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The generation of a hyperchaotic system based on a three-dimensional autonomous chaotic system*

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This paper reports a new four-dimensional hyperchaotic system obtained by adding a controller to a three-dimensional autonomous chaotic system. The new system has two parameters, and each equation of the system has one quadratic cross-product term. Some basic properties of the new system are analysed. The different dynamic behaviours of the new system are studied when the system parameter a or b is varied. The system is hyperchaotic in several different regions of the parameter b. Especially, the two positive Lyapunov exponents are both larger, and the hyperchaotic region is also larger when this system is hyperchaotic in the case of varying a. The hyperchaotic system is analysed by Lyapunov-exponents spectrum, bifurcation diagrams and Poincaré sections.

Keywords: chaos, chaos generation, hyperchaotic system, bifurcation analysis
PACC: 0545

1. Introduction

The well-known Lorenz chaotic attractor was discovered in a three-dimensional autonomous system in 1963.[1] Another well-known three-dimensional chaotic system, the Chen system, was constructed via a state feedback controller to the Lorenz system in 1999.[2] Later, the Lorenz system family[3] and the generalized Lorenz canonical form[4] were presented. In the past few years, chaos generation has become a hot point of study and chaotic systems have been extensively studied in engineering.[5−8] But, in secure communication, there exists a possibility that the messages, masked by a simple chaotic system with only a single positive Lyapunov exponent, can be easily extracted sometimes when they were intercepted. An example of how to extract messages that are masked by a chaotic signal in a system of two Lorenz oscillators was shown in Ref.[9].

Hyperchaotic systems with more than one positive Lyapunov exponent usually have more complex and richer dynamic behaviours than chaotic systems. The dynamics of hyperchaotic attractor expands in more than one direction, enhancing the security of the messages to be transmitted in secure communication. Hyperchaos was first reported by Rössler in 1979.[10] Having seen its great potential for applications in science and technology,[11−16] researchers pay more attention to the study and generation[17,18] of hyperchaos recently. A family of modified Rössler hyperchaotic systems was demonstrated in Ref.[19]. Many hyperchaos systems were generated by circuits.[20−23] Particularly, designing a hyperchaotic system from an originally chaotic but non-hyperchaotic system with some simple techniques on purpose is a hot topic for research.[24−26]

For hyperchaos, there are two aspects that should be mentioned here. One is that the minimal dimension of the phase space in which the hyperchaotic attractor is embedded should be at least four, which requires the minimum number of coupled first-order autonomous ordinary differential equations to be four. The other is that the number of terms in the coupled equations that give rise to instability should be at least two, of which one should be a nonlinear function.

Recently, in Ref.[27], Qi *et al* proposed a new three-dimensional autonomous chaotic system, modified from the Lorenz system, in which each equation contains a single quadratic cross-product term.

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This chaotic system has five equilibria and some larger chaotic regions.

This paper reports a new four-dimensional hyperchaotic system, which is obtained by adding one more state variable into the three-dimensional autonomous chaotic system proposed in Ref.[27]. It can have three or five equilibria. The different dynamic behaviours of the new system are studied, respectively, when the parameter a or b is varied. This system has a larger hyperchaotic region and the two positive Lyapunov exponents both of which are larger when the parameter a is varied. The system can be periodic, quasi-periodic, chaotic or hyperchaotic with b varying.

2. The new hyperchaotic system and its basic properties

2.1. Generating hyperchaos from chaos

2.1.1. The three-dimensional autonomous chaotic system

Consider the following three-dimensional autonomous chaotic system found by Qi *et al* in Ref.[27]:

$$\begin{cases} \dot{x} = a(y - x) + yz, \\ \dot{y} = cx - xz - y, \\ \dot{z} = xy - bz, \end{cases} \tag{1}$$

where $(x, y, z) \in \Re^3$, and $a, b, c \in \Re^1$. Add a cross-product term to the first equation of the Lorenz system, then we will obtain such a system, that each equation of the system contains a single quadratic cross-product term, which has five equilibria, some larger chaotic regions and more complex bifurcation behaviours.

When $a = 35$, $b = \dfrac{8}{3}$, and $c = 25$, system (1) is chaotic with a single positive Lyapunov exponent. The corresponding chaotic attractor is shown in Figs.1 (a) and (b), respectively. For more detailed analysis of this system, see Ref.[27].

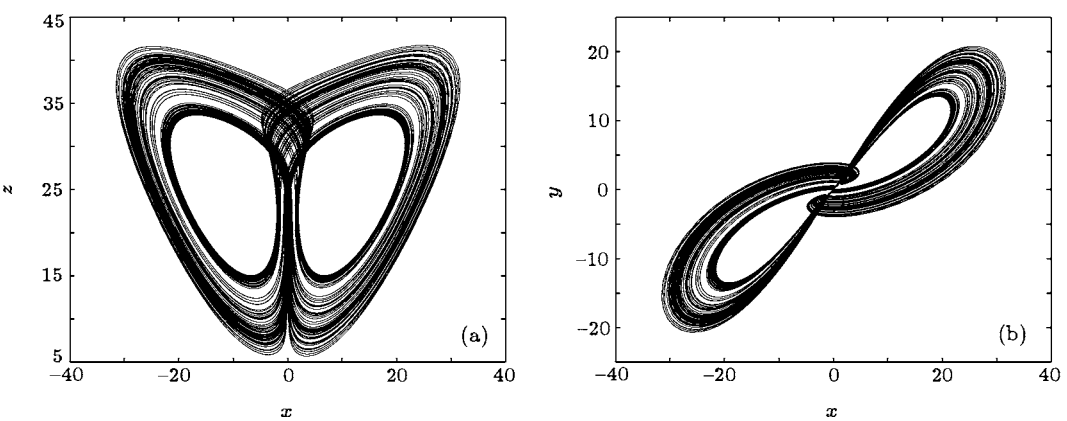


Fig.1. Phase portraits of chaotic system (1) with $a = 35$, $b = \dfrac{8}{3}$, and $c = 25$. (a) x-z plane; (b) x-y plane.

