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Steady-state analysis of a bistable system subject to a coloured multiplicative noise and a white additive noise with coloured cross-correlated noises*

Wang Can-Jun(王参军)[a)b)], Chen Shi-Bo(陈世波)[a)], and Mei Dong-Cheng(梅冬成)[a)†]

[a)] *Department of Physics, Yunnan University, Kunming* 650091, *China*

[b)] *Nonlinear Research Institute, Baoji University of Sciences and Arts, Baoji* 721007, *China*

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The steady-state properties of a bistable system are investigated when both the multiplicative noise and the coupling between additive and multiplicative noises are coloured with different values of noise correlation times τ_1 and τ_2. After introducing a dimensionless parameter $R(R = \alpha/D$, D is the intensity of the multiplicative noise and α is the intensity of the additive noise), and performing the numerical computations, we find the following points: (1) For the case of $R > 1$, λ (the intensity of correlation between additive and multiplicative noises), τ_1 and τ_2 can induce the stationary probability distribution (SPD) transition from bimodal to unimodal in structure, but for the cases of $R \leq 1$, the bimodal structure is preserved; (2) α can also induce the SPD transition from bimodal to unimodal in structure; (3) the bimodal structure of the SPD exhibits a symmetrical structure as D increases.

Keywords: bistable system, correlated noise, stationary probability distribution
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1. Introduction

In recent years, there has been an increasing interest in studying the effects of correlated noise in stochastic systems. Correlated noises have been applied to many fields, such as, biology system,[1,2] laser system,[3−6] bistable system,[7−20] and so on. Those investigations show that the coloured nature of noise can induce many new interesting phenomena, for example, when coloured correlated pump noise and quantum noise are introduced to a gain–noise model of the single-mode laser, there appear two kinds of stochastic resonance (SR), one is in the single-peak form SR, the other is in the form of stochastic multiresonance, both of which depend on the noise correlation time.[6] Correlated noises can change the form of the stationary probability distribution (SPD) from bimodal to unimodal in structure[8] and induce nonequilibrium phase transitions.[21,22]

Since the bistable model is of generic interest in science, it has been widely studied by many scientists. In 1991, Fulinski and Telejko first introduced a white correlation between a multiplicative white noise and an additive white noise.[7] Soon after that, Wu *et al* studied the SPD of the bistable system driven by coupled noises in the case of zero cross-correlation time.[8] Physically, the correlation time of a real noise, though it is small, is never exactly equal to zero. For a noise with zero correlation time, its power spectral distribution, which is given by the Fourier transform of its correlation function, is independent of frequency. Thus the total power dissipated in all frequencies is infinite, but the actual power dissipated would be somewhat less than infinite. In other words, it appears ideal, and it is valid only when the time scale for its correlation is much shorter than that for the relaxation of the driven process. On the other hand, the assumption that the correlation time is zero is usually adopted as a first step in studying the system driven by noises. Afterwards it is reasonable to relax this condition and include the finite correlation time.[23] In 1995, Cao *et al* generalized the unified coloured noise approximation to study a non-Markovian bistable system with a coloured multiplicative noise in white correlation with

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†Corresponding author. E-mail: kmdcmei@public.km.yn.cn

a white additive noise.[9] In 1997, Jia and Li studied a system driven by a white multiplicative noise and a white additive noise, the time of correlation between noises was nonzero. They discovered that the coloured cross-correlation led to reentrant transformation.[11] In 1998, Jia and Li investigated a stochastic dynamic system with a coloured multiplicative noise in coloured correlation with a white additive noise and derived an approximate Fokker–Planck equation for the system.[12] The stochastic resonance[19] and the mean first-passage time[20]of a bistable system driven by two different kinds of coloured noises have been studied, respectively. However, the steady-state properties of the bistable system subject to a coloured multiplicative noise and the coloured correlation between an additive noise and a multiplicative noise have not been investigated yet. The steady-state probability distribution is important to describe the statistical properties of nonlinear systems.

