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On the linearization of the coupled Harry–Dym soliton hierarchy*

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This paper is devoted to the study of the underlying linearities of the coupled Harry–Dym (cHD) soliton hierarchy, including the well-known cHD equation. Resorting to the nonlinearization of Lax pairs, a family of finite-dimensional Hamiltonian systems associated with soliton equations are presented, constituting the decomposition of the cHD soliton hierarchy. After suitably introducing the Abel–Jacobi coordinates on a Riemann surface, the cHD soliton hierarchy can be ultimately reduced to linear superpositions, expressed by the Abel–Jacobi variables.

Keywords: soliton hierarchy, Hamiltonian systems, Riemann surface, Abel–Jacobi coordinates
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1. Introduction

The study of soliton equations is one of the central and highlighted topics in the realm of mathematical physics, because many mathematical models are described by nonlinear evolution equations. In revealing the basic structure of the Liouville integrability of soliton equations, the main aim is to appropriately choose the action-angle variables straightening out various flows on the invariant tours by using the celebrated Liouville–Arnold theory.[1] Actually, there exist many excellent examples in this framework, such as the Korteweg–de Vries equation $\bar{u}_t - 6\bar{u}\bar{u}_x + \bar{u}_{xxx} = 0$ equipped with the angular variable $\bar{\phi} = \bar{\Omega}_1 x + \bar{\Omega}_2 t + \bar{K}$, having the well-known explicit solution[2,3]

$$\bar{u}(x,t) = -2\partial_x^2 \ln\theta(\bar{\Omega}_1 x + \bar{\Omega}_2 t + \bar{K}) + \bar{C},$$

and the Kadomtsev–Petviashvili equation $\tilde{u}_t = \frac{1}{4}(\tilde{u}_{xx} + 3\tilde{u}^2)_x + \frac{3}{4}\partial_x^{-1}\tilde{u}_{yy}$ endowed with the angular variable $\tilde{\phi} = \tilde{\Omega}_1 x + \tilde{\Omega}_2 y + \tilde{\Omega}_3 t + \tilde{K}$, resulting in the beautiful formula[4]

$$\tilde{u}(x,y,t) = 2\partial_x^2 \ln\theta(\tilde{\Omega}_1 x + \tilde{\Omega}_2 y + \tilde{\Omega}_3 t + \tilde{K}) + \tilde{C}.$$

Analogously, another powerful approach, the so-called Hirota method,[5,6] also provides an effective way to transform the soliton equation into a type of a bilinear equation, and to find an exact solution by a perturbation method. In contrast to finding the action-angle variables, Hirota's method is relatively direct, systematic and easy to be treated. Based on the analysis of the Lax representations of soliton equations, it is interesting to see that the nonlinearization of Lax pairs[7–9] is also effective in linearizing soliton equations with the help of the theory of algebraic curves, and further in writing out the desired quasi-periodic solutions to exhibit the quasi-periodic behaviour or characteristic for Liouville integrability to soliton equations.[10–18] The concrete mechanism of our manipulation consists of three steps: the first step is a nonlinear separation of variables, which reduces infinite dimensional soliton models into finite-dimensional Hamiltonian systems (FDHSs) of ordinary differential equations using the nonlinearization of Lax pairs; the second step is to linearize the resulting FDHSs by Abel–Jacobi coordinates on the Jacobian of a Riemann surface; finally, we write out the explicit solutions in the original variables via the Jacobi inversion. To our knowledge, the key process of all the above-mentioned methods is to transform nonlinear differential equations into linear differential equations through a change of dependent variable, which are easier to be tackled with some available tools.[19–21]

In this paper our purpose is to disclose the linear behaviour of the coupled Harry–Dym (cHD) soliton hierarchy, including the well-known cHD equation on the Riemann surface. Recalling the cHD spectral

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problem of Antonowicz and Fordy, we readily recover the cHD soliton hierarchy. In a future step, the cHD soliton hierarchy is reduced into FDHSs, separating spatial and temporal variables by means of the nonlinearization of Lax pairs. An explicit relation between soliton equations and the resulting FDHSs is therefore established through the Bargmann constraint. Finally, using the Abel–Jacobi coordinates, the associated flows determined by the cHD soliton hierarchy are straightened as simple linear superpositions with respect to corresponding flow variables.

