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A unified symmetry of nonholonomic mechanical systems in phase space

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In this paper, we have studied the unified symmetry of a nonholonomic mechanical system in phase space. The definition and the criterion of a unified symmetry of the nonholonomic mechanical system in phase space are given under general infinitesimal transformations of groups in which time is variable. The Noether conserved quantity, the generalized Hojman conserved quantity and the Mei conserved quantity are obtained from the unified symmetry. An example is given to illustrate the application of the results.

Keywords: nonholonomic mechanical system, unified symmetry, conserved quantity, phase space
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1. Introduction

The research on symmetries and conserved quantities of a mechanical system possesses important mathematical and physical significance, and is also a new development direction in analytical mechanics. The modern methods of finding conserved quantities are mainly Noether symmetry,[1] Lie symmetry,[2] and Mei symmetry.[3]

The conserved quantities deduced respectively from the three symmetries mentioned above are mainly Noether conserved quantity, Hojman conserved quantity and Mei conserved quantity. The research on the three symmetries and corresponding conserved quantities has achieved lots of important results.[4−15]

Previous studies on symmetries and conserved quantities are all carried out for from a certain kind of symmetry to seek for a type of conserved quantity. Recently, Mei *et al*[16] gave a unified symmetry of Lagrange system, Xu *et al*[17−19] studied the unified symmetries of holonomic system and the Hamilton system.

In this paper, we generalize the unified symmetry to a nonholonomic system in phase space, study the unified symmetry of the nonholonomic system in phase space, give the definition and criterion of a unified symmetry of nonholonomic mechanical system in phase space, and obtain all the Noether conserved quantity, generalized Hojman conserved quantity and Mei conserved quantity from the unified symmetry.

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2. Definition and criterion of unified symmetry

Suppose that the configuration of a mechanical system is determined by n generalized coordinations $q_s(s = 1, \cdots, n)$. The system is subject to g ideal nonholonomic constraints of Chetaev's type

$$f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0\,, \quad (\beta = 1, \cdots, g)\,. \tag{1}$$

Suppose that the constraints are independent of each other. The limitation of constraints (1) imposed on the virtual displacement is

$$\sum_{s=1}^{n} \frac{\partial f_\beta}{\partial \dot{q}_s} \delta q_s = 0\,. \tag{2}$$

The equation of motion of the nonholonomic system can be written as[20]

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s + \Lambda_s,$$
$$(s = 1, 2, \cdots, n), \tag{3}$$

$$\Lambda_s = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum_{s=1}^{n} \lambda_\beta \frac{\partial f_\beta}{\partial \dot{q}_s}, \tag{4}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the Lagrangian of the system, $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are the generalized nonpotential forces, Λ_s are the generalized nonholonomic constraint reacting forces, $\lambda_\beta = \lambda_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are the constraint multipliers. According to Ref. [21], the canonical form of Eq.(3) in phase space is

$$\dot{q}_s = \partial H / \partial p_s,$$

$$\dot{p}_s = -\partial H/\partial q_s + \tilde{Q}_s + \tilde{\Lambda}_s, \quad (s = 1, 2, \cdots, n), \tag{5}$$

where $H = H(t, \boldsymbol{q}, \boldsymbol{p})$ is the Hamiltonian,

$$\tilde{Q}_s = \tilde{Q}_s(t, \boldsymbol{q}, \boldsymbol{p}) = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}(t, \boldsymbol{q}, \boldsymbol{p})),$$
$$\tilde{\Lambda}_s = \tilde{\Lambda}_s(t, \boldsymbol{q}, \boldsymbol{p}) = \Lambda_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}(t, \boldsymbol{q}, \boldsymbol{p})). \tag{6}$$

Expanding Eq.(5), we obtain

$$\dot{q}_s = g_s(t, \boldsymbol{q}, \boldsymbol{p}),$$
$$\dot{p}_s = h_s(t, \boldsymbol{q}, \boldsymbol{p}), \quad (s = 1, 2, \cdots, n). \tag{7}$$

The nonholonomic constraints (1) in phase space can be expressed as

$$\tilde{f}_\beta = \tilde{f}_\beta(t, \boldsymbol{q}, \boldsymbol{p}) = f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}(t, \boldsymbol{q}, \boldsymbol{p})) = 0. \tag{8}$$

Definition If symmetry is Noether symmetry that is also Lie symmetry and Mei symmetry at the same time, then the symmetry is called unified symmetry.

