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Mei symmetry of Tzénoff equations of holonomic system*

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The Mei symmetry of Tzénoff equations under the infinitesimal transformations of groups is studied in this paper. The definition and the criterion equations of the symmetry are given. If the symmetry is a Noether symmetry, then the Noether conserved quantity of the Tzénoff equations can be obtained by the Mei symmetry.

Keywords: Tzénoff equations, Mei symmetry, Noether symmetry, conserved quantity
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1. Introduction

The analytical mechanics in which the symmetry theory and the conserved quantity theory play important roles becomes more and more important in the area of modern physics, and it develops very rapidly.[1−3] It becomes an important means to solve practical problems of obtaining the conserved quantities of dynamical systems by investigating the symmetries. There are three symmetries available for seeking the conserved quantities of dynamical systems today, namely the Noether symmetry, the Lie symmetry and the Mei symmetry.

Presently, the symmetry theory of dynamical systems is developing rapidly. Mei Fengxiang has studied the non-Noether conserved quantities in phase space.[4,5] Extending his result, some authors[6−8] obtained the non-Noether Lutzky conserved quantities for nonconservative dynamical systems and the non-Noether conserved quantities of velocity-dependent symmetry for nonholonomic system, and also a new method to solve conserved quantities of electromechanical systems.[9] At the same time, the research on non-Noether symmetry of the constrained dynamical systems also has made good progress.[10−15]

Mei symmetry was put forward by Mei Fengxiang, and its corresponding Noether conserved quantities and non-Noether conserved quantities[16,17] in the constrained dynamical systems have been deduced. There are many dynamical equations describing in the mechanical systems, of which the Tzénoff equations have the simplest form.

Today, the Mei symmetry of Lagrange systems and Hamilton systems and Appell equations have been studied and applied.[18−20] But the Mei symmetry of Tzénoff equations has not been studied yet. Mei symmetry of Tzénoff equations under the infinitesimal transformations of groups is studied in this paper, and the conserved quantity of Tzénoff equations is given.

The Mei symmetry of Tzénoff equations under the infinitesimal transformations of groups is studied firstly, and then Tzénoff equations are transformed into Lagrange equations; then the conserved quantity of Tzénoff equations is obtained by Noether symmetry.

2. Tzénoff equations of holonomic system

Let the position of a dynamical system be determined by the n generalized coordinates $q_s (s = 1, \cdots, n)$, and let Tzénoff function be

$$K = \frac{1}{2}(\ddot{T} - 3\ddot{T}_0) - Q_s \ddot{q}_s, \tag{1}$$

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where T is the kinetic energy of the system, $\ddot{T}_0$ is the 2nd-order differential coefficient taking the generalized velocity in T as constant, Q_s is the generalized force, then the Tzénoff equations of holonomic system in the generalized coordinates are

$$\frac{\partial K}{\partial \ddot{q}_s}=0, \quad s=1,\cdots,n. \tag{2}$$

3. Mei symmetry of Tzénoff equations

Introducing the infinitesimal transformations of time and coordinates as

$$t^*=t+\Delta t,\ \ q_s^*(t^*)=q_s(t)+\Delta q_s, \quad s=1,\cdots,n, \tag{3}$$

or their expansion formulae

$$\begin{aligned} t^*&=t+\varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\\ q_s^*(t^*)&=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}), \end{aligned} \tag{4}$$

where ε is an infinitesimal parameter and ξ_0, ξ_s are infinitesimal generators, we then have

$$\begin{aligned} \frac{\mathrm{d}q_s^*}{\mathrm{d}t^*}=&\frac{\mathrm{d}q_s+\varepsilon d\xi_s}{\mathrm{d}t+\varepsilon d\xi_0}\\ =&\dot{q}_s+\varepsilon(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)+O(\varepsilon^2), \end{aligned}$$

$$\frac{\mathrm{d}^2q_s^*}{\mathrm{d}t^{*2}}=\ddot{q}_s+\varepsilon[(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)^{\cdot}-\ddot{q}_s\dot{\xi}_0]+O(\varepsilon^2). \tag{5}$$

Definition If the form of Tzénoff equations (2) keeps invariant, when replacing function K before transforming by Tzénoff function K^* after transforming, then the form invariance is called Mei symmetry of Tzénoff equations.