2.1.2. Generating the four-dimensional hyperchaotic system

Now, introduce an additional state, w, and couple it to the second equation of the chaotic system (1), then we will obtain the following four-dimensional system:

$$\begin{cases} \dot{x} = a(y - x) + yz, \\ \dot{y} = 25x - xz - y - w, \\ \dot{z} = xy - \frac{8}{3}z, \\ \dot{w} = bx + \frac{1}{2}yz + w, \end{cases} \tag{2}$$

where $(x, y, z, w) \in \Re^4$, $a, b \in \Re^1$. Due to the new state w, system (2) can be hyperchaotic when its parameters are taken appropriately.

2.2. The basic properties of the new hyperchaotic system

Now, some basic properties of system (2) are analysed as follows.

2.2.1. Symmetry

System (2) is symmetrical about the z-axis for its invariance under the coordinate transform $(x, y, z, w) \rightarrow (-x, -y, z, -w)$. Furthermore, the phase portrait of this system in $x - y - w$ three-dimensional space is symmetrical about the origin.

And the symmetry holds on for all values of system parameters.

2.2.2. Dissipation

For system (2), we have

$$\begin{aligned}\nabla V =&\frac{\partial \dot{x}}{\partial x}+\frac{\partial \dot{y}}{\partial y}+\frac{\partial \dot{z}}{\partial z}+\frac{\partial \dot{w}}{\partial w}\\ =&-a-1-\frac{8}{3}+1=-a-\frac{8}{3}.\end{aligned}$$

Therefore, to ensure system (2) being dissipative, it is required that $-a-\dfrac{8}{3}<0$, i.e. $a>-\dfrac{8}{3}$. Under this condition, system (2) converges exponentially:

$$\frac{\mathrm{d}V}{\mathrm{d}t}=\exp\left(-\left(a+\frac{8}{3}\right)\right).$$

That is, the volume element at time t is $V_0\exp(-(a+8/3)t)$ with the initial volume element being V_0. Furthermore, all orbits of this system are confined to a specific subset of zero volume as $t\to\infty$. Then, the existence of attractor is proved.

2.2.3. Equilibria

The equilibria of system (2) can be found by letting the left-hand side of Eq.(2) equal zero. Obviously, $S_0=(0,0,0,0)$ is an equilibrium point for all values of system parameters. Set

$$\begin{aligned}c=&a^4+284a^3+12a^3b+4a^2b^2+200a^2b+2500a^2,\\ d=&2a^2+4ab+92a+2\sqrt{c},\\ e=&-\frac{8}{3}\frac{d}{a+2},\quad f=\frac{1}{12}(8a-3e),\\ g=&2a^2+4ab+92a-2\sqrt{c},\\ h=&-\frac{8}{3}\frac{g}{a+2},\quad m=\frac{1}{12}(8a-3h),\\ x_1=&\frac{4}{3}\frac{a\sqrt{e}}{f},\quad y_1=\frac{1}{2}\sqrt{e},\quad z_1=\frac{1}{4}\frac{ae}{f},\\ w_1=&\sqrt{e}\frac{768a^2-3a^2e-64ab-72ae+6be}{64a^2+128ab+3200a-18ae-12be-300e},\\ x_2=&\frac{4}{3}\frac{a\sqrt{h}}{m},\quad y_2=\frac{1}{2}\sqrt{h},\quad z_2=\frac{1}{4}\frac{ah}{m},\\ w_2=&\sqrt{h}\frac{768a^2-3a^2h-64ab-72ah+6bh}{64a^2+128ab+3200a-18ah-12bh-300h}.\end{aligned}$$

It is easily verified that if $c>0$, $e\geq 0$ and $h\geq 0$, then system (2) has five equilibria:

$$\begin{aligned}S_0=&(0,0,0,0),\\ S_1=&(x_1,y_1,z_1,w_1),\\ S_2=&(-x_1,-y_1,z_1,-w_1),\\ S_3=&(x_2,y_2,z_2,w_2),\\ S_4=&(-x_2,-y_2,z_2,-w_2).\end{aligned}$$

Obviously, S_1 and S_2, S_3 and S_4 are symmetrical with respect to the z-axis.

Here, assume that

$$a>0,\quad \text{and}\quad b\geq -0.8. \tag{3}$$

Then, one can prove $c>0$, $e\leq 0$ and $h\geq 0$. System (2) has three equilibria:

$$\begin{aligned}S_0=&(0,0,0,0),\\ S_1=&(x_2,y_2,z_2,w_2),\\ S_2=&(-x_2,-y_2,z_2,-w_2).\end{aligned}$$

Also, S_1 and S_2 are symmetrical about the z-axis.

2.2.4. Stability of the equilibria

The Jacobian of the linearized system (2) at $S_0=(0,0,0,0)$ is

$$J=\begin{bmatrix}-a & a & 0 & 0\\ 25 & -1 & 0 & -1\\ 0 & 0 & -\dfrac{8}{3} & 0\\ b & 0 & 0 & 1\end{bmatrix}.$$

Then, the corresponding characteristic equation is

$$\begin{aligned}f_1(\lambda)=&\left(\lambda+\frac{8}{3}\right)(\lambda^3+a\lambda^2-(25a+1)\lambda+a(24+b))\\ =&0.\end{aligned}\tag{4}$$

Denote

$$f_2(\lambda)=\lambda^3+a\lambda^2-(25a+1)\lambda+a(24+b)=0. \tag{5}$$

According to the Routh–Hurwitz stability conditions, under the assumption (3), not all of the real parts of the roots λ of Eq.(5) are negative. So, not all of the roots λ of Eq.(4) have negative real parts. It means that the equilibrium S_0 is unstable.