Based on the Noether theory of nonholonomic mechanical systems, if there exists such a gauge function $G_{\mathrm{N}} = G_{\mathrm{N}}(t, \boldsymbol{q}, \boldsymbol{p})$ that the generators of infinitesimal transformations ξ_0, ξ_s, η_s satisfy

$$p_s \frac{\bar{\mathrm{d}}\,\xi_s}{\mathrm{d}\,t} - \frac{\partial H}{\partial t}\xi_0 - \frac{\partial H}{\partial q_s}\xi_s - H\frac{\bar{\mathrm{d}}\,\xi_0}{\mathrm{d}\,t} + (\tilde{Q}_s + \tilde{\Lambda}_s)(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}\,G_{\mathrm{N}}}{\mathrm{d}\,t} = 0 \tag{9}$$

and restriction equations[4]

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) = 0, \tag{10}$$

the symmetry is called the Noether symmetry of the nonholonomic mechanical system (5) with the nonholonomic constrains (8) in phase space, here $\partial f_\beta/\partial \dot{q}_s$ should be expressed as a function of $t, \boldsymbol{q}, \boldsymbol{p}$, where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}\,t} = \frac{\partial}{\partial t} + g_s\frac{\partial}{\partial q_s} + h_s\frac{\partial}{\partial p_s}. \tag{11}$$

Based on the Lie theory of nonholonomic mechanical systems, if the generators of infinitesimal transformations ξ_0, ξ_s, η_s satisfy

$$X^{(1)}\Big(\dot{q}_s - \frac{\partial H}{\partial p_s}\Big) = 0,$$
$$X^{(1)}\Big(\dot{p}_s + \frac{\partial H}{\partial q_s} - \tilde{Q}_s - \tilde{\Lambda}_s\Big) = 0, \tag{12}$$

or its equivalent expression

$$\frac{\bar{\mathrm{d}}\,\xi_s}{\mathrm{d}\,t} - g_s\frac{\bar{\mathrm{d}}\,\xi_0}{\mathrm{d}\,t} = X^{(0)}(g_s),$$
$$\frac{\bar{\mathrm{d}}\,\eta_s}{\mathrm{d}\,t} - h_s\frac{\bar{\mathrm{d}}\,\xi_0}{\mathrm{d}\,t} = X^{(0)}(h_s),$$
$$(s = 1, 2, \cdots, n), \tag{13}$$

and the restriction equation

$$X^{(0)}[\tilde{f}_\beta(t, \boldsymbol{q}, \boldsymbol{p})] = 0 \tag{14}$$

and additional restriction equation[22]

$$\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) = 0, \tag{15}$$

then the symmetry is called the Lie symmetry of the nonholonomic system (5) with the nonholonomic constraints (8) in phase space, where

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \eta_s\frac{\partial}{\partial p_s},$$
$$X^{(1)} = X^{(0)} + (\dot{\xi}_s - \dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s} + (\dot{\eta}_s - \dot{p}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{p}_s}. \tag{16}$$

Based on the Mei theory of nonholonomic mechanical systems, if the generators of infinitesimal transformations ξ_0, ξ_s, η_s satisfy

$$\frac{\partial}{\partial p_s}[X^{(0)}(H)] = 0,$$
$$\frac{\partial}{\partial q_s}[X^{(0)}(H)] - X^{(0)}(\tilde{Q}_s + \tilde{\Lambda}_s) = 0, \tag{17}$$

and the restriction equation

$$X^{(0)}[\tilde{f}_\beta(t, \boldsymbol{q}, \boldsymbol{p})] = 0, \tag{18}$$

then the symmetry is called the Mei symmetry of the nonholonomic system (5) with the nonholonomic constraints (8) in phase space.

