burgers equations, numerical tests, including the different dimensional Burgers equations, various parameters, comparisons with analytical solutions, are carried out. In simulations, the non-equilibrium extrapolation method[23] is adopted to deal with the boundary condition.

4.1. Problem 1

We first take the Burgers equation

$$u_t + uu_x = Du_{xx}, \quad \frac{1}{2} \le x \le \frac{3}{2}.$$

The diffusive coefficient D is written as $D = \dfrac{1}{R_\mathrm{e}}$. Here, R_e is a constant varying with D. The initial and boundary conditions are

$$u(x,0) = D\left(x + \mathrm{tg}\,\frac{x}{2}\right),$$

$$u(1/2,t) = \frac{1}{R_\mathrm{e}+t}\left[\frac{1}{2} + \mathrm{tg}\,\frac{R_\mathrm{e}}{4(R_\mathrm{e}+t)}\right],$$

$$u(3/2,t) = \frac{1}{R_\mathrm{e}+t}\left[3 + \mathrm{tg}\,\frac{3R_\mathrm{e}}{4(R_\mathrm{e}+t)}\right].$$

The exact solution of this problem is

$$u_\mathrm{e}(x,t) = \frac{1}{R_\mathrm{e}+t}\left[x + \mathrm{tg}\,\frac{R_\mathrm{e}x}{2(R_\mathrm{e}+t)}\right] \qquad (26)$$

The numerical solutions of the Burgers equation above reported in Ref.[24] and those found by the present LBGK model with different conditions are compared with analytic solution in Fig.4 and Fig.5. Accuracy is evaluated by measuring the absolute error at every point in Fig.5. The absolute error is given by

$$\mathrm{error}(u) = \max|u - u_\mathrm{e}|, \qquad (27)$$

where the u means computational solution and u_e is the analytic solution given by Eq.(26). These numerical solutions are for the case D=0.002 using time step $\Delta t = 0.002$, space step $\Delta x = 0.01$ and α=1.0 respectively. Moreover, the relative global error is also shown in Table 1 as the α and relaxation time varies, the relative global error is

$$\mathrm{err} = \frac{\sum|u - u_\mathrm{e}|}{\sum|u_\mathrm{e}|},$$

where the summation is over the entire system. We can find that the best choice of the parameters can preserve the better accuracy and stability.

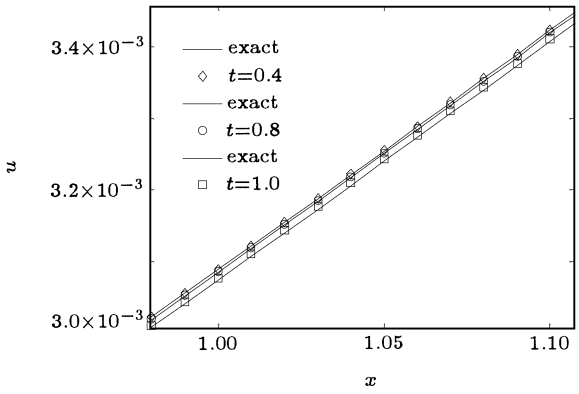


Fig.4. The comparisons between numerical and theoretical results of test 1.

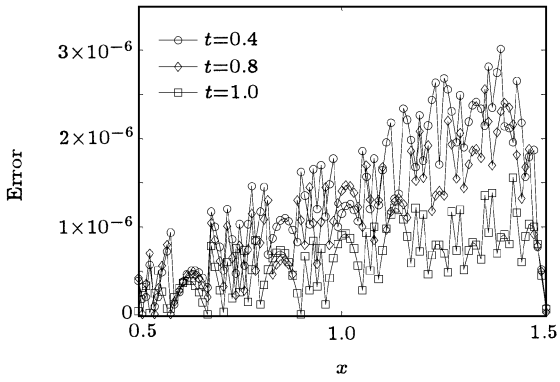


Fig.5. The error between numerical and analytical solutions of test 1.

Table 1. Comparison of results for problem 1 with D=0.002.

