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An exact solution of Fisher equation and its stability*

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In this paper, the Fisher equation is analysed. One of its travelling wave solution is obtained by comparing it with KdV–Burgers (KdVB) equation. Its amplitude, width and speed are investigated. The instability for the higher order disturbances to the solution of the Fisher equation is also studied.

Keywords: Fisher equation, KdVB equation
PACC: 0340K, 5235

1. Introduction

Modelling of gene frequencies is of particular relevance today, especially in light of the recent developments in the field of gene technology. Recently, there are many advances and problems in this field. For examples, agricultural crops and livestock can now be genetically modified to be more resistant to pests and produce a higher yield; some bacteria have developed a tolerance towards widely prescribed antibiotics; introduced species can develop immunity to viral pathogens which have been introduced to control the species; archaeologists hope that the human genome project will enable us to map geographic distributions of signature genes and to deduce historical patterns of migration. In order to make use of genetic population data, we need to understand the dynamics of gene patterns through a population. Therefore, the mathematical modelling of changing gene proportions is timely reexamined. In the 1930s, a number of authors proposed reaction–diffusion equations to model changing gene frequencies in a population. One of the earliest and best known was Fisher. In his paper,[1] Fisher proposed a reaction–diffusion equation with quadratic source term that models the spread of a recessive advantageous gene through a population that previously had only one allele at the locus in question. Fisher's equation is

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + mu(1-u),$$

where u is the frequency of the new mutant gene, k is the coefficient of diffusion, and m is the intensity of selection in favour of the mutant gene. This equation predicts a wave front of increasing allele frequency, which propagates through the population. Only original alleles are present in front of the wave, and behind the wave is an area taken over by the mutant allele. Fisher wrote his equation with no formal derivation and very little explanation. Several other authors had also developed reaction–diffusion equations with a similar quadratic source term (see Ref.[2], for example), and others claimed to have verified Fisher's results (see Ref.[3], for example), but many of these authors gave no convincing or formal derivation. Modelling changes in gene frequencies in a population was reexamined in the latter half of the last century. Some authors suggested that a cubic source term was more appropriate than a quadratic source term.[4,5] Although the cubic source term was suggested, a thorough derivation was not usually provided. Nagylaki,[6] Nagylaki and Crow,[7] however, derived the cubic source term by using Fick's laws and a local Hardy–Weinberg property. Although the cubic source term is implicit as one possibility in the general genetic dispersion equations derived by others, its significance has not been highlighted and the difference between cubic and quadratic source terms has not been examined. Because of these reasons, many authors are interested in the Fisher equation.[8−10]

In the present paper, we attempt to find an exact solution of Fisher equation. One solution has been ob-

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tained. It is likely that this solution can be useful in the many fields of science related to Fisher equation.

2. A solution of the Fisher equation

For a more general Fisher equation, it takes the following form[9]

$$\frac{\partial u}{\partial t} = au - bu^2 + c\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

this is a reaction diffusion equation which is used to model natural processes such as gene mutation, population growth, and flame propagation. In this equation u is the density of the informed particle, x is the location in consideration, and $\partial^2 u/\partial x^2$ is the partial derivative at location x. The equation states that the change of the density of informed particles at a given time depends on the infection rate $au - bu^2$ and the diffusion from neighbourhood. The first term measures the infection rate which is proportional to the product of the density of the infected and uninfected particles, it is an approximation in a continuum case. The second term shows how fast the infected particles are diffusing. If nearby places have a higher density, then the Laplacian will be positive and infected particles will diffuse toward this position. Otherwise, the infected particles will diffuse away from this position.

There are many methods, such as the homogeneous balance method,[11,12] the Jacobi Elliptic function expansion method,[13] to solve a given nonlinear evolution equation. In this paper, in order to solve Fisher equation (1), firstly we study the KdV–Burgers (KdVB) equation.

As is well known, the KdVB equation has been well studied.[11,14–17] For the following KdVB equation:

$$\frac{\partial z}{\partial t} + z\frac{\partial z}{\partial x} - \alpha\frac{\partial^2 z}{\partial x^2} + \beta\frac{\partial^3 z}{\partial x^3} = 0, \tag{2}$$

there is a solution[11]

$$z = \frac{3\alpha^2}{25\beta}[\mathrm{sech}^2\xi \pm 2\tanh\xi \pm 2], \tag{3}$$

where $\xi = \mp\alpha x/10\beta + 3\alpha^2 t/125\beta^2$. This is a travelling wave solution of the KdVB equation. If we let $z = z(\xi)$, then the KdVB equation (2) becomes

$$\frac{\partial z}{\partial \xi} \mp \frac{25}{6}\frac{\beta}{\alpha^2}z\frac{\partial z}{\partial \xi} - \frac{5}{12}\frac{\partial^2 z}{\partial \xi^2} \mp \frac{1}{24}\frac{\partial^3 z}{\partial \xi^3} = 0, \tag{4}$$