The present paper is divided into five sections. In Section 2, we briefly review some useful formulae and partly rewrite their expressions. In Section 3, the nonlinearization of Lax pairs with Bargmann constraint is discussed. The relation between soliton equations and FDHSs is established in Section 4. The whole cHD soliton hierarchy is straightened out to display the intrinsic linearities in the window of Abel-Jacobi coordinates in Section 5.

2. Preliminaries

To describe our results, a brief survey of the problem is needed. According to the known expression,[22] the cHD spectral problem is of the form

$$\partial_x^2\varphi + u\lambda\varphi + v\lambda^2\varphi = \alpha\varphi, \tag{1}$$

which is equivalent to a matrix form

$$\varphi_x = U\varphi, \quad U = \begin{pmatrix} 0 & 1 \\ \alpha - u\lambda - v\lambda^2 & 0 \end{pmatrix}, \quad \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{2}$$

where λ is a constant spectral parameter; u and v are two potentials. Considering the adjoint representation of Eq.(2), we have

$$V_x = [U, V], \quad V = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{j\geq 0} \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \lambda^{-j}, \tag{3}$$

which coincides with

$$a_j = -\frac{1}{2}\partial_x b_j, \tag{4a}$$

$$Kg_{j-1} = Jg_j, \tag{4b}$$

$$c_j = -\frac{1}{2}\partial_x^2 b_j + \alpha b_j - ub_{j+1} - vb_{j+2}, \tag{4c}$$

where $g_j = (b_j, b_{j+1})^{\mathrm{T}}$, and

$$K = \begin{pmatrix} 0 & \frac{1}{2}\partial_x^3 - 2\alpha\partial_x \\ \frac{1}{2}\partial_x^3 - 2\alpha\partial_x & u\partial_x + \partial_x u \end{pmatrix},$$

$$J = \begin{pmatrix} \frac{1}{2}\partial_x^3 - 2\alpha\partial_x & 0 \\ 0 & -(\partial_x v + v\partial_x) \end{pmatrix}. \tag{5}$$

Here $\{g_j\}$ is the Lenard sequence; K and J are the Lenard operator pairs. Assuming that $g_{-2} = (1, 0)^{\mathrm{T}}$ and $g_{-1} = (0, v^{-\frac{1}{2}})^{\mathrm{T}}$, it is clear that $\ker J = \{\rho_1 g_{-1} + \rho_2 g_{-2} | \forall \rho_1, \rho_2 \in \mathbb{R}\}$; thus $\{g_j\}$ $(j \geq 0)$ can be uniquely established (up to $\ker J$) by means of the recursive relation (4b). In particular, $g_0 = (v^{-\frac{1}{2}}, -\frac{1}{2}uv^{-\frac{3}{2}})^{\mathrm{T}}$.

To deduce the cHD soliton hierarchy, let us introduce an auxiliary spectral problem of Eq.(2),

$$\varphi_{t_n} = V^{(n)}\varphi, \quad V^{(n)} = \begin{pmatrix} V_{11}^{(n)} & V_{12}^{(n)} \\ V_{21}^{(n)} & -V_{11}^{(n)} \end{pmatrix}, \quad n \geq 0, \tag{6}$$

where

$$V_{11}^{(n)} = \sum_{j=0}^{n} a_{j-1}\lambda^{n+1-j}, \quad V_{12}^{(n)} = \sum_{j=0}^{n} b_{j-1}\lambda^{n+1-j},$$

$$V_{21}^{(n)} = \sum_{j=0}^{n} c_{j-1}\lambda^{n+1-j} + \lambda u b_n + \lambda^2 v b_n + \lambda v b_{n+1}, \tag{7}$$

with $a_{-1} = b_{-1} = c_{-1} = 0$. Then the compatibility condition of Eqs.(2) and (6), i.e. $\varphi_{xt_n} = \varphi_{t_n x}$, leads to the zero-curvature equation

$$U_{t_n} - V_x^{(n)} + [U, V^{(n)}] = 0, \tag{8}$$

which is the cHD soliton hierarchy[22]

$$(u, v)_{t_n}^{\mathrm{T}} = Jg_{n-1}, \qquad n \geq 0. \tag{9}$$

It is clear that the first nontrivial member of Eq.(9) is the cHD equation with $t = t_1$,

$$u_t = \frac{1}{2}(\frac{1}{\sqrt{v}})_{xxx} - 2\alpha(\frac{1}{\sqrt{v}})_x, \quad v_t = u_x(\frac{1}{\sqrt{v}}) + 2u(\frac{1}{\sqrt{v}})_x, \tag{10}$$

which results from the compatibility condition of Lax pairs (2) and

$$\varphi_t = V^{(1)}\varphi,$$

$$V^{(1)} = \begin{pmatrix} \frac{1}{4}v^{-\frac{3}{2}}\lambda & v^{-\frac{1}{2}}\lambda \\ -\frac{3}{8}v^{-\frac{5}{2}}\lambda + v^{-\frac{1}{2}}\lambda(\alpha - \lambda u - \lambda^2 v) & -\frac{1}{4}v^{-\frac{3}{2}}\lambda \end{pmatrix}. \tag{11}$$

