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Analysing chaos in fractional-order systems with the harmonic balance method

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In this paper, the fractional-order Genesio–Tesi system showing chaotic behaviours is introduced, and the corresponding one in an integer-order form is studied intensively. Based on the harmonic balance principle, which is widely used in the frequency analysis of nonlinear control systems, a theoretical approach is used to investigate the conditions of system parameters under which this fractional-order system can give rise to a chaotic attractor. Finally, the numerical simulation is used to verify the validity of the theoretical results.

Keywords: fractional-order system, harmonic balance principle, Genesio–Tesi system, chaos, Lur'e system

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1. Introduction

The dynamics of fractional-order systems has increasingly attracted attention in recent years. Many physical phenomena, including certain types of electric noise, relaxation behaviours of polarized impedances in dielectrics and interfaces, transmission lines, cardiac rhythm, and spectral densities of music are known to exhibit fractional-order (or called noninteger-order) dynamics.[1,2] The theory of derivatives of fractional-order was presented in Leibniz's note to L'Hopital, dated 30 September 1695, in which the meaning of derivative of order one half was discussed.[3] Although fractional calculus has a 300-years-old history, it has not been applied to physics and engineering long, especially to the study of chaos. It is well known that chaos cannot occur in continuous-time systems with total order less than 3. This assertion is based on the ordinary concepts of integer-order, such as the number of separate differentiations or integrations in a system. But, in fractional-order cases, there are many examples showing new results. For example, in Ref.[4], it was shown that the fractional-order Chua circuit with order as low as 2.7 could produce a chaotic attractor. In Ref.[5], it was indicated that the nonautonomous Duffing system with order less than 2 could still behave in a chaotic manner. In Ref.[6], chaotic behaviours of the fractional-order 'jerk' model were studied, in which a chaotic attractor could be generated with order as low as 2.1. In Ref.[7], chaos and hyperchaos in fractional-order Rössler systems were discussed, in which it was shown that chaos could exist in the fractional-order Rössler system with order as low as 2.4, and hyperchaos could also exist in the fractional-order Rössler hyperchaotic system with order as low as 3.8. More recently, in Ref.[8], Gao *et al* have found that chaos exists in a fractional-order electronic oscillator with an order being less than 3. Moreover, in Ref.[9], Lu *et al* have found that chaos exists in a fractional-order Chen system with order as low as 0.3, which is much smaller than the results reported in Refs.[10,11].

In this paper, we use a heuristic approach, which has been used to analyse the integer-order dynamical systems successfully,[12,13] to approximately determine the system parameters under which the fractional-order system can produce a chaotic attractor. The main idea is to use the well-known harmonic balance principle (HBP) as an operating tool that is especially familiar in the area of system and control theories for detecting nonlinear oscillations by means of describing function method.[14] Here, the fractional-order Generio–Tesi system with 2.7 order is analysed as an example to illustrate the analysis process. In

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the end, numerical simulations are used to verify the theoretical results.

2. Fractional derivative and its approximation

In what follows, we introduce the definition of fractional derivative in brief. Fractional calculus is a generalization of integration and differentiation to a noninteger-order integro-differential operator ${}_aD_t^\alpha$ defined by

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}, & R(\alpha) > 0, \\ 1, & R(\alpha) = 0, \\ \int_a^t (\mathrm{d}\tau)^{-\alpha}, & R(\alpha) < 0. \end{cases} \tag{1} $$

Two commonly used definitions for the general fractional differential-integral are Riemann–Liouville definition and Grunward–Letnikov definition.[3] In the present study, only the former is used, and its definition is given as follows:

$$ \frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau, \tag{2} $$

where $n = [\alpha]$, i.e. n is the first integer that is not less than α, and $\Gamma(\bullet)$ is the Gamma function. The geometric and the physical interpretations of the fractional derivatives are given in Ref.[15].