For the nonzero equilibria, their stabilities can be estimated similarly, which is not further discussed here.

3. The bifurcation analysis of the hyperchaotic system

The following investigation relies on a combination of numerical computation and analysis.

3.1. Bifurcation analysis when varying *b*

Fixing $a=35$ and varying b, the Lyapunov-exponent spectrum and the bifurcation diagram of state w versus the parameter b are shown in Figs.2(a) and (b), when $b\in[-0.8,145]$. It can be observed that the bifurcation diagram well coincides with the corresponding Lyapunov-exponent spectrum.

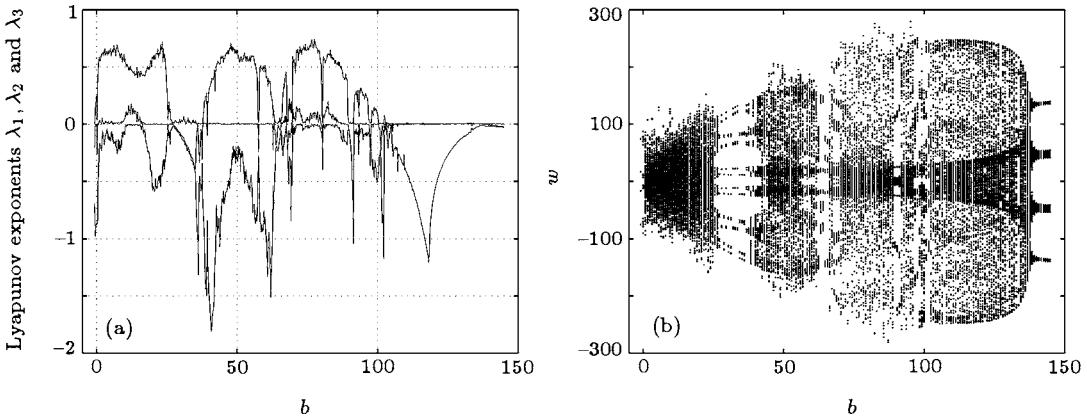


Fig.2. Lyapunov-exponent spectrum and bifurcation diagram versus b with $b \in [-0.8, 145]$. (a) Lyapunov-exponents spectrum; (b) Bifurcation diagram of w.

As we know, the relations between the orbits of the hyperchaotic system and Lyapunov exponents are described as follows:

For periodic orbits, there are one zero and three negative Lyapunov exponents.

For quasi-periodic orbits, there are two zero and two negative Lyapunov exponents.

For chaotic orbits, there are one positive, one zero and two negative Lyapunov exponents.

For hyperchaotic orbits, there are two positive, one zero and one negative Lyapunov exponents.

From the Lyapunov-exponent spectrum shown in Fig.2(a), we can see that system (2) is hyperchaotic in several different regions of parameter b.

In fact, as b increases, system (2) can be periodic, quasi-periodic, chaotic or hyperchaotic. There is no sharp boundary between two different orbits. Several typical attractors are shown as follows:

At first, assume the four Lyapunov exponents of system (2) to be λ_1, λ_2, λ_3, λ_4.

(i) System (2) is periodic when $\lambda_1 = 0$, λ_2, λ_3, $\lambda_4 < 0$. Some typical periodic attractors of system (2) are shown in Figs.3(a)–(d):

1) when $b = 37$, system (2) is periodic (Fig. 3 (a));

2) when $b = 68$, system (2) is periodic (Fig. 3 (b));

3) when $b = 91.4$, system (2) is periodic (Fig. 3 (c));

4) when $b = 145$, system (2) is periodic (Fig. 3 (d)).

(ii) System (2) is quasi-periodic when $\lambda_1 = \lambda_2 = 0$, λ_3, $\lambda_4 < 0$. Some typical quasi-periodic attractors of system (2) are shown in Figs.4(a)–(d):

1) when $b = 127$, system (2) is quasi-periodic (Fig. 4 (a));

2) when $b = 129.6$, system (2) is quasi-periodic (Fig. 4 (b));

3) when $b = 136.2$, system (2) is quasi-periodic (Fig. 4 (c));

4) when $b = 138$, system (2) is quasi-periodic (Fig. 4 (d)).

(iii) System (2) is chaotic when $\lambda_1 > 0$, $\lambda_2 = 0$, λ_3, $\lambda_4 < 0$. Some typical chaotic attractors of system (2) are shown in Figs.5(a)–(d):

1) when $b = 25.8$, system (2) is chaotic (Fig. 5 (a));

2) when $b = 41$, system (2) is chaotic (Fig. 5 (b));

3) when $b = 57.25$, system (2) is chaotic (Fig.5 (c));

4) when $b = 96$, system (2) is chaotic (Fig. 5 (d)).

(iv) System (2) is hyperchaotic when $\lambda_1, \lambda_2 > 0$, $\lambda_3 = 0$, $\lambda_4 < 0$. Some typical hyperchaotic attractors of system (2) are shown in Figs.6 and 7:

1) when $b = 13$, system (2) is hyperchaotic (Figs.6(a)–(f));

2) when $b = 71.2$, system (2) is hyperchaotic (Figs.7(a)–(f)).

And the Lyapunov exponents of the typical hyperchaotic systems with different b are listed in Table 1.