α	Relaxation time τ	The iteration time t	Relative global error
0.01	12.5	0.4	0.0104
		0.8	0.0155
0.1	1.7000	0.4	0.0017
		0.8	0.0031
1.0	0.6200	0.4	1.9230×10^{-4}
		0.8	3.6812×10^{-4}
1.3	0.5923	0.4	1.5028×10^{-4}
		0.8	2.8659×10^{-4}
1.5	0.5800	0.4	1.3163×10^{-4}
		0.8	2.5035×10^{-4}
1.7	0.5706	0.4	1.1724×10^{-4}
		0.8	2.2228×10^{-4}
2.0	0.5602	0.4	1.1013×10^{-4}
		0.8	1.9088×10^{-4}
2.2	0.5545	0.4	–
		0.8	–

Note: – means the computation is unstable.

4.2. Problem 2

The second test problem is the solution of two-dimensional unsteady Burgers equation which is dominated by moderate gradients. The equation with the following Dirichlet boundary conditions is solved by the proposed D2Q9 model:

$$\frac{\partial u}{\partial t}+u\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\right)=D\left(\frac{\partial^2 u}{\partial^2 x}+\frac{\partial^2 u}{\partial^2 y}\right),$$

$$u(0,y,t)=1/[1+\mathrm{e}^{y-t/(2D)}],$$
$$u(1,y,t)=1/[1+\mathrm{e}^{(1+y-t)/(2D)}],$$
$$u(x,0,t)=1/[1+\mathrm{e}^{x-t/(2D)}],$$
$$u(x,1,t)=1/[1+\mathrm{e}^{(1+x-t)/(2D)}].$$

And the initial condition is that $u(x,y,0)=1/[1+\mathrm{e}^{(x+y)/(2D)}]$, $0\le x\le 1, 0\le y\le 1$. The exact solution is $u_{\mathrm{e}}(x,y;t)=[1+\mathrm{e}^{(x+y-t)/(2D)}]^{-1}$. The errors at every point to compare analytic and numerical results at time t=2.5 and t=3, are shown in Fig.6, with grid size (128×128), and D=0.1. Moreover, the computed solutions using the new LBGK model at different time, with different grid sizes and the relative global errors defined as Eq.(28) are presented in Table 2.

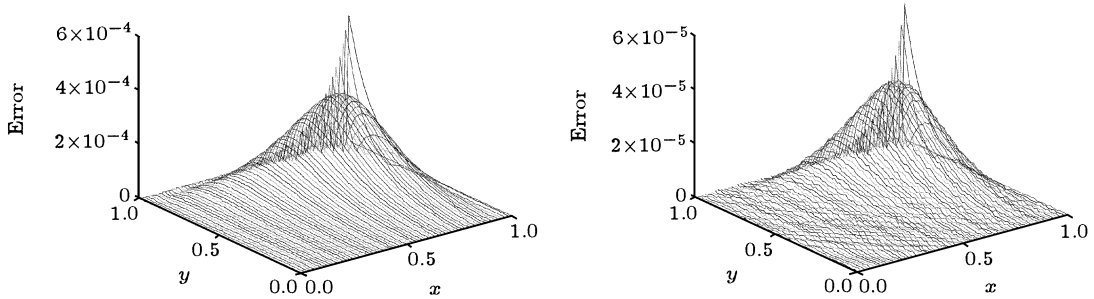


Fig.6. The error between numerical solution and analytical solution for all $x,y\in[0,1]$ at t=2.5 (left) and t=3 (right) respectively. With D=0.1, error=$|u-u_{\mathrm{e}}|$.

Table 2. The relative global error for test 2 with the D2Q9 model.

Grid size	Time t=1	Time t=2	Time t=3	Time t=4	Time t=5
32×32	0.0982	0.0104	1.0595×10^{-4}	1.0557×10^{-6}	1.8138×10^{-8}
64×64	0.0475	0.0047	4.4337×10^{-5}	6.9170×10^{-7}	1.4802×10^{-8}
128×128	0.0235	0.0022	1.7867×10^{-5}	6.6621×10^{-7}	1.5356×10^{-8}
256×256	0.0117	9.4303×10^{-4}	8.7065×10^{-6}	3.5405×10^{-6}	1.2704×10^{-8}

4.3. Problem 3

The last test problem is the three-dimensional Burgers equation.