By definition, Mei symmetry of Tzénoff equations can be written in the form

$$\frac{\partial K^*}{\partial \ddot{q}_s}=\frac{\partial K}{\partial \ddot{q}_s}=0, \quad s=1,\cdots,n, \tag{6}$$

where

$$K^*=K\left(t^*,\boldsymbol{q}^*,\frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*},\frac{\mathrm{d}^2\boldsymbol{q}^*}{\mathrm{d}t^{*2}}\right). \tag{7}$$

We have the following criterion:

Criterion For a given function K, if infinitesimal generators ξ_0 and ξ_s satisfy the formula

$$\frac{\partial}{\partial\ddot{q}_k}\left\{\frac{\partial K}{\partial t}\xi_0+\frac{\partial K}{\partial q_s}\xi_s+\frac{\partial K}{\partial\dot{q}_s}(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\right\}=0, \quad k=1,\cdots,n, \tag{8}$$

then the Tzénoff equations have a Mei symmetry.

Proof We have

$$\begin{aligned} K^*=&K\left(t^*,\boldsymbol{q}^*,\frac{\mathrm{d}\boldsymbol{q}^*}{\mathrm{d}t^*},\frac{\mathrm{d}^2\boldsymbol{q}^*}{\mathrm{d}t^{*2}}\right)=K\left(t,\boldsymbol{q},\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t},\frac{\mathrm{d}^2\boldsymbol{q}}{\mathrm{d}t^2}\right)\\ &+\varepsilon\left\{\frac{\partial K}{\partial t}\xi_0+\frac{\partial K}{\partial q_s}\xi_s+\frac{\partial K}{\partial\dot{q}_s}(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)+\frac{\partial K}{\partial\ddot{q}_s}[(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)^{\cdot}-\ddot{q}_s\dot{\xi}_0]\right\}+O(\varepsilon^2), \end{aligned}$$

then

$$\begin{aligned} \frac{\partial K^*}{\partial\ddot{q}_k}=&\frac{\partial K}{\partial\ddot{q}_s}+\varepsilon\frac{\partial}{\partial\ddot{q}_k}\left\{\frac{\partial K}{\partial t}\xi_0+\frac{\partial K}{\partial q_s}\xi_s+\frac{\partial K}{\partial\dot{q}_s}(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)+\frac{\partial K}{\partial\ddot{q}_k}[(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)^{\cdot}-\ddot{q}_s\dot{\xi}_0]\right\}+O(\varepsilon^2)\\ =&\varepsilon\frac{\partial}{\partial\ddot{q}_k}\left\{\frac{\partial K}{\partial t}\xi_0+\frac{\partial K}{\partial q_s}\xi_s+\frac{\partial K}{\partial\dot{q}_s}(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\right\}+O(\varepsilon^2). \end{aligned}$$

The equation (8) is obtained by neglecting ε^2 and higher order small terms.

4. Lagrange realization of Tzénoff equations

In order to seek the conserved quantities corresponding to the Mei symmetry of the Tzénoff equations, let us study the relationship between Mei symmetry and Noether symmetry; hence, it is necessary to transform the Tzénoff equations into Lagrange equations. For a given Tzénoff function K, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}=\frac{\partial K}{\partial\ddot{q}_s}\\ \equiv&A_{\mathrm{sk}}(t,\boldsymbol{q},\dot{\boldsymbol{q}})\ddot{q}_k+B_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0, \end{aligned} \tag{9}$$

where $L=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ is the Lagrangian. A necessary and sufficient condition for equations (9) to have the

ordered direct analytic representation of general Lagrange equations is that equations (9) are self-adjoint, i.e.,