Table 1. The Lyapunov exponents of the typical hyperchaotic systems of different b with $a = 35$.

b	λ_1	λ_2	λ_3	λ_4
13	0.4509	0.1443	0	−38.2593
71.2	0.6481	0.1535	0	−38.4680

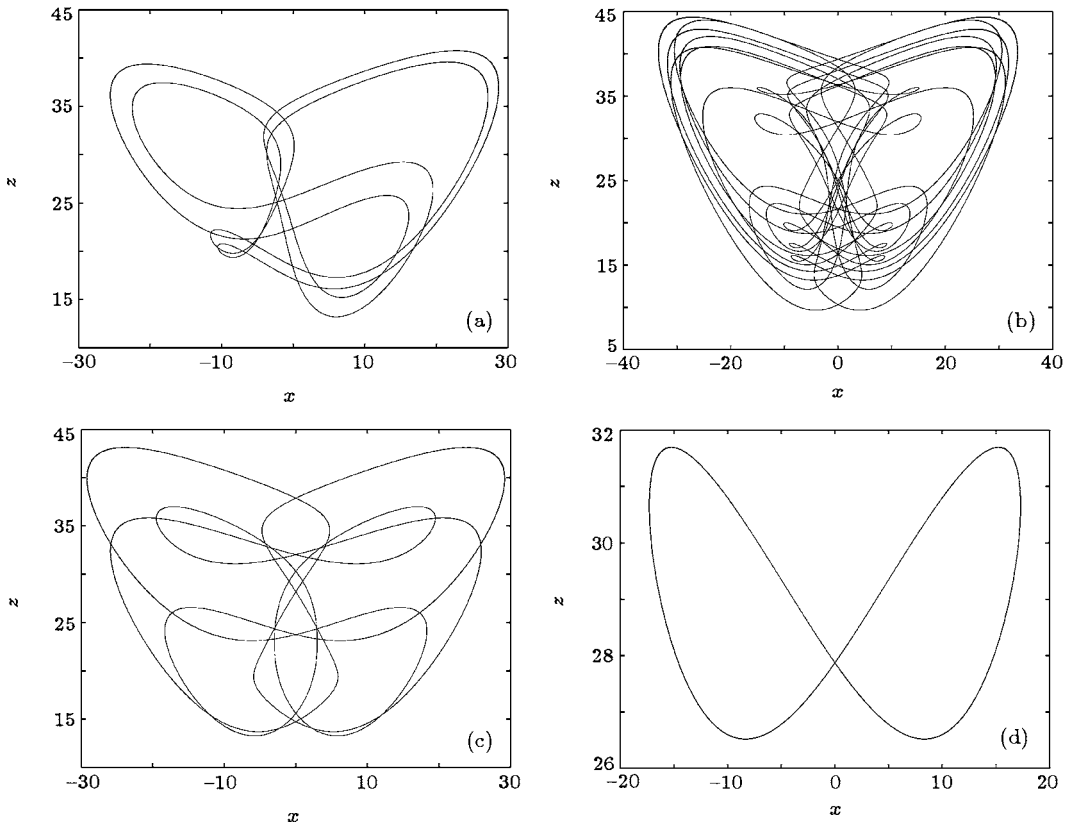


Fig.3. Phase portraits of periodic attractors of system (2) in the x–z plane, with b increasing and $a = 35$. (a) $b = 37$; (b) $b = 68$; (c) $b = 91.4$; (d) $b = 145$.

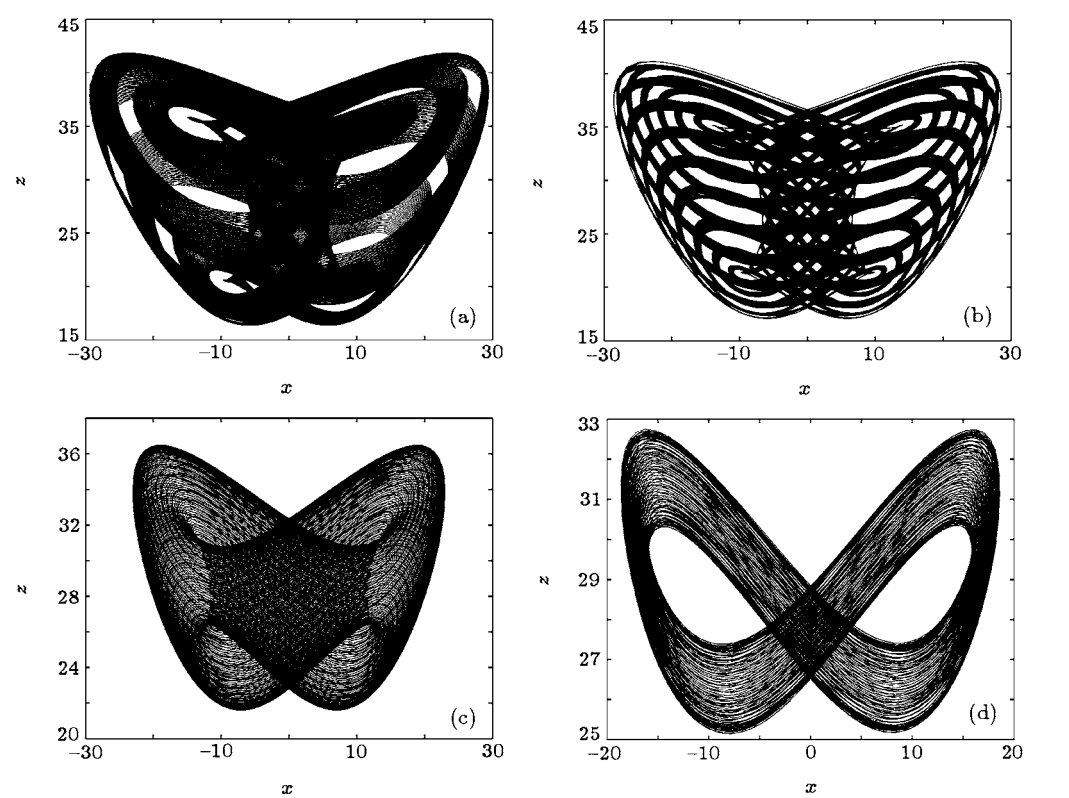


Fig.4. Phase portraits of quasi-periodic attractors of system (2) in the x–z plane, with b increasing and $a = 35$. (a) $b = 127$; (b) $b = 129.6$; (c) $b = 136.2$; (d) $b = 138$.