Let us introduce a generating function of $\{g_k\}$,

$$g_\lambda = g_{-1} + \sum_{k=0}^{\infty} g_k \lambda^{-k-1}. \tag{12}$$

It is easy to check that $(K-\lambda J)g_\lambda = 0$. Denoting

$$g = (g^{(1)}, g^{(2)})^{\mathrm{T}} = \sum_{j=0}^{n} g_j \lambda^{n+1-j},$$

an immediate consequence is that $V^{(n)}$ can be rewritten in a concise form,

$$V^{(n)} \triangleq \sigma[g] = \begin{pmatrix} -\frac{1}{2}\partial_x g^{(1)} & g^{(1)} \\ -\frac{1}{2}\partial_x^2 g^{(1)} + g^{(1)}(\alpha - \lambda u - \lambda^2 v) & \frac{1}{2}\partial_x g^{(1)} \end{pmatrix}. \tag{13}$$

After a direct calculation, we have

$$V_x - [U, V] = U_*[(K-\lambda J)g], \tag{14}$$

where

$$U_*(\xi) = \frac{\mathrm{d}}{\mathrm{d}\varepsilon} U(w+\varepsilon\xi)\,|_{\varepsilon=0}\,, \quad w = (u, v)^{\mathrm{T}}.$$

Then $(K-\lambda J)g = 0$ implies that $\partial_x \det \sigma[g] = 0$.[23] Whence, from Eqs.(12) and (13), it is easy to obtain

$$\det \sigma[g_\lambda] = 1. \tag{15}$$

3. The finite-dimensional Hamiltonian systems

Let $\lambda_1, \lambda_2, \cdots, \lambda_N$ be N distinct eigenvalues and $\varphi = (p_j, q_j)^{\mathrm{T}}$. According to the principle of nonlinearization of Lax pairs,[7,8] we take N copies of the spectral problem (2),

$$\begin{pmatrix} p_j \\ q_j \end{pmatrix}_x = \begin{pmatrix} 0 & 1 \\ \alpha - u\lambda_j - v\lambda_j^2 & 0 \end{pmatrix} \begin{pmatrix} p_j \\ q_j \end{pmatrix}, \quad 1 \le j \le N. \tag{16}$$

A direct calculation yields[8] (up to a constant factor)

$$(K - \lambda_j J)\nabla\lambda_j = 0,$$

$$\nabla\lambda_j = (\delta\lambda_j/\delta u, \delta\lambda_j/\delta v)^{\mathrm{T}} = (\lambda_j p_j^2, \lambda_j^2 p_j^2)^{\mathrm{T}}. \tag{17}$$

Taking into consideration the following Bargmann constraint:[7,8]

$$g_0 = \sum_{j=1}^{N} \nabla\lambda_j, \tag{18}$$

yields

$$u = -2\langle\Lambda^2 p, p\rangle\langle\Lambda p, p\rangle^{-3}, \quad v = \langle\Lambda p, p\rangle^{-2}, \tag{19}$$

where $p = (p_1, \cdots, p_N)^{\mathrm{T}}$, $\Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_N)$, and $\langle\cdot,\cdot\rangle$ stands for the standard inner product in $\mathbb{R}^N$. Substituting Eq.(19) into Eq.(16) generates the following FDHS:

$$\begin{cases} p_x = q = -\dfrac{\partial H_0}{\partial q}, \\ q_x = \alpha p + 2\langle\Lambda^2 p, p\rangle\langle\Lambda p, p\rangle^{-3}\Lambda p \\ \qquad -\langle\Lambda p, p\rangle^{-2}\Lambda^2 p = \dfrac{\partial H_0}{\partial p}, \end{cases} \tag{20}$$

where $q = (q_1, \cdots, q_N)^{\mathrm{T}}$, and

$$H_0 = -\frac{1}{2}(\langle q, q\rangle - \alpha\langle p, p\rangle + \langle\Lambda p, p\rangle^{-2}\langle\Lambda^2 p, p\rangle). \tag{21}$$