According to the above definition of the unified symmetry and the relations (9), (10), (12), (14), (15), (17) and (18), we can obtain the following criterion.

Criterion If there exists such a gauge function $G_{\mathrm{N}} = G_{\mathrm{N}}(t, \boldsymbol{q}, \boldsymbol{p})$ that the generators of the infinitesimal transformations ξ_0, ξ_s, η_s satisfy the condition

$$\Big\{p_s\frac{\bar{\mathrm{d}}\,\xi_s}{\mathrm{d}\,t} - \frac{\partial H}{\partial t}\xi_0 - \frac{\partial H}{\partial q_s}\xi_s - H\frac{\bar{\mathrm{d}}\,\xi_0}{\mathrm{d}\,t} + (\tilde{Q}_s + \tilde{\Lambda}_s)(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}\,G_{\mathrm{N}}}{\mathrm{d}\,t}\Big\}^2 + \Big\{X^{(1)}\Big(\dot{q}_s - \frac{\partial H}{\partial p_s}\Big)\Big\}^2 + \Big\{X^{(1)}\Big(\dot{p}_s + \frac{\partial H}{\partial q_s} - \tilde{Q}_s - \tilde{\Lambda}_s\Big)\Big\}^2 + \Big\{\frac{\partial}{\partial p_s}X^{(0)}(H)\Big\}^2 + \Big\{\frac{\partial}{\partial q_s}X^{(0)}(H) - X^{(0)}(\tilde{Q}_s + \tilde{\Lambda}_s)\Big\}^2 = 0, \tag{19}$$

and

$$\Big\{X^{(0)}[\tilde{f}_\beta(t, \boldsymbol{q}, \boldsymbol{p})]\Big\}^2 + \Big\{\frac{\partial f_\beta}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0)\Big\}^2 = 0, \tag{20}$$

then the symmetry is called the unified symmetry of the nonholonomic system (5) with the nonholonomic constraints (8) in phase space.

3. The conserved quantities deduced from the unified symmetry

Under some conditions, the unified symmetry can lead to the Noether conserved quantity, the generalized Hojman conserved quantity and the Mei conserved quantity.

Proposition 1 For a nonholonomic mechanical system in phase space, the unified symmetry leads to the Noether conserved quantity

$$I_{\rm N} = p_s\xi_s - H\xi_0 + G_{\rm N} = \text{const}\,. \tag{21}$$

Proof Because the unified symmetry is certainly a Noether symmetry, such a gauge function $G_{\rm N} = G_{\rm N}(t,\boldsymbol{q},\boldsymbol{p})$ makes Eq.(9) hold true. By using the Noether symmetry theory of the nonholonomic system, the unified symmetry can lead to the conserved quantity (21).

Proposition 2 If there exists a function $\mu = \mu(t,\boldsymbol{q},\boldsymbol{p})$ satisfying

$$\frac{\partial g_s}{\partial q_s} + \frac{\partial h_s}{\partial p_s} + \frac{\bar{\rm d}\ln\mu}{{\rm d}t} = 0, \tag{22}$$

then the unified symmetry leads to a generalized Hojman conserved quantity

$$\begin{aligned} I_{\rm H} = &\frac{1}{\mu}\frac{\partial(\mu\xi_0)}{\partial t} + \frac{1}{\mu}\frac{\partial(\mu\xi_s)}{\partial q_s} \\ &+ \frac{1}{\mu}\frac{\partial(\mu\eta_s)}{\partial p_s} - \frac{\bar{\rm d}\xi_0}{{\rm d}t} = \text{const}. \end{aligned} \tag{23}$$

Proof According to the definition, the unified symmetry must be Lie symmetry, using similar method to that in Ref.[9], we can obtain the generalized Hojman conserved quantity (23).