In this paper, the SPD of a bistable system is investigated when both the multiplicative noise and the coupling between additive and multiplicative noises are coloured with different correlation times τ_1 and τ_2. The effects of noise parameters on the SPD of the system are discussed separately.

2. The stationary probability distribution

Consider a conventional symmetric bistable system driven by a cross-correlated noise. Its Langevin equation reads

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x - x^3 + x\xi(t) + \eta(t), \tag{1}$$

where $\eta(t)$ and $\xi(t)$ are the Gaussian white noise with zero-mean and coloured noise with zero-mean, respectively. The two noise terms are characterized by their means and variances.

$$\langle \xi(t) \rangle = \langle \eta(t) \rangle = 0, \tag{2}$$

$$\langle \xi(t)\xi(t^{'}) \rangle = \frac{D}{\tau_1}\exp\left[- \frac{|t-t'|}{\tau_1} \right], \tag{3}$$

$$\langle \eta(t)\eta(t^{'}) \rangle = 2\alpha\delta(t-t^{'}), \tag{4}$$

$$\begin{aligned}\langle \xi(t)\eta(t') \rangle =& \langle \eta(t)\xi(t') \rangle \\ =& \frac{\lambda\sqrt{\alpha D}}{\tau_2}\exp\left[- \frac{|t-t'|}{\tau_2} \right],\end{aligned} \tag{5}$$

where τ_1 and D are the correlation time and intensity of the multiplicative coloured noise, respectively. α is the intensity of white noise, and τ_2 is the noise correlation time of the coupling between multiplicative and additive noise. λ denotes the intensity of correlation between $\xi(t)$ and $\eta(t)$.

When the limit $\tau_1 \to 0$ and $\tau_2 \to 0$, Eq.(1) degenerates to the bistable model driven by two correlated Gaussian white noises.[8] When either $\tau_1 \to 0$ or $\tau_2 \to 0$, Eq.(1) degenerates to the bistable system driven by cross-correlated Gaussian white noises[10] and correlated multiplicative coloured noise and additive white noise,[11] separately. From Eq.(1) we can obtain the potential $\phi(x) = -\dfrac{x^2}{2} + \dfrac{x^4}{4}$, which has two stable states $x_{s1} = 1$, and $x_{s1} = -1$ and an unstable state $x_u = 0$.

We assume that Eq.(1) is a Stratomovich stochastic differential equation. According to the Novikov theorem and Fox approach, the approximate Fokker–Planck equation corresponding to Eq.(1) with Eq.(3) is obtained[9,12] as follows:

$$\frac{\partial P(x,t)}{\partial t} = L_{FP}P(x,t), \tag{6}$$

$$L_{FP} = -\frac{\partial}{\partial x}A(x) + \frac{\partial^2}{\partial x^2}B(x), \tag{7}$$

where

$$A(x) = \frac{x - x^3}{1 + 2\tau_1 x^2} + \frac{1}{2}B'(x), \tag{8}$$

and

$$B(x) = \frac{1}{(1+2\tau_1 x^2)^2}\left(Dx^2 + \frac{2\lambda\sqrt{D\alpha}}{1+2\tau_2}x + \alpha \right). \tag{9}$$

Note that this approximate Fokker–Planck equation holds under the condition $1 + 2\tau_1 > 0$ and $1 + 2\tau_2 > 0$ for all τ_1 and τ_2. Thus, there are no restrictions on τ_1 and τ_2 so that there is not any restriction on all the parameters treated in this case. For the case ($\lambda < 0$) of negative correlation between the multiplicative noise and the additive noise, the diffusion function $B(x)$ will be taken as zero value for some parameters. For simplicity, we consider only the case of positive correlation between the multiplicative noise and the additive noise. The SPD of the system can be obtained from Eq.(6) with Eqs.(7)–(9)[19]