Similarly, we can also apply the nonlinearization of Lax pairs to the temporal parts of soliton hierarchy. From Eqs.(11) and (19), we obtain the Hamiltonian system associated with the temporal part of cHD equation

$$\begin{cases} p_t = -\langle\Lambda p, q\rangle\Lambda p + \langle\Lambda p, p\rangle\Lambda q = -\dfrac{\partial H_1}{\partial q}, \\ q_t = -\langle\Lambda q, q\rangle\Lambda p + \langle\Lambda p, q\rangle\Lambda q - \dfrac{2\langle\Lambda^2 p, p\rangle^2}{\langle\Lambda p, p\rangle^3}\Lambda p + \\ \qquad \dfrac{\langle\Lambda^3 p, p\rangle}{\langle\Lambda p, p\rangle^2}\Lambda p + \dfrac{2\langle\Lambda^2 p, p\rangle}{\langle\Lambda p, p\rangle^2}\Lambda^2 p - \dfrac{1}{\langle\Lambda p, p\rangle}\Lambda^3 p = \dfrac{\partial H_1}{\partial p}, \end{cases} \tag{22}$$

where

$$H_1 = \frac{1}{2}\left(\frac{\langle\Lambda^2 p, p\rangle^2}{\langle\Lambda p, p\rangle^2} - \frac{\langle\Lambda^3 p, p\rangle}{\langle\Lambda p, p\rangle} - \langle\Lambda p, p\rangle\langle\Lambda q, q\rangle + \langle\Lambda p, q\rangle^2\right). \tag{23}$$

Only for the convenience, we introduce a bilinear function $Q_\lambda(\xi, \eta) = \sum_{j=1}^{N} \dfrac{\xi_j\eta_j}{\lambda - \lambda_j}$. Denoting

$$G_\lambda = g_{-1} + \sum_{j=1}^{N} \frac{\nabla\lambda_j}{\lambda - \lambda_j} = \begin{pmatrix} Q_\lambda(\Lambda p, p) \\ \langle \Lambda p, p\rangle + Q_\lambda(\Lambda^2 p, p) \end{pmatrix}, \tag{24}$$

it is not difficult to obtain $(K - \lambda J)G_\lambda = 0$. Substituting Eq.(24) into Eq.(13) yields

$$V_\lambda \triangleq \sigma[G_\lambda] = \begin{pmatrix} -Q_\lambda(\Lambda p, q) & Q_(\Lambda p, p) \\ \dfrac{\langle \Lambda^2 p, p\rangle}{\langle \Lambda p, p\rangle^2} - \dfrac{\lambda}{\langle \Lambda p, p\rangle} - Q_\lambda(\Lambda q, q) & Q_\lambda(\Lambda p, q) \end{pmatrix}, \tag{25}$$

in view of

$$Q_\lambda(\Lambda^l \xi, \eta) = \lambda Q_\lambda(\Lambda^{l-1}\xi, \eta) + \langle \Lambda^{l-1}\xi, \eta\rangle, \quad 1 \le l \in \mathbb{Z}.$$

Supposing that $F_\lambda = \det V_\lambda$, it is evident that F_λ is invariant under the action of $x-$flow. Hence the generating function of integrals of motion for Eq.(20) can be written as

$$\begin{aligned} F_\lambda =& \frac{\lambda Q_\lambda(\Lambda p, p)}{\langle \Lambda p, p\rangle} - \frac{\langle \Lambda^2 p, p\rangle Q_\lambda(\Lambda p, p)}{\langle \Lambda p, p\rangle^2} \\ &+ \begin{vmatrix} Q_\lambda(\Lambda p, p) & Q_\lambda(\Lambda p, q) \\ Q_\lambda(\Lambda p, q) & Q_\lambda(\Lambda q, q) \end{vmatrix} \\ =& 1 + \sum_{m=1}^{\infty} F_m \lambda^{-m-1}, \end{aligned} \tag{26}$$

where

$$\begin{aligned} F_m =& \frac{\langle \Lambda^{m+2} p, p\rangle}{\langle \Lambda p, p\rangle} - \frac{\langle \Lambda^2 p, p\rangle \langle \Lambda^{m+1} p, p\rangle}{\langle \Lambda p, p\rangle^2} \\ &+ \sum_{j=0}^{m-1} \begin{vmatrix} \langle \Lambda^{j+1} p, p\rangle & \langle \Lambda^{m-j} p, q\rangle \\ \langle \Lambda^{j+1} p, q\rangle & \langle \Lambda^{m-j} q, q\rangle \end{vmatrix}. \end{aligned} \tag{27}$$