The Laplace transform of the Riemann–Liouville fractional derivative (2) is

$$ \int_0^\infty {}_0D_t^\alpha f(t) \mathrm{e}^{-st} \mathrm{d}t = s^\alpha \mathcal{L}\{f(t)\} - \sum_{k=0}^{n-1} s^k \, {}_0D_t^{\alpha-k-1} f(t)|_{t=0}, \tag{3} $$

for $n-1 \le \alpha < n$, where $s = \mathrm{j}\omega$ denotes the Laplace operator. Considering all the initial conditions to be zero, Eq.(3) can be reduced to

$$ \mathcal{L}\left\{ \frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} \right\} = s^\alpha \mathcal{L}\{f(t)\}. \tag{4} $$

Therefore, the fractional integral operator of order 'α' can be represented by the transfer function $F(s) = 1/s^\alpha$ in the frequency domain. In the present paper, the associative law and commutative law of the fractional-order derivative will be used. They can be expressed as

$$ \frac{\mathrm{d}^\beta}{\mathrm{d}t^\beta}({}_aD_t^\alpha f(t)) = {}_aD_t^\alpha \left(\frac{\mathrm{d}^\beta f(t)}{\mathrm{d}t^\beta} \right) = {}_aD_t^{\alpha+\beta} f(t). \tag{5} $$

The standard definitions of fractional differintegral do not allow direct implementation of fractional operators in time-domain simulations. An efficient method to solve this problem is to approximate fractional operators by using standard integer order operators. In Ref.[1], an effective algorithm was developed to approximate fractional order transfer functions. Basically, the idea is to approximate the system behaviour in a frequency domain. By utilizing frequency domain techniques based on Bode diagrams, one can obtain a linear approximation of the fractional order integrator, the order of which depends on the desired bandwidth and discrepancy between the actual and the approximate magnitude Bode diagrams. This approximate approach has been adopted in Refs.[7–9]. In Table 1 of Ref.[4], approximations for $1/s^\alpha$ with $\alpha = 0.1 - 0.9$ in steps 0.1 were given, with errors of approximately 2 dB. We mainly use these approximations in the following simulations.

3. Fractional-order Lur'e system

In Refs.[12,13], the Generio–Tesi system, which is described by

$$ \begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_2, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_3, \\ \frac{\mathrm{d}x_3}{\mathrm{d}t} &= -\beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3 + \beta_4 x_1^2, \end{aligned} \tag{6} $$

has been studied intensively. When the system parameters $\beta_1, \beta_2, \beta_3$ and β_4 satisfy some conditions, its behaviour will be in a chaotic manner. In the previous work,[16] we found that the lowest order for chaos to appear is 2.4 in the fractional-order Generio–Tesi system. It is described by

$$ \begin{aligned} \frac{\mathrm{d}^{\alpha_1} x_1}{\mathrm{d}t^{\alpha_1}} &= x_2, \\ \frac{\mathrm{d}^{\alpha_2} x_2}{\mathrm{d}t^{\alpha_2}} &= x_3, \\ \frac{\mathrm{d}^{\alpha_3} x_3}{\mathrm{d}t^{\alpha_3}} &= -\beta_1 x_1 - \beta_2 x_2 - \beta_3 x_3 + \beta_4 x_1^2, \end{aligned} \tag{7} $$

where $0 < \alpha_1, \alpha_2, \alpha_3 < 1$. By applying the associative law and commutative law of the fractional-order derivative, it is easy to obtain the input–output model

$$ y^{(\alpha_1+\alpha_2+\alpha_3)} + \beta_3 y^{(\alpha_1+\alpha_2)} + \beta_2 y^{(\alpha_1)} + \beta_1 y = \beta_4 y^2 \tag{8} $$

with the output variable $y = x_1$. Then, we can define a fractional-order Lur'e system that is similar to the integer-order one,[17] as shown in Fig. 1.

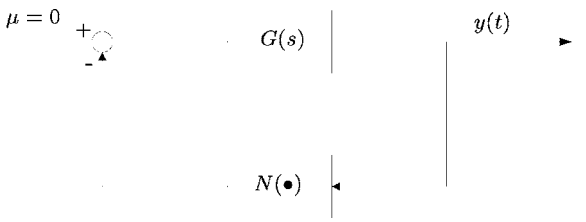


Fig.1. Fractional-order Lur'e system

Assuming all the initial conditions to be zero and the system to be unforced, i.e. $u = 0$, we can acquire the linear subsystem represented by the fractional-order transfer function

$$G(s) = \frac{1}{s^{(\alpha_1+\alpha_2+\alpha_3)} + \beta_3 s^{(\alpha_1+\alpha_2)} + \beta_2 s^{(\alpha_1)} + \beta_1}, \quad (9)$$

and the nonlinear one described by a single-value real function

$$N(y) = -\beta_4 y^2. \quad (10)$$

To make calculation easy, we let $\alpha_1 = \alpha_2 = \alpha_3 = 0.9$ and only the system parameter β_1 be variable, the other system parameters are fixed to be $\beta_2 = 1, \beta_3 = \frac{\sqrt{2}}{4}$ and $\beta_4 = 1$.