$$\begin{aligned}P_{\mathrm{st}}(x) =& N\frac{1 + 2\tau_1 x^2}{\sqrt{Dx^2 + \dfrac{2\lambda\sqrt{D\alpha}}{1+2\tau_2}x + \alpha}} \\ & \times \exp[-U(x)/D], \quad (0 \le \lambda \le 1)\end{aligned} \tag{10}$$

by virtue of the generalized potential $U(x)$ defined as

$$\begin{aligned}U(x) =& -\int^{x}\frac{(1+2\tau_1 y^2)(y-y^3)}{y^2+\dfrac{2\lambda\sqrt{\alpha/D}}{1+2\tau_2}y+\dfrac{\alpha}{D}}\mathrm{d}y\\ =&\frac{\tau_1}{2}x^4-\frac{4\lambda\tau_1}{3(1+2\tau_2)}\sqrt{\frac{\alpha}{D}}x^3+\alpha_1 x^2\\ &+\alpha_2 x+\beta\ln\left[Dx^2+\frac{2\lambda\sqrt{D\alpha}}{1+2\tau_2}+\alpha\right]\\ &+\frac{\gamma_1}{\sqrt{1-(\dfrac{\lambda}{1+2\tau_2})^2}}\\ &\times\arctan\left[\frac{\sqrt{\dfrac{\alpha}{D}}x+\dfrac{\lambda}{1+2\tau_2}}{\sqrt{1-\left(\dfrac{\lambda}{1+2\tau_2}\right)^2}}\right],\end{aligned}\tag{11}$$

where

$$\begin{aligned}\alpha_1 =&\frac{\alpha}{D}\tau_1\left[4\left(\frac{\lambda}{1+2\tau_2}\right)^2-1\right]-\frac{2\tau_1-1}{2},\\ \alpha_2 =&8\sqrt{\left(\frac{\alpha}{D}\right)^3}\left(\frac{\lambda\tau_1}{1+2\tau_2}\right)\left[1-2\left(\frac{\lambda}{1+2\tau_2}\right)^2\right]\\ &+2\sqrt{\frac{\alpha}{D}}\frac{\lambda(2\tau_1-1)}{1+2\tau_2},\\ \beta =&-\frac{1}{2}-\frac{\alpha}{2D}(2\tau_1-1)\left[4\left(\frac{\lambda}{1+2\tau_2}\right)^2-1\right]\\ &+\frac{\alpha}{D}\tau_1\left[16\left(\frac{\lambda}{1+2\tau_2}\right)^4-12\left(\frac{\lambda}{1+2\tau_2}\right)^2+1\right],\\ \gamma_1 =&\frac{\lambda}{1+2\tau_2}+\frac{\alpha}{D}\left[\frac{\lambda(2\tau_1-1)}{1+2\tau_2}\right]\left[4\left(\frac{\lambda}{1+2\tau_2}\right)^2-3\right]\\ &+2\left(\frac{\alpha}{D}\right)^2\left(\frac{\lambda\tau_1}{1+2\tau_2}\right)\left[-16\left(\frac{\lambda}{1+2\tau_2}\right)^4\right.\\ &\left.+2\left(\frac{\lambda}{1+2\tau_2}\right)^2-5\right],\end{aligned}\tag{12}$$

N in Eq.(10) is the normalization constant.

3. Discussion and conclusions

According to Eq.(10) with the expressions of (11) and (12), the effects of λ, τ_1 , τ_2, α and D on SPD of the system can be studied by the numerical computation. Figures 1–5 show their effects. And the conclusions that can be drawn from these figures are as follows.