Now, we let $\eta = kx - \omega t$, $u = u(\eta)$, then Fisher equation (1) becomes

$$\frac{\partial u}{\partial \eta} - \frac{2b}{a}u\frac{\partial u}{\partial \eta} + \frac{\omega}{a}\frac{\partial^2 u}{\partial \eta^2} + \frac{ck^2}{a}\frac{\partial^3 u}{\partial \eta^3} = 0. \tag{5}$$

Comparing two equations (4) and (5) we get

$$\omega = -\frac{5}{12}a, \tag{6}$$

$$k^2 = \frac{a}{24c}, \tag{7}$$

$$\beta = -\frac{12}{25}\frac{b}{a}\alpha^2. \tag{8}$$

Finally one exact solution of Fisher equation (1) is obtained as follows:

$$u(x,t) = -\frac{1}{4}\frac{a}{b}\Big[\mathrm{sech}^2\Big(\pm\sqrt{\frac{a}{24c}}x + \frac{5a}{12}t\Big) - 2\tanh\Big(\pm\sqrt{\frac{a}{24c}}x + \frac{5a}{12}t\Big) - 2\Big]. \tag{9}$$

It is noted that this solution is a shock-like travelling wave. The amplitude is proportional to a/b. It means that the amplitude increases as coefficient a increases, but decreases as b increases. We define the width of this wave as $\sqrt{24c/a}$. It indicates that the width increases as the coefficient c increases, but decreases as the a increases. The speed of wave is $(5/\sqrt{6})\sqrt{ca}$. It means that the wave speed increases as the both coefficients a and c increase. This result can be useful to some works related to the Fisher equation. The solution (9) is shown in the Fig.1, where we take $a = 1$, $b = 1$, and $c = 1$. Further study of the Fisher equation, and its applications will be continued in the future.

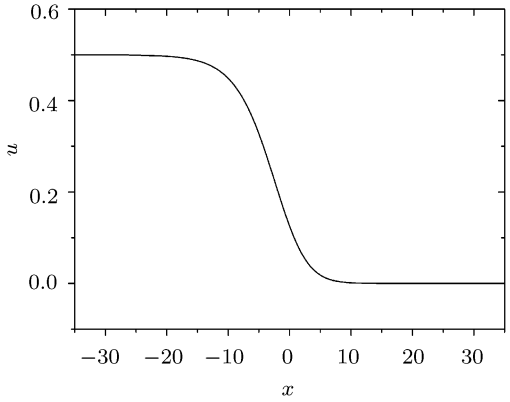


Fig.1. A solution of Fisher equation u vs $\eta = kx - \omega t$, where we let $a = 0.5, b = 1, c = 1$ and $t = 0$.

3. Stability of the solution

Now we study the stability of the solution of the Fisher equation. Firstly let u_0 be a solution of the Fisher equation, then we assume that there are some

higher order disturbances $u'(x,t)$ compared with u_0. We assume that $u = u_0(x,t) + u'(x,t)$ is a solution of the Fisher equation, then we can obtain the following equation:

$$\frac{\partial u'}{\partial t} = au' - 2bu_0u' + c\frac{\partial^2 u'}{\partial x^2}. \tag{10}$$

It is noted that $u_0 = a/b$ is a solution of the Fisher equation. It is found that, for this case

$$\frac{\partial u'}{\partial t} = -au' + c\frac{\partial^2 u'}{\partial x^2}. \tag{11}$$

If the initial disturbances is in the form $u'(x,0) = \epsilon_0(x)$, we can obtain the solution by letting $u'(x,t) = \mathrm{e}^{-at}v(x,t)$ since $v(x,t)$ becomes a typical heat conduction equation. The solution is

$$u'(x,t) = \frac{1}{2\sqrt{c\pi t}}\mathrm{e}^{-at}\int_{-\infty}^{+\infty}\epsilon_0(\xi)\mathrm{e}^{-(\xi-x)^2/(4ct)}\mathrm{d}\,\xi. \tag{12}$$

Now we study the following three cases.

Firstly we consider the case, $u_0 = a/b$. Assuming that $\epsilon_0(x) = A\cos(kx)$, we find that $u'(x,t) = A\mathrm{e}^{-k^2ct-at}\cos(kx)$. It seems that the disturbances are unstable if $k^2 < -a/c$. On the other hand, it is stable if $k^2 > -a/c$. It is also found numerically that the solution is stable if we let $\epsilon_0(x) = A\mathrm{sech}(k(x+x_0))$. We know that $a > 0$ and $c > 0$ for the real Fisher equation, therefore, for real wave number k, $k^2 > -a/c$ is always satisfied. So the wave is always stable.

Secondly we discuss the case $u_0 = 0$. If $\epsilon_0 = A\cos(kx)$, similarly we found that $u'(x,t) = A\cos(kx)\mathrm{e}^{-k^2ct+at}$. It indicates that the disturbances are stable if $k^2 > a/c$, otherwise, it is unstable if $k^2 < a/c$. If $\epsilon_0 = A\mathrm{sech}(k(x+x_0))$, we find numerically that the waves are unstable.