Assuming that F_λ is the Hamiltonian function of the symplectic space $(\mathbb{R}^{2N}, dp \wedge dq)$, we recall that the Poisson bracket of two functions F and G is defined as[1]

$$\begin{aligned} \{F, G\} =& \sum_{j=1}^{N} \left(\frac{\partial F}{\partial q_j}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_j}\frac{\partial G}{\partial q_j} \right) \\ =& \langle \frac{\partial F}{\partial q}, \frac{\partial G}{\partial p}\rangle - \langle \frac{\partial F}{\partial p}, \frac{\partial G}{\partial q}\rangle. \end{aligned}$$

Denoting the conjugate variable of F_λ by τ_λ, a direct calculation leads to the following canonical equation:

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}\begin{pmatrix} p_k \\ q_k \end{pmatrix} = I\nabla_k F_\lambda = \begin{pmatrix} -\partial F_\lambda/\partial q_k \\ \partial F_\lambda/\partial p_k \end{pmatrix} = W(\lambda, \lambda_k)\begin{pmatrix} p_k \\ q_k \end{pmatrix}, \tag{28}$$

where

$$I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$W(\lambda, \mu) = -\frac{2\mu}{\lambda - \mu} V_\lambda - \frac{2\mu Q_\lambda(\Lambda p, p)}{\langle \Lambda p, p\rangle^2} \begin{pmatrix} 0 & 0 \\ \lambda + \mu + \dfrac{2\langle \Lambda^2 p, p\rangle}{\langle \Lambda p, p\rangle} & 0 \end{pmatrix}. \tag{29}$$

Similar to the previous calculation,[16] we derive a crucial Lax equation as follows.

Proposition 1 The Lax matrix V_μ along with $\tau_\lambda-$flow satisfies Lax equation

$$\frac{\mathrm{d}V_\mu}{\mathrm{d}\tau_\lambda} = [W(\lambda, \mu), V_\mu] \tag{30}$$

and

$$\begin{aligned} &\{F_\mu, F_\lambda\} = 0, \qquad \lambda, \mu \in \mathbb{C}, \\ &\{F_j, F_k\} = 0, \qquad j, k = 1, 2, \cdots. \end{aligned} \tag{31}$$

4. Relation between the cHD soliton hierarchy and FDHSs

In this section, the relation between soliton equations and FDHSs is explicitly established, such that the solutions of soliton equations can be obtained via solving FDHSs of ordinary differential equations. For this purpose, defining $F_0 = -2H_0$, a direct calculation shows that $\{F_0, F_m\} = 0$ $(m \ge 1)$ and the variable x is indeed identical to the variable t_0 simultaneously. Next we introduce a set of polynomial integrals $\{H_m\}$ for Eq.(20),

$$H_0 = -\frac{1}{2}F_0, \quad H_1 = -\frac{1}{2}F_1,$$

$$H_m = -\frac{1}{2}F_m + \frac{1}{2}\sum_{j=0}^{m-1} H_j H_{m-1-j}, \qquad m \ge 2, \tag{32}$$

which are in agreement with

$$F_\lambda = (1 - H_\lambda)^2, \tag{33}$$

where

$$H_\lambda = \sum_{m=0}^{\infty} H_m \lambda^{-m-1}. \tag{34}$$

A simple application of Eq.(33) results in

$$\{H_\lambda, H_\mu\} = \frac{1}{4\sqrt{F_\lambda F_\mu}}\{F_\lambda, F_\mu\} = 0.$$