$$\frac{\partial u}{\partial t}+u\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$
$$=\nu\left(\frac{\partial^2 u}{\partial^2 x}+\frac{\partial^2 u}{\partial^2 y}+\frac{\partial^2 u}{\partial^2 z}\right),$$
$$0\leq x,y,z\leq 1$$

with the analytical solution $u_{\rm e}(x,y,z;t)=[1+{\rm e}^{(x+y+z-t)/(3\nu)}]^{-1}$. The computed solution at z=20/128 is shown in Fig.7. Furthermore, the absolute error for the entire system is presented at the same time in Fig.8. The grid size is 128×128 and the ν=0.1. In order to show the simulation more exactly, the relative global error for the whole (x, y) system varies at different point z can be observed in Fig.9. The relative global error defined by Eq.(28) can be written as

$$\mathrm{err}(z)=\frac{\sum_{x,y}|u(x,y,z)-u_{\rm e}(x,y,z)|}{\sum_{x,y}|u_{\rm e}(x,y,z)|}.$$

The results show that the numerical simulations by the new LBGK model are in good agreement with the analytical solution.

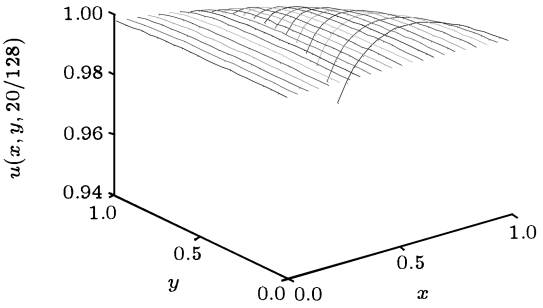


Fig.7. The computed solutions of test 3, for z=20/128, time t=3.

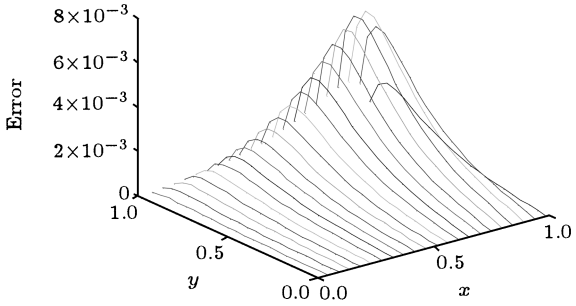


Fig.8. The error between numerical solution and analytical solution for all $x,y\in[0,1]$ at z=20/128, time t=3.

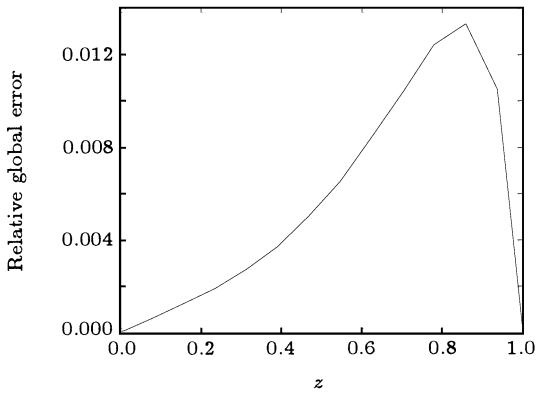


Fig.9. The relative global error at different point z for whole (x, y) domain at t=3.

5. Conclusions

In this paper, we have successfully proposed a general LBGK model for a class of generalized Burgers equations. We have numerically validated the model by using the model to simulate several benchmark problems, and have obtained satisfactory results. In contrast to the previous work on Burgers equations, the model is more general, simple, and can solve the high-dimensional problems successfully. It should also be emphasized that, the work is in part motivated by previous work,[19] the new model can recover the macroscopic equations with source term without any assumption. Through the analysis of the stability, we confirm that the LBGK model has good stability to simulate the class of generalized Burgers equations.

One difficulty encountered in the LBGK model is due to the nonlinear convective term, which makes corresponding LBGK models complex and hard to extend to high dimension space directly in the previous work. It is completely solved in our new LBGK model by adopting the vector constant E related to dimensions. However, not all nonlinear equations have been included by the general LBGK model. This is our direction in the future work.

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