Proposition 3 If there exists a function $G_{\rm M} = G_{\rm M}(t,\boldsymbol{q},\boldsymbol{p})$ satisfying

$$\begin{aligned} &X^{(0)}(p_s)\frac{\bar{\rm d}\xi_s}{{\rm d}t} + \frac{\bar{\rm d}}{{\rm d}t}[X^{(0)}(p_s)]\xi_s \\ &- X^{(0)}[X^{(0)}(H)] - X^{(0)}(H)\frac{\bar{\rm d}\xi_0}{{\rm d}t} \\ &+ [X^{(0)}(\tilde{Q}_s + \tilde{\Lambda}_s)](\xi_s - \dot{q}_s\xi_0) \\ &+ \frac{\bar{\rm d}G_{\rm M}}{{\rm d}t} = 0, \end{aligned} \tag{24}$$

then the unified symmetry leads to the Mei conserved quantity

$$I_{\rm M} = X^{(0)}(p_s)\xi_s - X^{(0)}(H)\xi_0 + G_{\rm M} = \text{const}. \tag{25}$$

Proof We have

$$\begin{aligned} \frac{\bar{\rm d}I_{\rm M}}{{\rm d}t} =& X^{(0)}(p_s)\frac{\bar{\rm d}\xi_s}{{\rm d}t} + \frac{\bar{\rm d}}{{\rm d}t}[X^{(0)}(p_s)]\xi_s \\ &- X^{(0)}(H)\frac{\bar{\rm d}\xi_0}{{\rm d}t} - \Big[\frac{\partial}{\partial t}X^{(0)}(H) \\ &+ \dot{q}_s\frac{\partial}{\partial q_s}X^{(0)}(H) + \dot{p}_s\frac{\partial}{\partial p_s}X^{(0)}(H)\Big]\xi_0 \\ &+ \frac{\bar{\rm d}G_{\rm M}}{{\rm d}t}, \end{aligned} \tag{26}$$

Based on the definition, the unified symmetry must be Mei symmetry. Substituting Eqs.(17) and (24) into Eq.(26), we can obtain

$$\frac{\bar{\rm d}I_{\rm M}}{{\rm d}t} = 0. \tag{27}$$

4. Example

Take an Appell–Hamel example. The Lagrangian of the problem is

$$L = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2) - mgq_3, \tag{28}$$

and the constraint equation is

$$f = \dot{q}_1^2 + \dot{q}_2^2 - \dot{q}_3^2 = 0. \tag{29}$$

Now let us study the unified symmetry and its conserved quantity of the nonholonomic mechanical system in phase space.

We can have the Hamiltonian of the system as follows:

$$H = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + mgq_3. \tag{30}$$

So Eq.(7) can be written as

$$\begin{aligned} &\dot{q}_1 = \frac{p_1}{m}, \qquad \dot{q}_2 = \frac{p_2}{m}, \\ &\dot{q}_3 = \frac{p_3}{m}, \qquad \dot{p}_1 = 2\lambda\frac{p_1}{m}, \\ &\dot{p}_2 = 2\lambda\frac{p_2}{m}, \quad \dot{p}_3 = -mg - 2\lambda\frac{p_3}{m}. \end{aligned} \tag{31}$$

The constraint equation (29) can be expressed as

$$\tilde{f} = p_1^2 + p_2^2 - p_3^2 = 0\,. \tag{32}$$

Combining Eqs.(31) with Eq.(32), we can obtain

$$\lambda = -\frac{m^2 g}{4p_3}. \tag{33}$$

Then we can obtain

$$\begin{aligned} &g_1 = \dot{q}_1 = \frac{p_1}{m}, \quad g_2 = \dot{q}_2 = \frac{p_2}{m}, \quad g_3 = \dot{q}_3 = \frac{p_3}{m}, \\ &h_1 = \dot{p}_1 = -\frac{mgp_1}{2p_3}, \quad h_2 = \dot{p}_2 = -\frac{mgp_2}{2p_3}, \\ &h_3 = \dot{p}_3 = -\frac{1}{2}mg, \quad \tilde{\Lambda}_1 = -\frac{mgp_1}{2p_3}, \\ &\tilde{\Lambda}_2 = -\frac{mgp_2}{2p_3}, \quad \tilde{\Lambda}_3 = \frac{1}{2}mg. \end{aligned} \tag{34}$$