Acting with $J^{-1}K$ upon Eq.(18) $(k-1)-$times, we have

$$\sum_{j=1}^{N} \lambda_j^{k-1}\nabla\lambda_j = g_{k-1} + c_3 g_{k-3} + \cdots + c_{k+1}g_{-1}, \quad k \geq 2, \tag{35}$$

where c_j is a constant of integral. Together with Eq.(35), Eq.(24) can be rewritten as

$$G_\lambda = c_\lambda g_\lambda, \tag{36}$$

where $c_\lambda = 1 + \sum_{k=0}^{\infty} c_{k+3}\lambda^{-k-2}$. From Eqs.(15) and (25), we have

$$V_\lambda = \sigma[G_\lambda] = \sigma[c_\lambda g_\lambda], \tag{37}$$

$$F_\lambda = c_\lambda^2. \tag{38}$$

The combination of Eqs.(33) and (38) gives $c_\lambda = 1 - H_\lambda$. Denoting the variables of H_λ and H_k by t_λ and t_k respectively, we have

$$\frac{d}{\mathrm{d}t_\lambda} = -\frac{1}{2c_\lambda}\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}. \tag{39}$$

Theorem 1 Let $(p(x,t_n), q(x,t_n))^{\mathrm{T}}$ be compatible solutions of H_0 and H_n, i.e.

$$\begin{pmatrix} p \\ q \end{pmatrix}_x = I\nabla H_0, \quad \begin{pmatrix} p \\ q \end{pmatrix}_{t_n} = I\nabla H_n, \quad n \geq 1, \tag{40}$$

then $u(x,t_n)$ and $v(x,t_n)$ determined by Eq.(19) is a solution of the n-th cHD equation.

Proof: Noticing Eq.(28), on one hand we have

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}\tau_\lambda} =& \frac{24\langle\Lambda^2 p, p\rangle}{\langle\Lambda p, p\rangle^4}(\langle\Lambda p, q\rangle Q_\lambda(\Lambda p, p) - \langle\Lambda p, p\rangle Q_\lambda(\Lambda p, q)) + \frac{8}{\langle\Lambda p, p\rangle^3}(\lambda\langle\Lambda p, p\rangle Q_\lambda(\Lambda p, q) - \lambda\langle\Lambda p, q\rangle Q_\lambda(\Lambda p, p) \\ &+ \langle\Lambda^2 p, p\rangle Q_\lambda(\Lambda p, q) - \langle\Lambda^2 p, q\rangle Q_\lambda(\Lambda p, p)),\end{aligned} \tag{41}$$

$$\frac{\mathrm{d}v}{\mathrm{d}\tau_\lambda} = \frac{8}{\langle\Lambda p, p\rangle^3}(\langle\Lambda p, p\rangle Q_\lambda(\Lambda p, q) - \langle\Lambda p, q\rangle Q_\lambda(\Lambda p, p)). \tag{42}$$

On the other hand, from Eq.(20), Eqs.(41) and (42) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}(u, v)^{\mathrm{T}} = -\frac{2}{\lambda}JG_\lambda.$$

Recalling Eq.(39), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t_\lambda}\begin{pmatrix} u \\ v \end{pmatrix} =& -\frac{1}{2c_\lambda}\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}\begin{pmatrix} u \\ v \end{pmatrix} = -\frac{1}{2c_\lambda}(-\frac{2}{\lambda})JG_\lambda \\ =& \frac{1}{\lambda}Jg_\lambda = \sum_{k=1}^{\infty} Jg_{k-1}\lambda^{-k-1},\end{aligned}$$

which indicates that Eq.(19) is a solution of the n-th equation of Eq.(9)

Corollary 1 Let $(p(x,t), q(x,t))^{\mathrm{T}}$ be compatible solutions of H_0 and H_1 with respect to $x-$flow and $t-$flow respectively, thus $(u(x,t), v(x,t))^{\mathrm{T}}$ given by Eq.(19) is the solution of the cHD equation (10).

5. Linearizing the cHD soliton hierarchy

In this section, the cHD soliton hierarchy are reduced into simple linear superpositions described by Abel–Jacobi variables on a Riemann surface. From Eqs.(26) and (27), we define

$$F_\lambda = -V_\lambda^{12}V_\lambda^{21} - (V_\lambda^{11})^2 = \frac{b(\lambda)}{a(\lambda)} = \frac{R(\lambda)}{a(\lambda)^2}, \tag{43}$$

$$V_\lambda^{12} = Q_\lambda(\Lambda p, p) = <\Lambda p, p> \frac{m(\lambda)}{a(\lambda)}, \tag{44}$$

where

$$a(\lambda) = \prod_{k=1}^{N}(\lambda - \lambda_k), \quad b(\lambda) = \prod_{k=1}^{N}(\lambda - \lambda_{N+k}), \tag{45}$$

$$m(\lambda) = \prod_{k=1}^{N-1}(\lambda - \mu_k),$$

$$R(\lambda) = a(\lambda)b(\lambda) = \prod_{k=1}^{2N}(\lambda - \lambda_k). \tag{46}$$