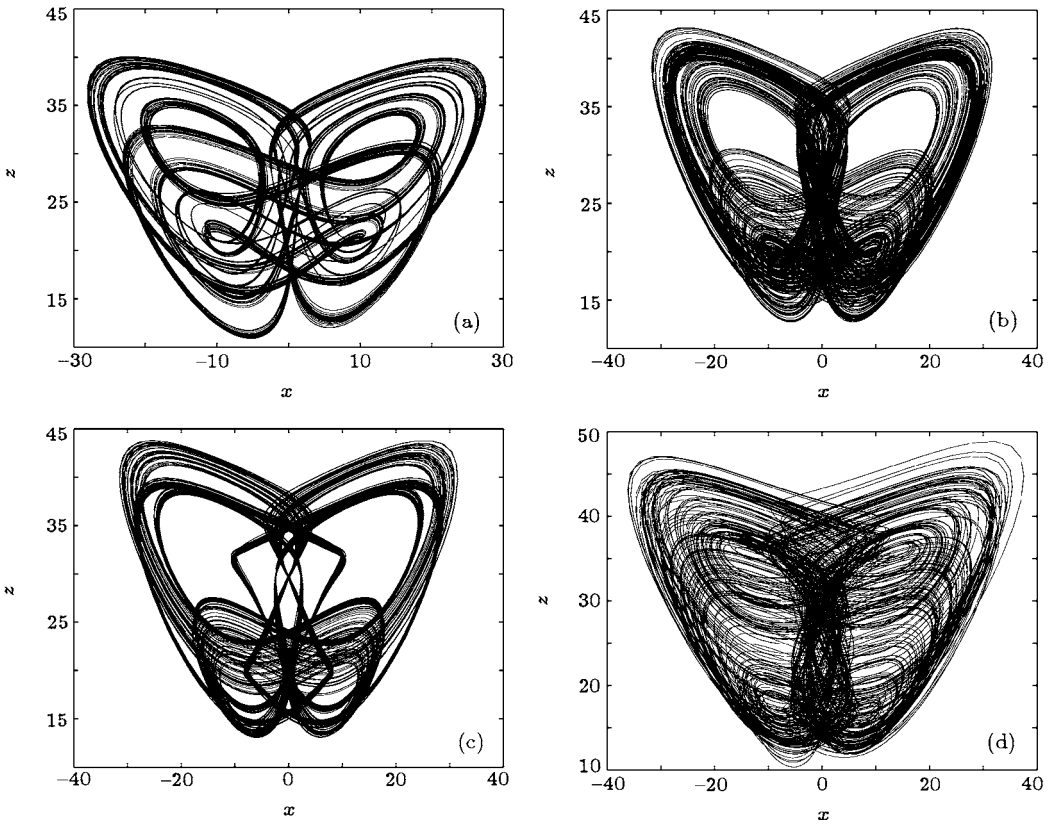
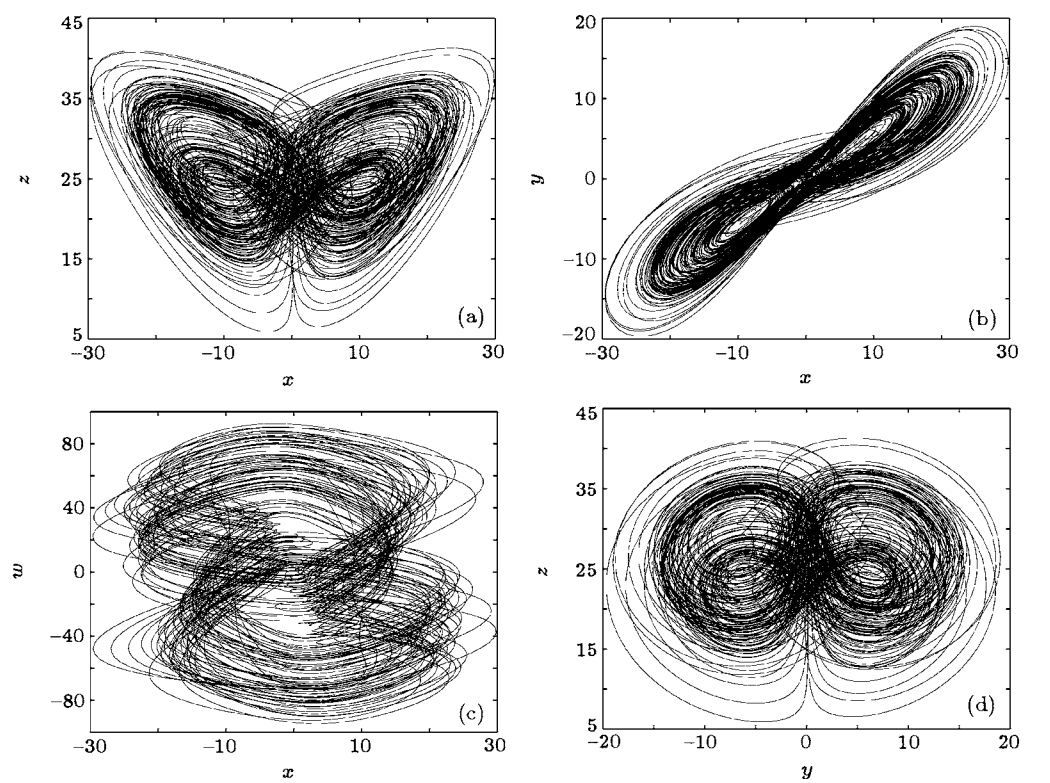


Fig.5. Phase portraits of chaotic attractors of system (2) in the x–z plane, with b increasing and $a = 35$. (a) $b = 25.8$; (b) $b = 41$; (c) $b = 57.25$; (d) $b = 96$.