4. Chaos prediction in fractional-order Lur'e system

In this section, based on HBP, we determine the range of parameter β_1 to chaotify the fractional-order Lur'e system. In a lot of literature, the harmonic balance theory has been used to detect conditions for the onset of bifurcation phenomena in nonlinear circuits and systems.[13,18] By making a suitable filtering hypotheses (i.e. along the loop the subsystems $G(s)$ and $N(\bullet)$ produce a strong attenuation of the components at frequencies that are multiple fundamental frequency), a predicted limit cycle (PLC) will be detected in the simple Lur'e system. Then, chaos will occur when weak conditions for the existence of a limit cycle hold and a perturbing equilibrium point (EP) is present.[12] Actually, the chaotic behaviour can be considered as a kind of noisy limit cycle (more complex behaviours can be explained as coupling between two or among more of these situations).

Based on a first harmonic study, assume that the system output is

$$y_0(t) = A + B\sin\omega t, \quad B, \omega > 0, \quad (11)$$

and that the corresponding output of the nonlinearity $N(y_0(t))$ is expanded in Fourier series as

$$N(y_0(t)) = N_0(A,B)A + N_1(A,B)B\sin(\omega t) + \cdots. \quad (12)$$

The nonlinear system $N(y_0(t))$ is characterized, in an approximate form related to the periodic regime, by the bias and ω frequency real gains

$$N_0(A,B) = \frac{1}{2\pi A}\int_{-\pi}^{\pi} N(y_0(t))\mathrm{d}\omega t, \quad (13)$$

$$N_1(A,B) = \frac{1}{\pi B}\int_{-\pi}^{\pi} N(y_0(t))\sin\omega t\mathrm{d}\omega t, \quad (14)$$

which are the well-known describing function terms.[17]

The predicted existence of chaos can be synthesized, with the following conditions satisfied:[12] (i) existence of a PLC: $y = y_0(t)$; (ii) existence of a separate EP: $y = E$; (iii) stability properties of PLC and EP: $y_0(t)$ being stable and not generated by the equilibrium point in (ii), E having at least one unstable real eigenvalue; (iv) interaction between PLC and EP: $y_0(t) = E$ for some t; (v) suitable filtering effect along the system. So, the following five steps must be performed.

4.1. Limit cycle detection

In order to establish the conditions under which a stable limit cycle exists, two equations of the HBP must be satisfied:[12]

$$A[1 + G(0)N_0(A,B,\omega)] = 0, \quad (15)$$

$$1 + G(\mathrm{j}\omega)N_1(A,B,\omega) = 0. \quad (16)$$

The solutions of Eqs.(13) and (14) with Eq.(10) are described as

$$N_0(A,B) = \frac{1}{2\pi A}\int_{-\pi}^{\pi} -(A + B\sin(\omega t))^2\mathrm{d}\omega t = -A - \frac{B^2}{2A}, \quad (17)$$

$$N_1(A,B) = \frac{1}{\pi B}\int_{-\pi}^{\pi} -(A + B\sin(\omega t))^2\sin\omega t\mathrm{d}\omega t = -2A. \quad (18)$$

Inserting Eqs.(17) and (18) into Eqs.(15) and (16) respectively, we can obtain

$$A + \frac{B^2}{2A} = \beta_1,$$

$$B = \sqrt{2(\beta_1 - A)A} \quad (\text{supposing } B > 0), \tag{19}$$

and

$$\begin{aligned} 2A =& (\mathrm{j}\omega)^{2.7} + \frac{\sqrt{2}}{4}(\mathrm{j}\omega)^{1.8} + (\mathrm{j}\omega)^{0.9} + \beta_1 \\ =& (\omega)^{2.7}\sqrt[10]{-\mathrm{j}} + \frac{\sqrt{2}}{4}(\omega)^{1.8}\sqrt[10]{-1} \\ &+ (\omega)^{0.9}\sqrt[10]{\mathrm{j}} + \beta_1. \end{aligned} \tag{20}$$