$$\begin{aligned}
&A_{sk}=A_{ks},\\
&\frac{\partial A_{sk}}{\partial\dot{q}_r}=\frac{\partial A_{rk}}{\partial\dot{q}_s},\\
&\frac{\partial B_s}{\partial\dot{q}_k}+\frac{\partial B_k}{\partial\dot{q}_s}=2\left(\frac{\partial}{\partial t}+\dot{q}_r\frac{\partial}{\partial q_r}\right)A_{sk},\\
&\frac{\partial B_s}{\partial q_k}-\frac{\partial B_k}{\partial q_s}=\frac{1}{2}\left(\frac{\partial}{\partial t}+\dot{q}_r\frac{\partial}{\partial q_r}\right)\left(\frac{\partial B_s}{\partial\dot{q}_k}-\frac{\partial B_k}{\partial\dot{q}_s}\right);\\
&(s,k,r=1,\cdots,n).
\end{aligned}\tag{10}$$

Under the conditions (10), Tzénoff equations can be expressed by Lagrange equations.

Let the general structure of the Lagrange function be

$$L(t,\boldsymbol{q},\dot{\boldsymbol{q}})=E(t,\boldsymbol{q},\dot{\boldsymbol{q}})+D_k(t,\boldsymbol{q})\dot{\boldsymbol{q}}_k+C(t,\boldsymbol{q}),\tag{11}$$

where $n+2$ functions E, D_k and C are a solution of the following over-determined system of partial differential equations:

$$\frac{\partial^2E}{\partial\dot{q}_s\partial\dot{q}_k}=A_{sk}(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\tag{12}$$

$$\frac{\partial D_s}{\partial q_k}-\frac{\partial D_k}{\partial q_s}=\frac{1}{2}\left(\frac{\partial B_s}{\partial\dot{q}_k}-\frac{\partial B_k}{\partial\dot{q}_s}\right)+\frac{\partial^2E}{\partial q_s\partial\dot{q}_k}-\frac{\partial^2E}{\partial\dot{q}_s\partial q_k}\equiv Z_{sk}(t,\boldsymbol{q}),\tag{13}$$

$$\frac{\partial C}{\partial q_s}=\frac{\partial D_s}{\partial t}-B_s-\frac{\partial E}{\partial q_s}+\frac{\partial^2E}{\partial\dot{q}_s\partial t}+\left[\frac{\partial^2E}{\partial q_s\partial\dot{q}_k}+\frac{1}{2}\left(\frac{\partial B_s}{\partial\dot{q}_k}-\frac{\partial B_k}{\partial\dot{q}_s}\right)\right]\dot{q}_k\equiv W_s(t,\boldsymbol{q}).\tag{14}$$

The solution of the above equations is

$$E(t,\boldsymbol{q},\dot{\boldsymbol{q}})=\dot{q}_s\int_0^1\mathrm{d}\tau'\left\{\left[\int_0^1\mathrm{d}\tau A_{sk}(t,\boldsymbol{q},\tau\dot{\boldsymbol{q}})\right]\dot{q}_k\right\}(t,\boldsymbol{q},\tau'\dot{\boldsymbol{q}}),\tag{15}$$

$$D_s=\int_0^1[\mathrm{d}\tau\tau Z_{sk}(t,\tau q)]q_k,\tag{16}$$

$$C=\left[\int_0^1\mathrm{d}\tau W_k(t,\tau q)\right]q_k.\tag{17}$$

The Tzénoff equations can be transformed into Lagrange equations by (11)–(17).