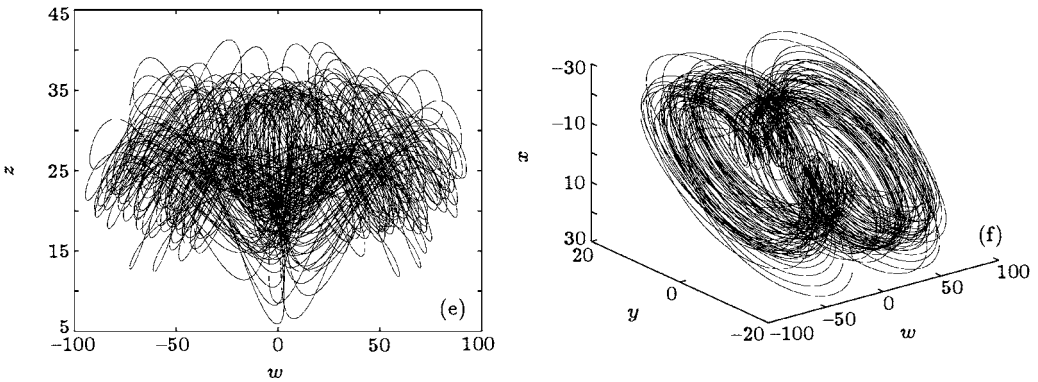


Fig.6. Phase portraits of hyperchaotic attractors of system (2) with $a = 35$, $b = 13$. (a) x–z plane; (b) x–y plane; (c) x–w plane; (d) y–z plane; (e) w–z plane; (f) w, y Vs x.

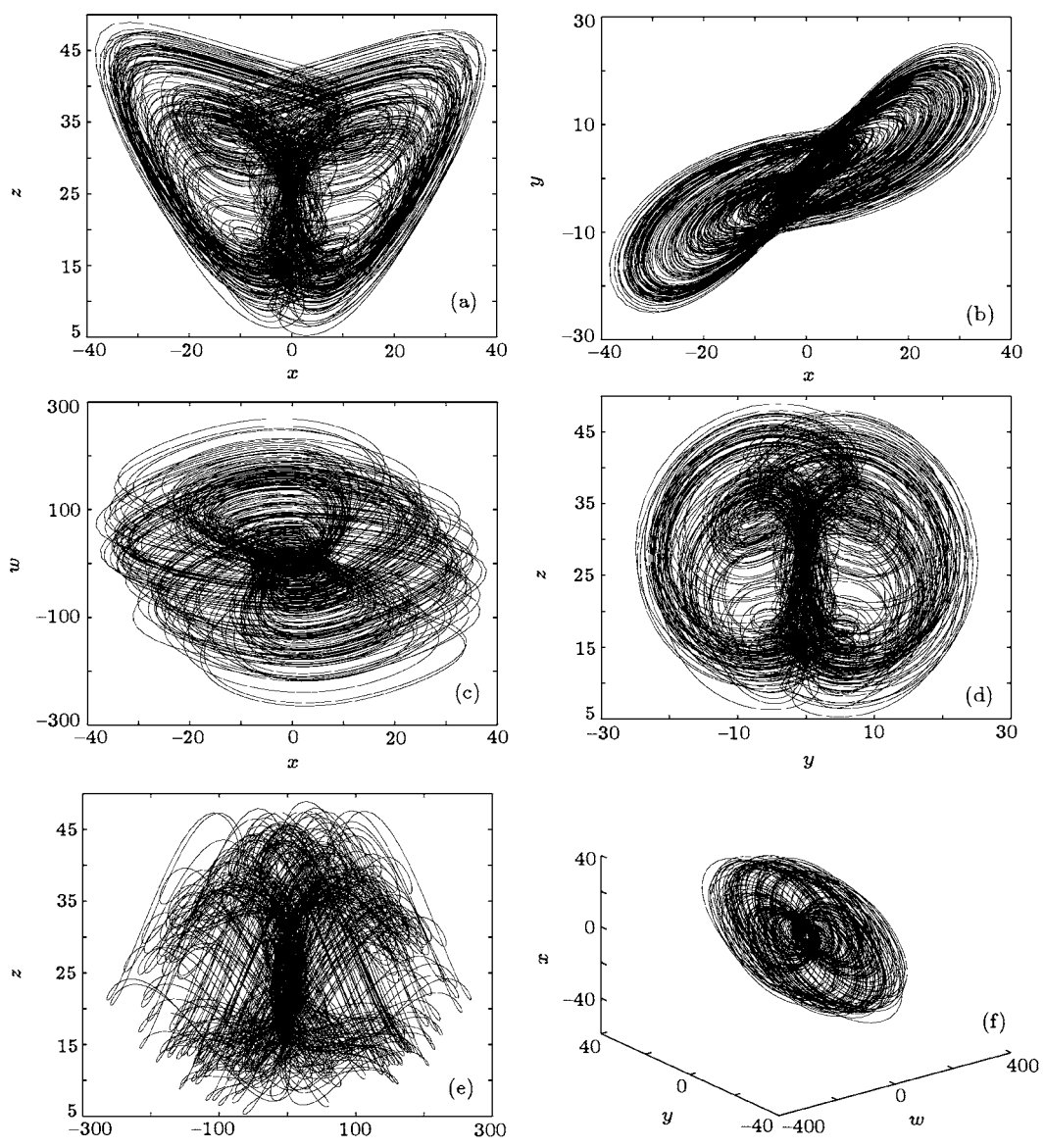


Fig.7. Phase portraits of hyperchaotic attractors of system (2) with $a = 35$, $b = 71.2$. (a) x–z plane; (b) x–y plane; (c) x–w plane; (d) y–z plane; (e) w–z plane; (f) w, y Vs x.