Because $\sqrt[10]{-\mathrm{j}} = \pm\frac{\sqrt{2}}{2}(1-\mathrm{j})$, $\sqrt[10]{-1} = \pm\mathrm{j}$ and $\sqrt[10]{\mathrm{j}} = \pm\frac{\sqrt{2}}{2}(1+\mathrm{j})$, the Eqs.(20) can be in eight cases

$$\begin{aligned} 2A =& \pm\frac{\sqrt{2}}{2}\omega^{2.7}(1-\mathrm{j}) \pm \frac{\sqrt{2}}{4}\omega^{1.8}\mathrm{j} \\ &\pm \frac{\sqrt{2}}{2}\omega^{0.9}(1+j) + \beta_1. \end{aligned} \tag{21}$$

Use the complex principle, i.e. let the real and imaginary parts be equal to zero respectively, then the four equations of imaginary part will be independent. Their solutions can be expressed as follows:

$$\begin{aligned} &\frac{\sqrt{2}}{2}\omega^{2.7} - \frac{\sqrt{2}}{4}\omega^{1.8} - \frac{\sqrt{2}}{2}\omega^{0.9} = 0, \\ &\omega^{0.9}(2\omega^{1.8} - \omega^{0.9} - 2) = 0, \\ &\omega = 0,\ \omega = 1.3, \end{aligned} \tag{22}$$

$$\begin{aligned} &\frac{\sqrt{2}}{2}\omega^{2.7} - \frac{\sqrt{2}}{4}\omega^{1.8} + \frac{\sqrt{2}}{2}\omega^{0.9} = 0, \\ &\omega^{0.9}(2\omega^{1.8} - \omega^{0.9} + 2) = 0, \\ &\omega = 0,\ \omega = 0.11 \pm 0.99\mathrm{j}, \end{aligned} \tag{23}$$

$$\begin{aligned} &\frac{\sqrt{2}}{2}\omega^{2.7} + \frac{\sqrt{2}}{4}\omega^{1.8} - \frac{\sqrt{2}}{2}\omega^{0.9} = 0, \\ &\omega^{0.9}(2\omega^{1.8} + \omega^{0.9} - 2) = 0, \\ &\omega = 0,\ \omega = 0.77, \end{aligned} \tag{24}$$

and

$$\begin{aligned} &\frac{\sqrt{2}}{2}\omega^{2.7} + \frac{\sqrt{2}}{4}\omega^{1.8} + \frac{\sqrt{2}}{2}\omega^{0.9} = 0, \\ &\omega^{0.9}(2\omega^{1.8} + \omega^{0.9} + 2) = 0, \\ &\omega = 0,\ \omega = -0.44 \pm 0.89\mathrm{j}. \end{aligned} \tag{25}$$

Since $\omega \in R^+$, two equations are available. So we have two real part equations corresponding to $\omega = 1.3$ and $\omega = 0.77$ respectively:

$$2A = \frac{\sqrt{2}}{2}\omega^{2.7} + \frac{\sqrt{2}}{2}\omega^{0.9} + \beta_1,$$

$$2A = 2.33 + \beta_1, \quad (\omega = 1.3), \tag{26}$$

$$2A = 0.89 + \beta_1, \quad (\omega = 0.77), \tag{27}$$

and

$$2A = -\frac{\sqrt{2}}{2}\omega^{2.7} - \frac{\sqrt{2}}{2}\omega^{0.9} + \beta_1,$$

$$2A = -2.33 + \beta_1, \quad (\omega = 1.3), \tag{28}$$

$$2A = -0.89 + \beta_1, \quad (\omega = 0.77). \tag{29}$$

Combining the above with Eq.(19) yields

$$B = \sqrt{\frac{1}{2}(\beta_1^2 - 0.89^2)} \quad \text{or} \quad B = \sqrt{\frac{1}{2}(\beta_1^2 - 2.33^2)}.$$

The inequality

$$\beta_1 > 0.89 \quad \text{or} \quad \beta_1 > 2.33 \tag{30}$$

must hold true.

Next, the stability property of the PLC will be investigated. Assume

$$G(\mathrm{j}\omega) = U(\omega) + \mathrm{j}V(\omega), \tag{31}$$

then the Loeb condition for a stable limit cycle can be described as

$$\left(\frac{\partial V}{\partial \omega}\frac{\partial(-1/N_1)}{\partial A}\right)\Big|_s < 0, \tag{32}$$

where $|_s$ denotes that the partial derivative is evaluated at the solution (A, B, ω). Since the $N_1(A, B, \omega)$ is real and $\frac{\partial(-1/N_1)}{\partial A} = -\frac{1}{2A^2}$, the Eq.(32) can be reduced further to

$$\frac{\partial V}{\partial \omega}\Big|_s > 0. \tag{33}$$

Through calculation, if the inequality (30) can be satisfied with $\omega = 1.3$ and $\omega = 0.77$ respectively, we can conclude that the PLC is stable.