Here $\{\mu_k\}$ are called the elliptic coordinates. Noting that the expressions of u and v in Eq.(19) only depend

on the canonical variable p, we know that $\{\mu_k\}$ indeed meet the whole requirements in our calculation. From the Lax equation (30), we arrive at

$$\frac{\mathrm{d}V_\mu^{12}}{\mathrm{d}\tau_\lambda} = 2(W^{11}(\lambda,\mu)V_\mu^{12} - W^{12}(\lambda,\mu)V_\mu^{11}). \quad (47)$$

Choosing $\lambda = \mu_k$, it follows from Eqs.(43) and (44) that

$$V_{\mu_k}^{11} = \frac{\sqrt{-R(\mu_k)}}{a(\mu_k)}. \quad (48)$$

From Eqs.(29), (44), (47) and (48), it is not difficult to obtain

$$\frac{1}{4\mu_k\sqrt{-R(\mu_k)}}\frac{\mathrm{d}\mu_k}{\mathrm{d}\tau_\lambda} = -\frac{m(\lambda)}{a(\lambda)(\lambda-\mu_k)m'(\mu_k)}. \quad (49)$$

By applying the Lagrange interpolation formula for polynomials to Eq.(49), we derive

$$\sum_{k=1}^{N-1}\frac{\mu_k^{N-j-1}}{4\sqrt{-R(\mu_k)}}\frac{\mathrm{d}\mu_k}{\mathrm{d}\tau_\lambda} = -\frac{\lambda^{N-j}}{a(\lambda)}, \quad 1\le j\le N-1. \quad (50)$$

These formulae naturally lead to the consideration of the Riemann surface Γ given by the affine equation $\xi^2 + R(\lambda) = 0$, which is of genus $N-1$. It is known that

$$\tilde{\omega}_j = \frac{\lambda^{N-j-1}\mathrm{d}\lambda}{4\sqrt{-R(\lambda)}}, \qquad 1\le j\le N-1, \quad (51)$$

are $N-1$ linearly independent homomorphic differentials of Γ. Denoting an arbitrary point $P(\lambda) = (\lambda, \xi = \sqrt{-R(\lambda)})$ and a fixed point P_0 on Γ, we introduce the quasi-Abel–Jacobi coordinates,

$$\tilde{\phi}_j = \sum_{k=1}^{N-1}\int_{P_0}^{P(\mu_k)}\tilde{\omega}_j, \quad 1\le j\le N-1. \quad (52)$$

Therefore, Eq.(50) is of the form

$$\frac{\mathrm{d}\tilde{\phi}_j}{\mathrm{d}\tau_\lambda} = -\frac{\lambda^{N-j}}{a(\lambda)}, \qquad 1\le j\le N-1. \quad (53)$$

In what follows, we introduce a set of regular cycle paths on Γ: $a_1, b_1, a_2, b_2, \cdots, a_{N-1}, b_{N-1}$, which are independent and have intersection numbers as

$$a_i\circ a_j = b_i\circ b_j = 0, \quad a_i\circ b_j = \delta_{ij}, \quad i,j = 1,2,\cdots,N-1.$$

Now we may take[24,25]

$$A_{ij} = \int_{a_j}\tilde{\omega}_i, \quad C = (A_{ij})^{-1}, \quad 1\le i,j\le N-1, \quad (54)$$

then $\tilde{\omega}_l$ can be normalized into a new basis ω_j,

$$\omega_j = \sum_{l=1}^{N-1} C_{jl}\tilde{\omega}_l,$$

$$\int_{a_i}\omega_j = \sum_{l=1}^{N-1} C_{jl}\int_{a_i}\tilde{\omega}_l = \sum_{l=1}^{N-1} C_{jl}A_{li} = \delta_{ji},$$

$$j = 1,2,\cdots,N-1.$$

Additionally, the Abel map $\mathcal{A}(P)$ and Abel–Jacobi coordinates are defined as

$$\mathcal{A}(P) = \int_{P_0}^{P}\omega, \quad \mathcal{A}\left(\sum n_kP_k\right) = \sum n_k\mathcal{A}(P_k), \quad (55)$$

$$\phi = \mathcal{A}\left(\sum_{k=1}^{N-1}P(\mu_k)\right) = \sum_{k=1}^{N-1}\int_{P_0}^{P(\mu_k)}\omega = C\tilde{\phi}. \quad (56)$$