3.2. Bifurcation analysis when varying a

Fixing $b = 3.5$ and varying a, the corresponding Lyapunov-exponent spectrum and bifurcation diagram with $a \in [17.2, 80]$ are shown in Figs.8(a) and (b).

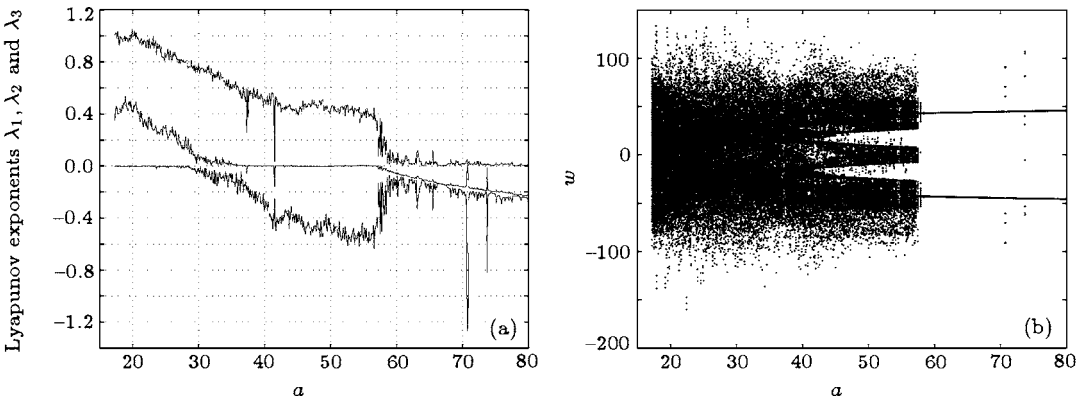


Fig.8. Lyapunov-exponent spectrum and bifurcation diagram versus a with $a \in [17.2, 80]$. (a) Lyapunov-exponent spectrum; (b) Bifurcation diagram of w.

As a increases, system (2) can be hyperchaotic, chaotic or periodic. From the Lyapunov-exponent spectrum, one can find that the two positive Lyapunov exponents are both larger when system (2) is hyperchaotic. The hyperchaotic region is also larger. Many hyperchaotic attractors in this situation are shown in Figs.9 and 10. And the Lyapunov exponents of the typical hyperchaotic systems with different a are shown in Table 2.

Table 2. The Lyapunov exponents of the typical hyperchaotic systems of different a with $b = 3.5$.

a	λ_1	λ_2	λ_3	λ_4
17.4	1.0181	0.4180	0	−21.5008
25	0.8332	0.3131	0	−28.8135

The discussion about periodic and quasi-periodic attractors is similar to that in the case of varying b, which is not mentioned here.

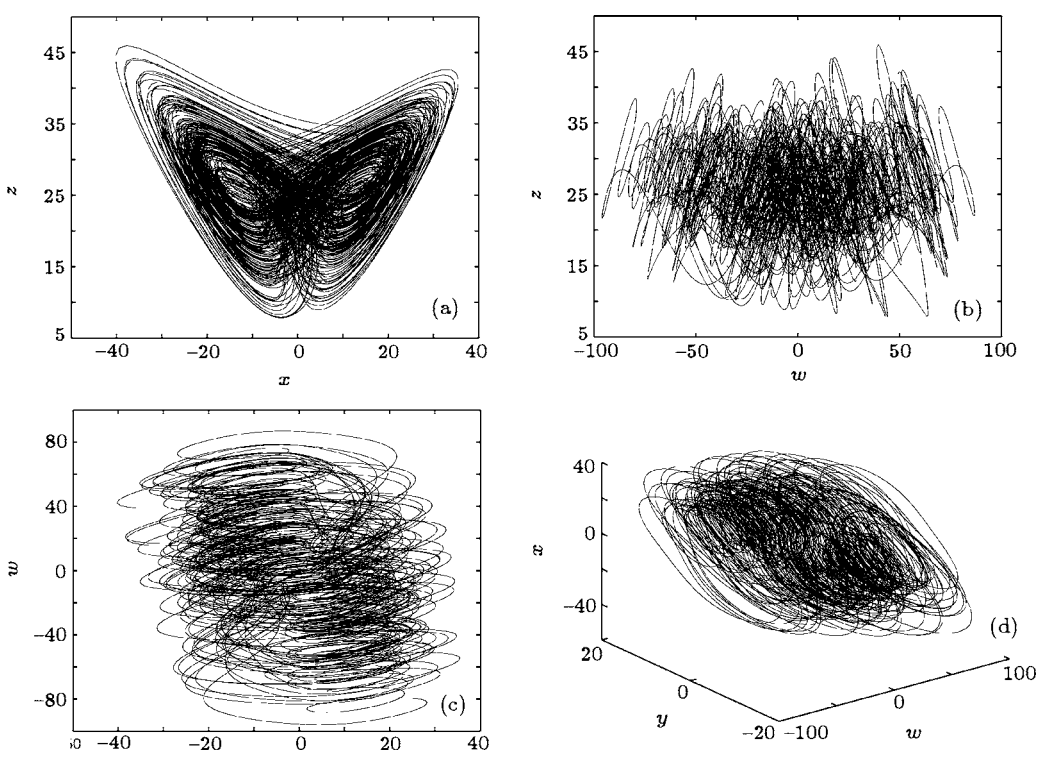


Fig.9. Phase portraits of hyperchaotic attractors of system (2) with $a = 17.4$ and $b = 3.5$. (a) x–z plane; (b) w–z plane; (c) x–w plane; (d) w, y Vs x.