4.2. Equilibrium point analysis

According to the theory in Ref.[19], if all of equations (7) have the same number of fractional-order, it can be proven that a given fractional-order system contains all the EPs of the corresponding integer-order system. By HBP, EPs can be obtained by solving the follow equation:

$$y + G(0)N(y) = 0. \tag{34}$$

Using the expressions (9) and (10), i.e. $G(0) = 1/\beta_1$ and $N(y) = -y^2$, we can have EPs $y = 0$ and $y = \beta_1$. In general, when B approaches zero, Eq.(15) expresses the Hopf bifurcation existence[20] and leads to a bias value $A = \beta_1$, with β_1 being the EP where the bifurcation occurs. Such a point can be viewed as a generator of a family of periodic solutions and does not fit for the condition (iii) . Thus, the stability property of the EP with $y = 0$ can be easily checked by studying the Jacobian matrix J of the system at the point $x_1 = y = 0$ as follows:

$$J_{y=0} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\beta_1 & -1 & -\dfrac{\sqrt{2}}{4} \end{bmatrix}. \tag{35}$$

The characteristic equation of J is

$$s^3 + \frac{\sqrt{2}}{4}s^2 + s + \beta_1 = 0. \tag{36}$$

In terms of the Routh–Hurwitz criterion, when $\beta_1 \geq \dfrac{\sqrt{2}}{4}$, the EP with $y = 0$ is an unstable point. Hence, the relation (30) is satisfied well.

4.3. Interaction between the limit cycle and the equilibrium point

Since output of the system is assumed to be the expression (11), it is easy to obtain the condition of interaction between the limit cycle and the equilibrium point as follows:

$$B \geq |E - A|, \tag{37}$$

where the equilibrium point is $E = 0$. Using the expressions (19) and (26) yields

$$\begin{aligned} &\sqrt{\frac{1}{2}(\beta_1^2 - 2.33^2)} \geq \left|\beta_1 - \frac{2.33 + \beta_1}{2}\right|, \\ &\beta_1^2 + 2 \times 2.33\beta_1 - 3 \times 2.33^2 \geq 0, \\ &\beta_1 \leq -6.99 \quad \text{or} \quad \beta_1 \geq 2.33. \end{aligned} \tag{38}$$

Similarly, we can acquire

$$\beta_1 \leq -2.33 \quad \text{or} \quad \beta_1 \geq 6.99 \tag{39}$$

obtained from the expressions (19) and (27),

$$\beta_1 \leq -2.31 \quad \text{or} \quad \beta_1 \geq 0.77 \tag{40}$$

obtained from the expressions (19) and (28),

$$\beta_1 \leq -0.77 \quad \text{or} \quad \beta_1 \geq 2.31 \tag{41}$$

obtained from the expressions (19) and (29).

Combining the above with the result (30) yields the ranges of the parameter β_1 as follows:

$$\beta_1 > 0.89, \quad \beta_1 > 2.31, \quad \beta_1 > 2.33 \quad \text{or} \quad \beta_1 > 6.99. \tag{42}$$

4.4. Filtering characteristics

As to the accuracy of the method, the results obtained in equilibrium are exact, while those concerning limit cycles are approximate, due to the first harmonic analysis being made by using the system dynamics. The reliability of a PLC is based on a strong attenuation (filtering hypothesis) of the higher frequency components $2\omega, 3\omega, \cdots$ along the loop of system, as shown in Fig.1. The amount of the neglected higher harmonics concerning a PLC with $y_0(t)$ of frequency ω can be expressed by a quantity that is so-called distortion index

$$\Delta = \frac{\|\tilde{y}_0(t) - y_0(t)\|_2}{\|y_0(t)\|_2}. \tag{43}$$

Here, the symbol $\|\bullet\|_2$ denotes the L_2 norm of the period $\dfrac{2\pi}{\omega}$, and $\tilde{y}_0(t)$ is the steady-state periodic output of the system that is obtained when the closed loop is broken just before the nonlinear subsystem $N(\bullet)$ and the signal $y_0(t)$ is injected into $N(\bullet)$.[13] But, in the fractional-order system, it is too difficult to calculate the undistorted signal $\tilde{y}_0(t)$. Since there does not exist time delay, we use the $\lim\limits_{t\to\infty} y_0(t)$ and $\lim\limits_{t\to\infty} \tilde{y}_0(t)$ to replace the $y_0(t)$ and $\tilde{y}_0(t)$. According to the final value theorem of Laplace transform, we can have