We have

$$X^{(0)}(H)=\frac{1}{m}(\eta_1 p_1+\eta_2 p_2+\eta_3 p_3)+\xi_3 mg,$$
$$X^{(0)}(\tilde{Q}_1+\tilde{\Lambda}_1)=-\eta_1\frac{mg}{2p_3}+\eta_3\frac{mgp_1}{2p_3^2},$$
$$X^{(0)}(\tilde{Q}_2+\tilde{\Lambda}_2)=-\eta_2\frac{mg}{2p_3}+\eta_3\frac{mgp_1}{2p_3^2},$$
$$X^{(0)}(\tilde{Q}_3+\tilde{\Lambda}_3)=0, \tag{35}$$
$$X^{(0)}(\tilde{f}_\beta)=2\eta_1 p_1+2\eta_2 p_2-2\eta_3 p_3,$$
$$\frac{\partial f_\beta}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)=2\xi_1\dot{q}_1+2\xi_2\dot{q}_2-2\xi_3\dot{q}_3. \tag{36}$$

Choosing the generators of the infinitesimal transformations as

$$\xi_0=1,\quad \xi_1=p_1,\quad \xi_2=p_2,\quad \xi_3=p_3,$$
$$\eta_1=-\frac{m^2gp_1}{2p_3},\quad \eta_2=-\frac{m^2gp_2}{2p_3},$$
$$\eta_3=-\frac{1}{2}m^2g, \tag{37}$$

and considering expressions (34), then we will have

$$X^{(0)}(H)=0,\quad X^{(0)}(\tilde{Q}_1+\tilde{\Lambda}_1)=0,$$
$$X^{(0)}(\tilde{Q}_2+\tilde{\Lambda}_2)=0,\quad X^{(0)}(\tilde{Q}_3+\tilde{\Lambda}_3)=0,$$
$$X^{(0)}(\tilde{f}_\beta)=0,\quad \frac{\partial f_\beta}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)=0. \tag{38}$$

It is easy to verify that when the gauge function is $G_{\rm N}=2m^2gq_3$. Equations (19) and (20) hold true. So, the generators (37) are unified symmetrical.

From Proposition 1 we can have the Noether conserved quantity deduced from the unified symmetry of the nonholonomic mechanical system in phase space as follows:

$$I_{\rm N}=(2m-1)\Big(\frac{p_3^2}{m}+mgq_3\Big). \tag{39}$$

From Eq.(21), we obtain

$$\frac{mg}{p_3}=\frac{\bar{\rm d}}{{\rm d}\,t}\ln\mu \tag{40}$$

with

$$\mu=\Big(p_3+\frac{1}{2}mgt\Big)p_3^{-2}. \tag{41}$$

According to Proposition 2, we have

$$I_{\rm H}=\frac{-m^2g+mg}{2p_3+mgt}, \tag{42}$$

which is the generalized Hojman conserved quantity deduced from the unified symmetry of the nonholonomic system in phase space.

From Eq.(24) we obtain

$$G_{\rm M}=-\frac{1}{2}m^3g^2t+p_1^2+p_2^2+m^2gq_3. \tag{43}$$

From Proposition 3, we have

$$I_{\rm M}=p_3^2+m^2g(q_3-p_3)-\frac{1}{2}m^3g^2t, \tag{44}$$

which is the Mei conserved quantity deduced from the unified symmetry of the nonholonomic system in phase space.

It is obvious that both the generalized Hojman conserved quantity (42) and the Mei conserved quantity (44) are non-Noether conserved quantities.

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