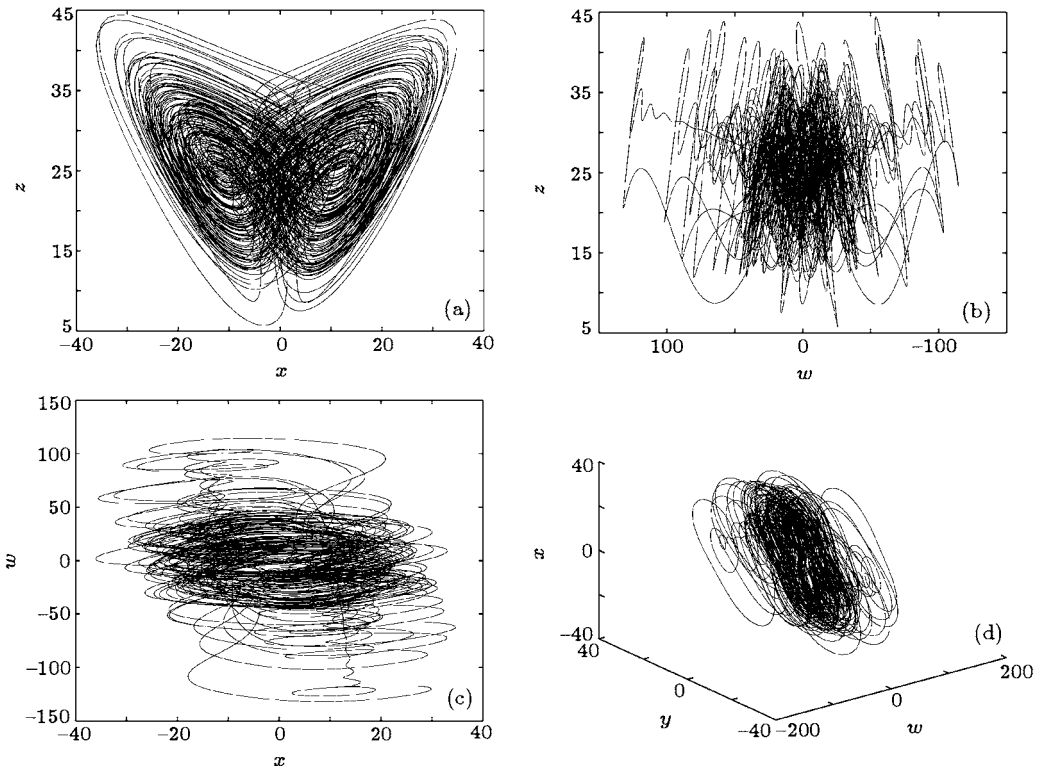


Fig.10. Phase portraits of hyperchaotic attractors of system (2) with $a = 25$ and $b = 3.5$. (a) x–z plane; (b) w–z plane; (c) x–w plane; (d) w, y Vs x.

It is clearly seen that the hyperchaotic orbits of system (2) in Figs.6 and 7 and Figs.9 and 10 are very different from each other in shape.

When $a = 19.9$ and $b = 3.5$, system (2) is hyperchaotic, which has two positive Lyapunov exponents, $\lambda_1 = 1.0508$ and $\lambda_2 = 0.4163$, the rest are $\lambda_3 = 0$ and $\lambda_4 = -24.0338$. The Lyapunov dimension of system (2) is $D_L = 3.061$. With these parameters, the three equilibria of system (2) are $s_0 = (0, 0, 0, 0)$, $s_1 = (-15.84, -5.80, 34.44, 155.29)$ and $s_2 = (15.84, 5.80, 34.44, -155.29)$. Some Poincaré sections are shown in Fig.11.

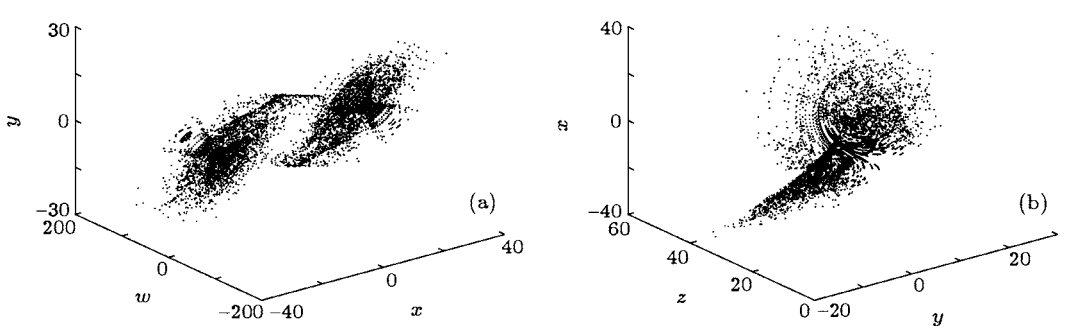


Fig.11. Poincaré sections of the hyperchaotic attractor of system (2) with $a = 19.9$ and $b = 3.5$ at various positions in the three-dimensional phase space. (a) z=34; (b) w=0.

4. Conclusions

This paper briefly reports a new hyperchaotic system by introducing a new state to a chaotic system. It has two parameters and each equation of the system has one quadratic cross-product term. Some basic properties of it are analysed, such as the symmetry, dissipation, equilibria and the stability of the zero equilibrium point. The complex dynamic behaviours of the system are shown when the system parameter a or b is varied. From the two Lyapunov-exponent spectrum diagrams, we can notice that when the system is hyperchaotic, the largest Lyapunov exponent in the case of varying a is larger than that in the case of

varying b. The system can be periodic, quasi-periodic, chaotic or hyperchaotic with b varying. Especially, it is hyperchaotic in several different regions of the parameter b.

It has more complex and richer dynamic behaviours, some larger hyperchaotic regions and larger positive Lyapunov exponents. The new hyperchaotic system will be very useful for secure communication.

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