$$\begin{aligned} \lim_{t\to\infty} \tilde{y}_0(t) &= \lim_{s\to 0} s\mathcal{L}(y_0(t)^2)G(s) \\ &= \lim_{s\to 0} s\left(\frac{A^2}{s} + \frac{2B^2\omega^2}{s^3 + 4s\omega^2} + \frac{2AB\omega}{s^2 + \omega^2}\right)G(s) \\ &= \frac{A^2}{\beta_1} + \frac{B^2}{2\beta_1} \end{aligned} \tag{44}$$

and

$$\begin{aligned} \lim_{t\to\infty} y_0(t) &= \lim_{s\to 0} s\mathcal{L}(y_0(t)) \\ &= \lim_{s\to 0} s\left(\frac{A}{s} + \frac{B\omega}{s^2 + \omega^2}\right) \\ &= A. \end{aligned} \tag{45}$$

Inserting the expressions (44) and (45) into the expression (43), we obtain the values of distortion Δ for different values of parameter β_1, as shown in Table.1.

Table 1. Distortion indices Δ for different values of the parameter β_1.

	$\beta_1 = 0.9$	$\beta_1 = 1.2$	$\beta_1 = 2.4$	$\beta_1 = 2.6$
$\omega = 0.77$	$\Delta = 0.0721702$	$\Delta = 0.0977917$	$\Delta = 0.1161$	$\Delta = 0.13957$
$\omega = 1.3$	none	none	$\Delta = 0.0212587$	$\Delta = 0.0117228$

Small values of Δ indicate that an open loop system is an efficient low-pass filter and that the corresponding PLC is reliable. Indeed, rigorous arguments can be used to guarantee the existence of a true periodic solution in a defined neighbourhood of the predicted one under suitable conditions of Δ.[17]

4.5. Validating ranges of the parameter by using numerical simulation

The HBP is an approximate method of analysing a nonlinear system, but it can just give some instructional results. Now we use numerical simulation to validate ranges of the parameter β_1 that have been given in the previous section. From Fig.2, the trajectories of states, x_1 versus x_2, show dynamic behaviours of the fractional-order Genesio–Tesi system as one period, double period, chaos and instability respectively when $\beta_1 = 0.9, 1.1, 1.31$ and 1.32.