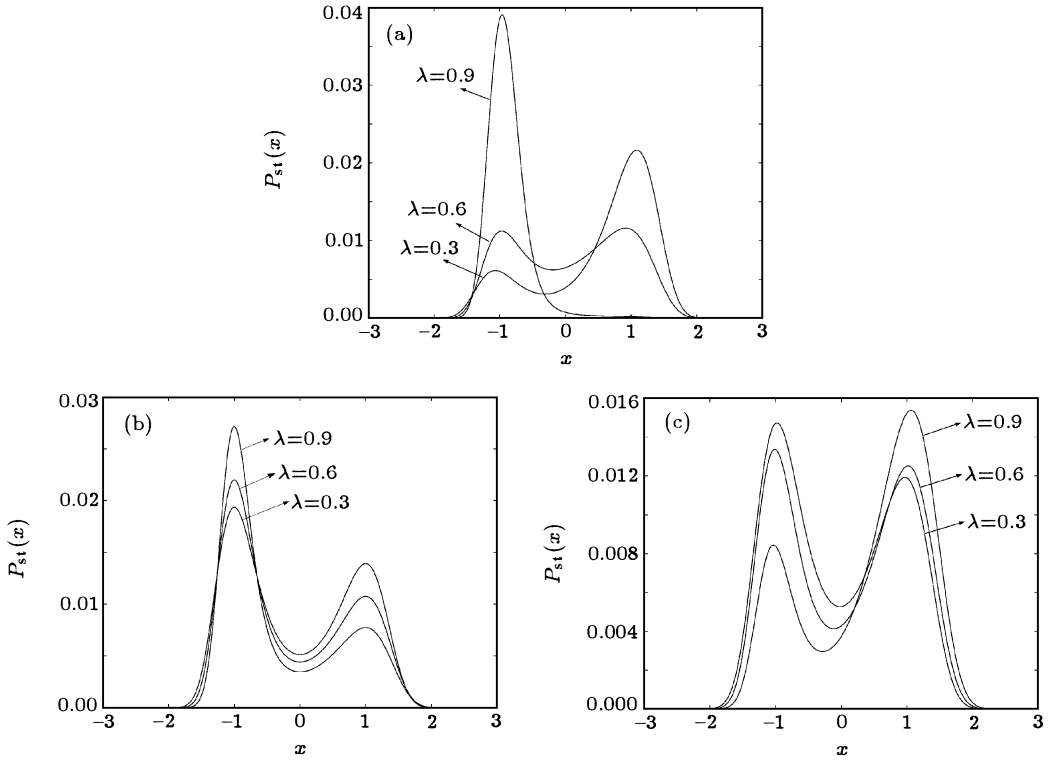


Fig.1. The stationary probability distribution $P_{\mathrm{st}}(x)$ as a function of x at several values of λ. τ_1=0.3 τ_2=0.3 and λ takes 0.3, 0.6 and 0.9, respectively.(a)D=0.2, and α=0.3 ($R=\alpha/D>1$); (b)D=0.2, and α=0.2 ($R=1$); (c)D=0.3, and α=0.2 ($R<1$).

The SPD as a function of x at several values λ is plotted in Fig.1 where the τ_1 and τ_2 are fixed. It is interesting to note that the SPD exhibits very different behaviours for $R > 1(R = \alpha/D)$ and for $R \leq 1$. For the case of $R > 1$, we can see that the SPD of the bistable system experiences the transition from bimodal to unimodal in structure and the SPD of the bistable system shifts from $x > 0$ to $x < 0$ as λ increases (see Fig.1(a)). Figure 1(b) shows the case of $R = 1$, where the SPD of the bistable system does not experience the transition from bimodal to unimodal in structure but shifts from $x > 0$ to $x < 0$ like the case of $R > 1$ as λ increases, and the peak of SPD becomes narrow and grows in height while the minimum of SPD drops in height. Figure 1(c) shows the case of $R < 1$, where the bimodal structure is preserved, but the SPD of the bistable system shifts from $x < 0$ to $x > 0$ as λ increases, and the λ exerts different effects on the SPD of the bistable system for the case of $R \geq 1$ and $R < 1$. When the noises are uncorrelated with each other ($\lambda = 0$), the SPD exhibits a symmetric bimodal structure.[8]