Designating $S_k = \lambda_1^k + \cdots + \lambda_{2N}^k$ and $\tilde{R}(\lambda^{-1}) = \prod_{j=1}^{2N}(1-\lambda_j\lambda^{-1})$, we know that[10]

$$\frac{1}{\sqrt{\tilde{R}(\lambda^{-1})}} = \sum_{k=0}^{\infty}\Lambda_k\lambda^{-k}, \quad (57)$$

where

$$\Lambda_0 = 1, \quad \Lambda_1 = \frac{1}{2}S_1, \quad \Lambda_k = \frac{1}{2k}(S_k + \sum_{i+j=k,i,j\ge1}S_i\Lambda_j).$$

Appealing to Eqs.(38) and (43), a simple calculation gives

$$\sqrt{R(\lambda)} = a(\lambda)c_\lambda. \quad (58)$$

Denoting the kth column vector of C by C_k, we are ready to formulate the main result in this article.

Theorem 2 The associated flows in the Abel–Jacobi coordinates are linearized as

$$\frac{\mathrm{d}\phi}{\mathrm{d}t_\lambda} = \sum_{k=0}^{\infty}\Omega_k\lambda^{-k-1}, \quad (59a)$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t_k} = \Omega_k, \qquad k\ge0, \quad (59b)$$

where Ω_k is designated by

$$\begin{aligned}
\Omega_0 &= \frac{1}{2}\Lambda_0C_1, \quad \Omega_1 = \frac{1}{2}(\Lambda_1C_1 + \Lambda_0C_2),\\
\Omega_k &= \frac{1}{2}(\Lambda_kC_1 + \cdots + \Lambda_1C_k + \Lambda_0C_{k+1}), \qquad k\le N-2,\\
\Omega_k &= \frac{1}{2}(\Lambda_kC_1 + \cdots + \Lambda_{k-N+1}C_N), \qquad k\ge N-1.
\end{aligned}$$

Proof: From Eqs.(39), (56), (53) and (58), we

have

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}t_\lambda} &= -\frac{1}{2c_\lambda}\frac{\mathrm{d}\phi}{\mathrm{d}\tau_\lambda} = -\frac{1}{2c_\lambda}C\frac{\mathrm{d}\tilde{\phi}}{\mathrm{d}\tau_\lambda}\\ &= \frac{\lambda^N}{2c_\lambda a(\lambda)}(C_1\lambda^{-1}+\cdots+C_{N-1}\lambda^{-N+1})\\ &= \frac{\lambda^N}{2\sqrt{R(\lambda)}}(C_1\lambda^{-1}+\cdots+C_{N-1}\lambda^{-N+1})\\ &= \frac{1}{2\sqrt{\tilde{R}(\lambda^{-1})}}(C_1\lambda^{-1}+\cdots+C_{N-1}\lambda^{-N+1})\\ &= \frac{1}{2}\sum_{k=0}^{\infty}\Lambda_k\lambda^{-k}\sum_{j=1}^{N-1}C_j\lambda^{-j} = \sum_{k=0}^{\infty}\Omega_k\lambda^{-k-1}.\end{aligned}$$

Comparing the coefficient with λ^{-k-1} of (59a), we see that (59b) holds immediately.

On application of the Theorem 2, it is easy to see that

$$\phi = \phi_0 + \sum_{k=0}^{\infty}\Omega_k t_k. \tag{60}$$

As a concrete application of Eq.(60), the kth cHD equation is explicitly presented in the window of Abel–Jacobi coordinates,

$$\phi_k = \phi_0 + \Omega_0 x + \Omega_k t_k. \tag{61}$$

In particular, the evolution picture of the cHD equation (10) in the Abel–Jacobi coordinates is described as

$$\phi_1 = \phi_0 + \Omega_0 x + \Omega_1 t. \tag{62}$$

In conclusion, we have decomposed the cHD soliton hierarchy into FDHSs, and further linearized the resulting Hamiltonian flows to display the underlying linear behaviour of soliton equations on a Riemann surface. To some extent, the given expressions (61) and (62) provide the specially explicit solutions to the kth cHD equation and the cHD equation in the Abel–Jacobi coordinates, respectively. Analogously, some other known soliton hierarchies can be studied in this framework as well. For this aspect, we will provide some other examples in the near future.

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