5. The relationship between Mei symmetry and Noether symmetry

For a system determined by Lagrange L, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}_s}-\frac{\partial L}{\partial q_s}=0.\quad(s=1,\cdots,n)\tag{18}$$

Noether theory points out that if the infinitesimal generators ξ_0, ξ_s and the gauge function $G=G(t,q)$ satisfy the Noether identity

$$L\dot{\xi}_0+\frac{\partial L}{\partial t}\xi_0+\frac{\partial L}{\partial q_s}\xi_s+\frac{\partial L}{\partial\dot{q}_s}(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)=-\dot{G},\tag{19}$$

then the system possesses the following Noether conserved quantity:

$$I=L\xi_0+\frac{\partial L}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)+G=\mathrm{const}.\tag{20}$$

Proposition For the generators ξ_0, ξ_s of Mei symmetry of Tzénoff equations, if there exists a gauge function $G=G(t,q)$ satisfying identity (19), then the conserved quantity of equations (20) will be deduced by Mei symmetry of Tzénoff equations.

6. Example

The Tzénoff function of a dynamical system is

$$K=\frac{1}{2}\ddot{q}^2+\ddot{q}\dot{q}.\tag{21}$$

Let us study its Mei symmetry and the conserved quantities.

Firstly, we study Mei symmetry of Tzénoff equations. Equation (8) gives

$$\frac{\partial}{\partial\ddot{q}}\left\{\ddot{q}(\dot{\xi}-\dot{q}\dot{\xi}_0)\right\}=0,$$

i.e.,

$$\dot{\xi}-\dot{q}\dot{\xi}_0=0.\tag{22}$$

Equation (22) has the following solutions:

$$\xi_0 = 0, \quad \xi = 1; \tag{23}$$

$$\xi_0 = 1, \quad \xi = 0; \tag{24}$$

$$\xi_0 = t, \quad \xi = q. \tag{25}$$

Now we transform Tzénoff equations into Lagrange equations. The equations of motion of the system can be written in following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \frac{\partial K}{\partial \ddot{q}} = \ddot{q} + \dot{q} = 0. \tag{26}$$

But Eq.(26) is not self-adjoint, it can be rewritten as

$$e^t(\ddot{q} + \dot{q}) = 0, \tag{27}$$

then Eq.(27) is self-adjoint. In this case, the Tzénoff function becomes

$$K = e^t\left(\frac{1}{2}\ddot{q}^2 + \ddot{q}\dot{q}\right).$$

The criterion equation (8) of Mei symmetry of the Tzénoff equations gives

$$e^t(\ddot{q} + \dot{q})\xi_0 + e^t(\dot{\xi} - \dot{q}\dot{\xi}_0) = 0,$$

i.e.,

$$\dot{\xi} - \dot{q}\dot{\xi}_0 = 0.$$

This is identical to Eq.(22); then by Eqs.(11)–(17) we can seek the Lagrangian

$$L = \frac{1}{2}e^t\dot{q}^2. \tag{28}$$

Substituting Eq.(28) into Eq.(19), we obtain

$$e^t\left(\frac{1}{2}\dot{q}^2\xi_0 - \frac{1}{2}\dot{q}^2\dot{\xi}_0 + \dot{q}\dot{\xi}\right) = -\dot{G}. \tag{29}$$

Substituting generators (23)–25) into Eq.(29), we respectively obtain

$$-\dot{G} = 0, \tag{30}$$

$$\frac{1}{2}e^t\dot{q}^2 = -\dot{G}, \tag{31}$$

$$\frac{1}{2}e^t\dot{q}^2(t+1) = -\dot{G}. \tag{32}$$

By Eq.(30), we have

$$G = 0. \tag{33}$$

We cannot find gauge function G by Eq.(31) and Eq.(32), therefore only the generator (23) and gauge function (33) satisfy Noether symmetry; then the corresponding conserved quantity obtained by Eq.(20) is

$$I = e^t\dot{q} = \text{const.} \tag{34}$$

7. Conclusions

The investigation in this paper indicates that if the generators ξ_0 are ξ_s of the Tzénoff function of a holonomic mechanical system satisfy criterion (8) under the infinitesimal transformations of groups, then the Tzénoff equations are Mei symmetrical, and if the infinitesimal generators ξ_0 and ξ_s of Mei symmetry satisfy the condition of Noether symmetry after transforming the Tzénoff equations into Lagrange equations, then the corresponding conserved quantity of the Tzénoff equations can be deduced.

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