In order to show the effect of the self-correlation time τ_1 on SPD of a bistable system, we plot Fig.2 where λ and τ_2 are fixed. For the case of $R > 1$ (Fig.2(a)), it is obvious that the SPD experiences the transition from bimodal to unimodal in structure, the maximum of the SPD of the bistable system shifts from left to right and the peak of SPD becomes narrow and grows in height at $x = 1$ with τ_1 increasing. For the cases of $R = 1$ (Fig.2(b)) and $R < 1$ (Fig.2(c)), we can see that the bimodal structure is preserved, and the positions of the maximum and the minimum of the SPD do not change, only the peak of SPD becomes narrow and grows in height at $x = 1$ as τ_1 decreases.

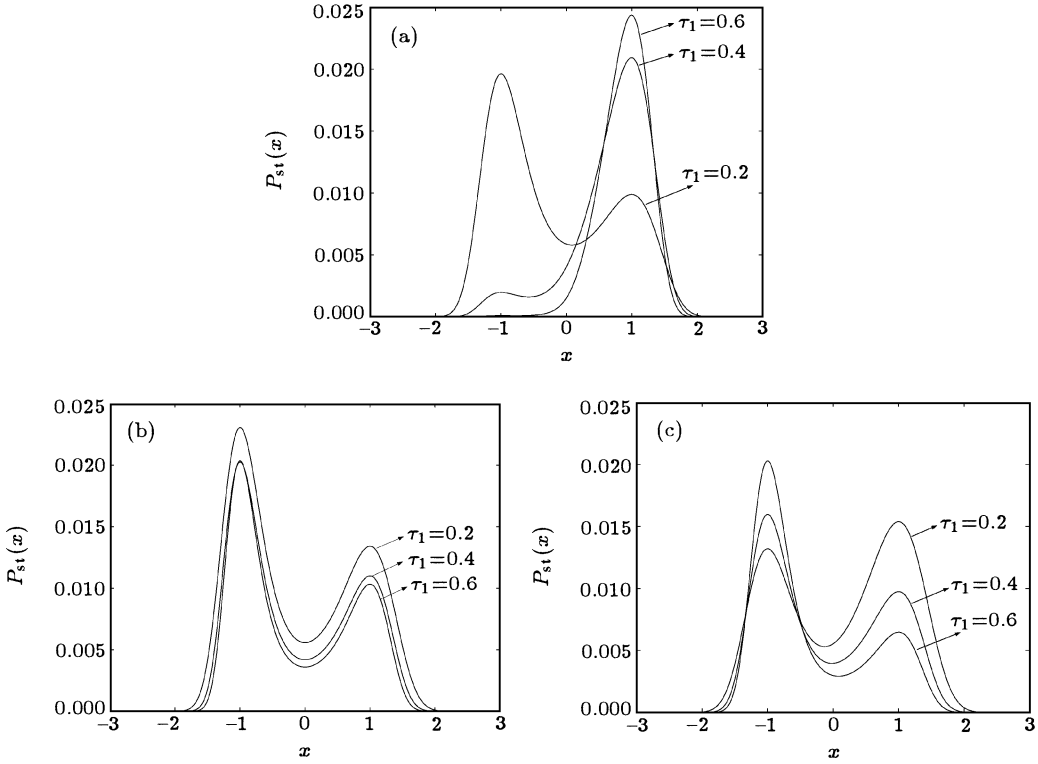


Fig.2. The stationary probability distribution $P_{\rm st}(x)$ vs x at several values of τ_1. λ=0.5, τ_2=0.3 and τ_1 takes 0.2, 0.4 and 0.6, respectively. (a) D=0.2, and α=0.3 ($R = \alpha/D > 1$); (b)D=0.2, and α=0.2 ($R = 1$); (c)D=0.3, and α=0.2 ($R < 1$).