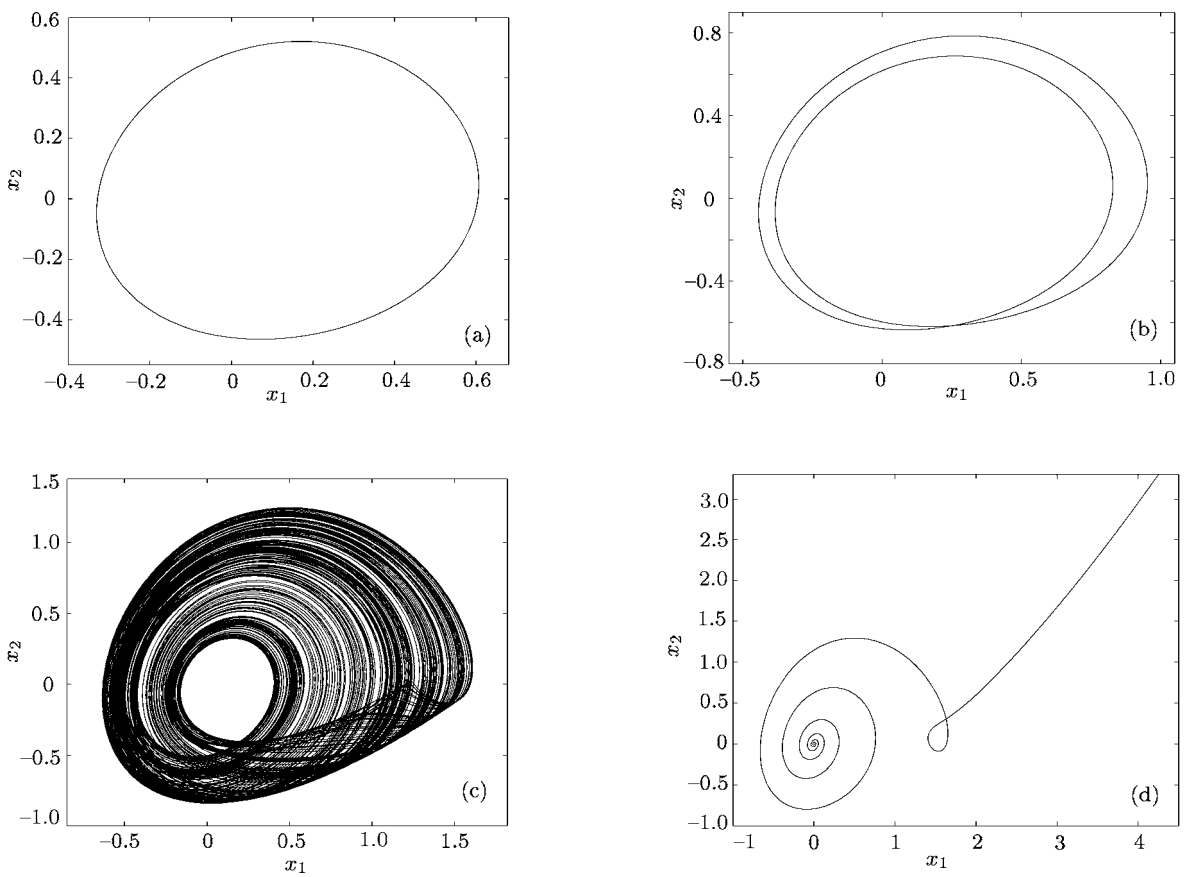


Fig.2. Projection on (x_1, x_2) plane of the trajectories corresponding to the $\beta_1 = 0.9, 1.1, 1.31, 1.32$.

So from the results of numerical simulation, we can conclude that the range of $\beta_1 > 0.89$ is correct answer for the route to chaos in the fractional-order Genesio–Tesi system. And when $\beta_1 > 1.318$, the states of the dynamic system are always unstable.

Remark 1. An interesting thing has been found when we analyse the system by HBP. If we change the value of β_4 in the nonlinear subsystem $N(\bullet)$ such as $\beta_4 = 2$, results (42) will not be affected. Numerical simulation results show that this action changes only the size of the chaotic attractor. When the parameter β_4 is in a range [0.59, 2.5], the simulation results are shown in Fig.3.

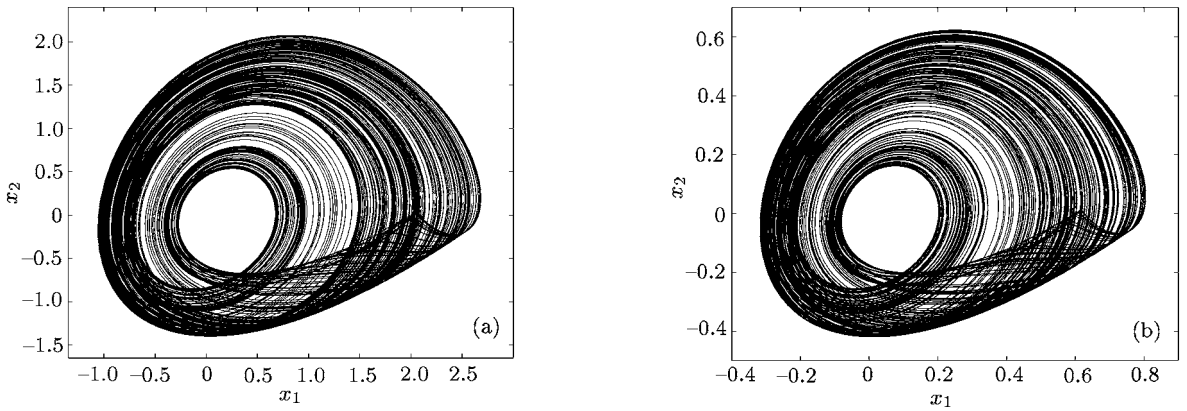


Fig.3. Chaotic attractor sizes for $\beta_4 = 0.6$ and 2.

5. Conclusion

In this letter, the fractional-order Genesio–Tesi system is investigated. By the HBP, we obtain the condition of the parameters under which this fractional-order system can produce a chaotic attractor. Simulation results have illustrated the effectiveness of the proposed method.

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