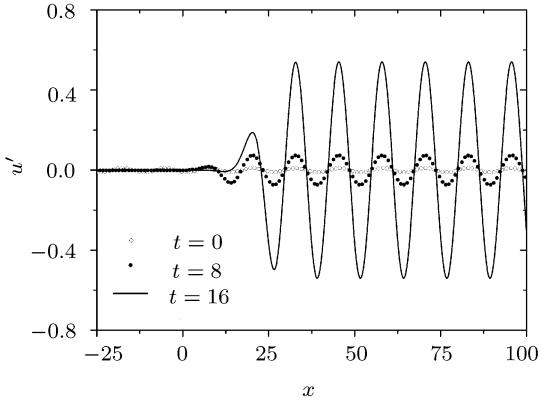


Fig.2. The stability of disturbances for $k^2 < a/c$. It is stable for the left-side of the wave and unstable for the right-side of the wave, where we take $a = 0.5, b = 1, c = 1$, $u'(x,0) = A\cos(k(x+x_0))$, $A = 0.01$, $k = 0.5$ and $x_0 = 30$.

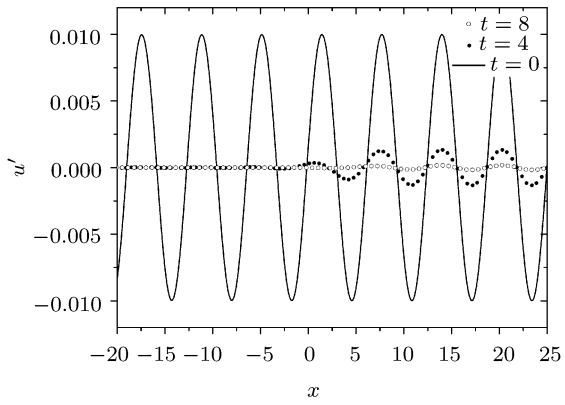


Fig.3. The stability of disturbances for $k^2 > a/c$. It is stable, where we take $a = 0.5, b = 1, c = 1$, $u'(x,0) = A\cos(k(x+x_0))$, $A = 0.01$, $k = 1$ and $x_0 = 30$.

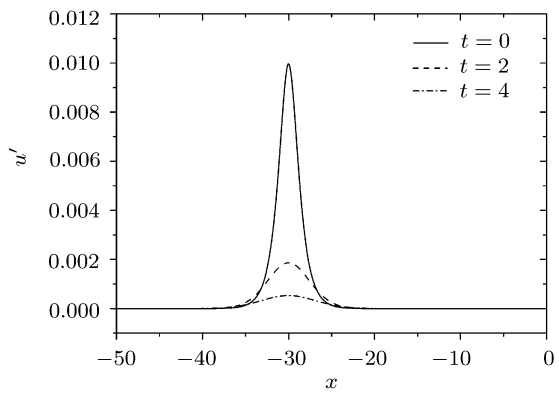


Fig.4. The stability of disturbances. It is stable, where we take $a = 0.5, b = 1, c = 1$, $u'(x,0) = A\mathrm{sech}(k(x+x_0))$, $A = 0.01$, $k = 0.5$ and $x_0 = 30$.

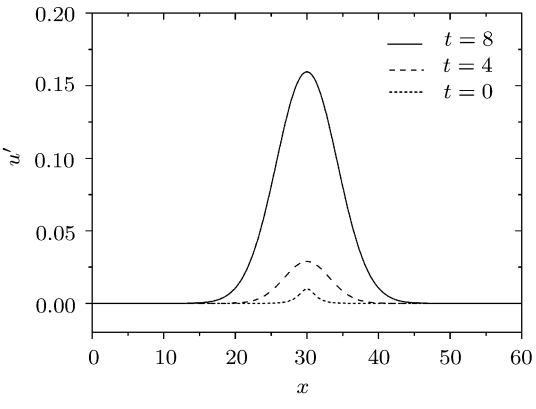


Fig.5. The stability of disturbances. It is unstable, where we take $a = 0.5, b = 1, c = 1$, $u'(x,0) = A\mathrm{sech}(k(x+x_0))$, $A = 0.01$, $k = 0.5$ and $x_0 = -30$.

Now we study the third case, where u_0 is in the form of Eq.(9). It is noted that $u_0(\xi) \to a/b$ if $\xi \to -\infty$, and $u_0(\xi) \to 0$ if $\xi \to +\infty$ where $\xi = -\sqrt{a/24c}\,x + 5at/12$. It is found from our numerical simulation that if $\epsilon_0(x) = A\cos(kx)$ the disturbances are stable when $k^2 > a/c$, namely it is stable for the short wavelength disturbances. But for the long wavelength disturbances, $k^2 < a/c$, it is stable for the left-side of the wave and it is unstable for the right-side of the wave, which is shown in Fig.2 and Fig.3. If the initial condition is in the form of $\epsilon_0 = A\mathrm{sech}(k(x + x_0))$, as shown in the Fig.4 and Fig.5, we find numerically that the disturbances are stable when $x_0 > 0$, but it is unstable when $x_0 < 0$.

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