This shows that the self-correlation time τ_1 plays opposing roles for the case of $R \leq 1$ and $R > 1$. When $\tau_1 = 0$, $\lambda \neq 0$ and $\tau_2 \neq 0$, both λ and τ_2 play opposing roles in the transition of the SPD of the bistable system.[11]

In order to show the effect of correlation time τ_2 on SPD of a bistable system, we plot Fig.3 where λ and τ_1 are fixed. For the case of $R > 1$ (Fig.3(a)), we can see that the transition from bimodal to unimodal in structure is not obvious, the peak of SPD becomes narrow and grows in height at $x = 1$ and the SPD of the bistable system shifts from $x < 0$ to $x > 0$ with τ_2 increasing.

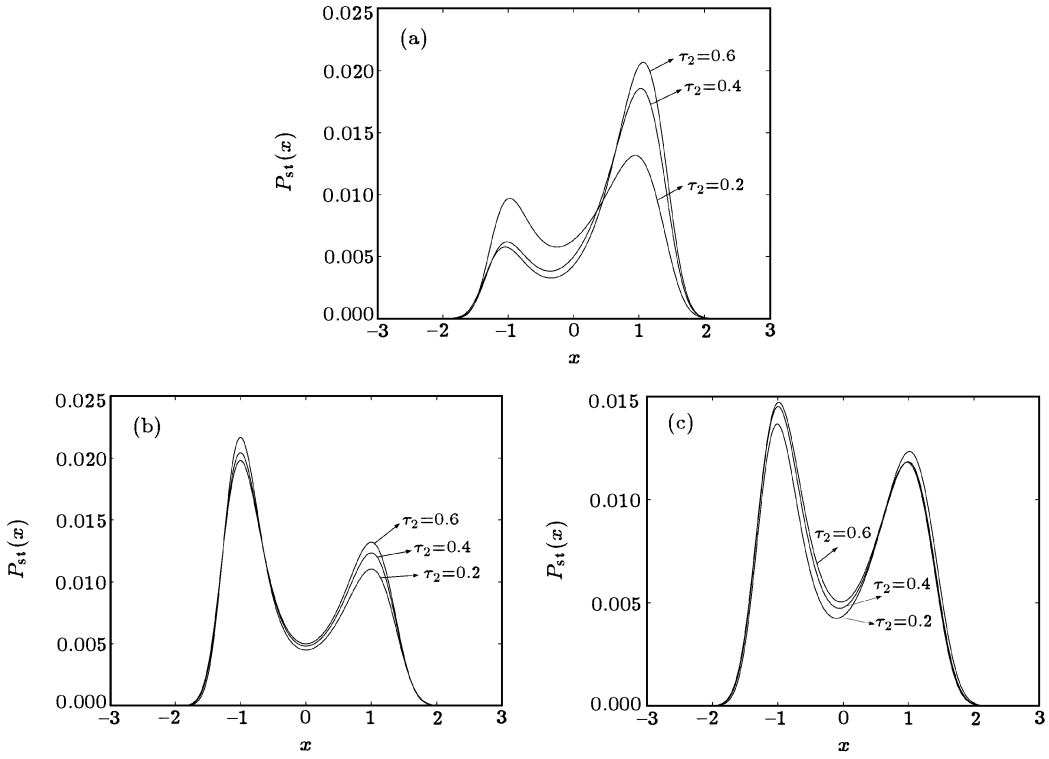


Fig.3. The stationary probability distribution $P_{\rm st}(x)$ vs x at several values of τ_2. λ=0.5, τ_1=0.3 and τ_2 takes 0.2, 0.4 and 0.6, respectively. (a) D=0.2, and α=0.3 ($R = \alpha/D > 1$); (b) D=0.2, and α=0.2 ($R = 1$); (c)D=0.3, and α=0.2 ($R < 1$).

For the cases of $R = 1$ (Fig.3(b)) and $R < 1$ (Fig.3(c)), the bimodal structure is preserved, and the positions of the extrema are not affected by the correlation time τ_2. It must be pointed out that the peak of the SPD of the bistable system becomes narrow and grows in height as τ_2 increases at $x = 1$ for the case of $R = 1$. However, the peak of the SPD of the bistable system becomes narrow and grows in height as τ_2 decreases at $x = 1$ for the case of $R < 1$. This shows that the correlation time τ_2 plays opposite roles for the cases of $R \geq 1$ (Fig.3(a) and Fig.3(b)) and $R < 1$ (Fig.3(c)). When the $\tau_1 \neq 0$, $\lambda \neq 0$ and $\tau_2 = 0$, the self-correlation time τ_1 and λ cause the transition of the SPD of the bistable system.

The SPD as a function of x at several values of α is plotted in Fig.4 where the τ_1, τ_2, λ and D are fixed. We can clearly see that the SPD of the bistable system experiences the transition from bimodal to unimodal in structure and the SPD of the bistable system shifts from $x > 0$ to $x < 0$ as α increases (see Fig.4). At the same time, the maximum of the SPD shifts from $x = -1$ to $x = 1$.

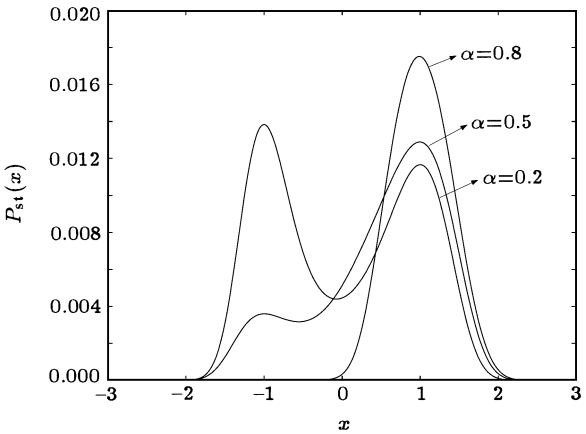


Fig.4. The stationary probability distribution $P_{\rm st}(x)$ vs x. D=0.3, τ_1=0.3, τ_2=0.3 and $\lambda = 0.5$, α takes 0.2, 0.5, 0.8 respectively.

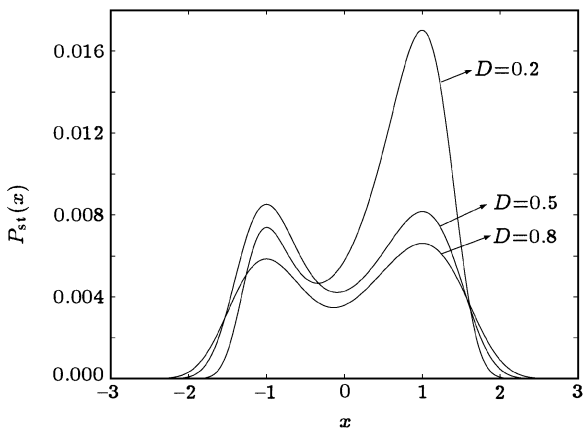


Fig.5. The stationary probability distribution $P_{\rm st}(x)$ vs x. $\alpha = 0.3$, τ_1=0.3, τ_2=0.3 and $\lambda = 0.5$, D takes 0.2, 0.5, 0.8 respectively.

The SPD as function of x at several values of D is plotted in Fig.5 where the τ_1, τ_2, λ and α are fixed. It is obvious that the bimodal structure is preserved as shown in Fig.4. At the same time, with the multiplicative noise intensity D increasing, the positions of the extrema of the SPD are not changed, but the extrema of the SPD all drop in height. We emphasize that the bimodal structure of the SPD of the bistable system will exhibit a symmetrical structure as D increases.

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