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Finite-time control of chaotic systems with nonlinear inputs*

Gao Tie-Gang(高铁杠)[a], Chen Zeng-Qiang(陈增强)[c],
Chen Guan-Rong(陈关荣)[b], and Yuan Zhu-Zhi(袁著祉)[c]

[a] *College of Software, Nankai University, Tianjin* 300070, *China*
[b] *Department of Electronic Engineering, City University of Hong Kong,* 83 *Tat Chee Avenue, Kowloon Hong Kong, China*
[c] *Department of Automation, Nankai University, Tianjin* 300070, *China*

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A finite-time controller is designed for a class of nonlinear systems subject to sector nonlinear inputs. A novel and simple approach is suggested based on the finite-time control principle. The designed sliding-mode controller can drive a chaotic system to track a smooth target signal in a finite time. The chaotic Duffing–Holmes oscillator is used for verification and demonstration.

Keywords: Finite-time control, sliding mode, nonlinear input, chaotic system
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1. Introduction

Chaos control and synchronization both have attracted increasing attention in recent years. A chaotic system is a very complex dynamical nonlinear system and possesses some special features such as extreme sensitivity to initial conditions, broadband spectrum of Fourier transform, and bounded trajectories in the phase-space with positive Lyapunov exponents, and so on[1]. Due to its special applications in liquid mixing, chemical reactions, power converters, biological systems, information processing, secure communications, etc, controlling these kinds of complex chaotic dynamics for engineering and technological practice has become a new and attractive direction of research in interdisciplinary fields of engineering, applied physics, applied mathematics, biology and beyond, and many profound new theories and methodologies have been developed to date.

A wide variety of approaches has been proposed to synchronization of chaotic systems, including linear and nonlinear feedback controls,[2−4] time-delay feedback control,[5,6] adaptive control,[7−9] and impulsive control[10,11] etc. Notice that in many engineering systems, due to physical limitations, there often exist nonlinearities in the system input. Their presence usually causes chaotic vibrations if the controller is not well designed. Therefore, their effect cannot be ignored in the design and analysis of controllers, particularly for chaotic systems. Consequently, the problem of controlling chaotic system with input nonlinearity is important as discussed in Ref.[12].

This paper suggests a sliding-mode (or variable structure) control design for a class of nonlinear systems subject to sector nonlinear inputs. To achieve faster convergence in a control system, a finite-time sliding-mode controller is designed and used to guide a chaotic system with nonlinear input to a target trajectory. In general, finite-time stability means the optimality in settling time.[13] Moreover, the finite-time control techniques have demonstrated better robustness and disturbance rejection properties.[14] Recently, Amato *et al*[15] considered finite-time control problems for linear systems subject to time-varying parametric uncertainties and to exogenous constant disturbances, and obtained a sufficient condition for robust finite-time stabilization via state feedback. Hong *et al*[16] studied the problem of finite-time output feedback sta-

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bilization for the double integrator system based on a finite-time separation principle, and obtained the condition of robust finite-time stabilization regardless of some nonlinear perturbations for a large class of nonlinear second-order systems. Li and Tian[17] developed a continuous state feedback control law to solve the synchronization problem of two chaotic systems using finite time control techniques

In this paper, more precisely, a finite-time controller is designed for a class of chaotic systems subject to sector nonlinear inputs. The designed sliding-mode controller can drive a chaotic system to track a smooth target trajectory in a finite time. Finally, the representative chaotic Düffing–Holmes oscillator is used for verification and demonstration.

2. Problem formulation

Consider a nonlinear system with input nonlinearities as follows:

$$\begin{aligned} &\dot{x}_i(t) = x_{i+1}(t), \qquad i = 1, 2, \cdots, n-1, \\ &\dot{x}_n(t) = f(X(t)) + \phi(u(t)), \end{aligned} \tag{1}$$

where

$$\begin{aligned} X(t) =& [x_1(t), x_2(t), \cdots, x_n(t)]^{\mathrm{T}} \\ =& [x_1(t), \dot{x}_1(t), \ldots, x_1^{(n-1)}(t)]^{\mathrm{T}} \in R^n \end{aligned}$$

is the state vector of the system, $f(X(t)) \in R$ is a given nonlinear function of X and t, $u(t) \in R$ is the control input, $\phi(u(t))$ is a continuous nonlinear function, where $\phi : R \to R$ with $u(t) \to \phi(u(t))$ inside the sector $[k_1, k_2]$, i.e

$$k_1 u^2(t) \le \phi(u(t))u(t) \le k_2 u^2(t), \tag{2}$$

with k_1 and k_2 being nonzero positive constants.

Notice that many chaotic systems are in the form of the expression (1), such as the Duffing–Holmes oscillator,[18] the Moore–Spiegel flow equation,[19] and so on.

The control problem is to drive the system output to track a target n-dimensional trajectory, a vector $X_d(t)$, which is in the form of

$$\begin{aligned} X_d(t) =& [x_{d1}(t), x_{d2}(t), \ldots, x_{dn}(t)]^{\mathrm{T}} \\ =& [x_d(t), \dot{x}_d(t), \ldots, x_d^{(n-1)}(t)]^{\mathrm{T}}. \end{aligned}$$

Define the tracking error as follows:

$$\begin{aligned} E(t) =& X(t) - X_d(t) \\ =& [x_1(t) - x_{d1}(t), x_2(t) - x_{d2}(t), \ldots, \\ & x_n(t) - x_{dn}(t)]^{\mathrm{T}} \\ =& [x_1(t) - x_d(t), \dot{x}_1(t) - \dot{x}_d(t), \ldots, \\ & x_1^{(n-1)}(t) - x_d^{(n-1)}(t)]^{\mathrm{T}} \\ =& [e_1(t), e_2(t), \ldots, e_n(t)]^{\mathrm{T}} \\ =& [e_1(t), \dot{e}_1(t), \ldots, e_1^{(n-1)}(t)]^{\mathrm{T}}. \end{aligned} \tag{3}$$

The control goal considered in this paper is that for any given target orbit $X_d(t)$, a sliding-mode controller is designed in spite of the nonlinearity existing in the control input, such that the resulting tracking error vector satisfies

$$\lim_{t\to\infty} \|E(t)\| = \lim_{t\to\infty} \|X(t) - X_d(t)\| \to 0. \tag{4}$$

First, it follows from expression (3) that

$$\begin{aligned} &\dot{e}_i(t) = e_{i+1}(t), \qquad i = 1, 2, \ldots, n-1, \\ &\dot{e}_n(t) = f(X(t)) + \phi(u(t)) - \dot{x}_{\mathrm{d}n}(t). \end{aligned} \tag{5}$$

3. Design of the finite-time controller

Before introducing a new controller, a sliding surface is first constructed.

Lemma 1[20] The fast sliding mode is given by

$$s = \dot{x} + \alpha x + \beta x^{q/p}, \tag{6}$$

where $x \in R$, α and β are positive constants, and q and p are positive odd integers. Then, for any initial value of the state $x(t)$ at $t = 0$, i.e. $x(0)$, the solution of Eq.(6) will reach $x = 0$ in a finite time, t_s, determined by

$$t_s = \frac{p}{\alpha(p-q)} \ln \frac{\alpha x(0)^{(p-q)/p} + \beta}{\beta}. \tag{7}$$

Now, consider the following sliding surface:

$$\begin{aligned} &s_0 = e_1(t), \\ &s_1 = \dot{s}_0 + \alpha_0 s_0 + \beta_0 s_0^{q_0/p_0}, \\ &s_2 = \dot{s}_1 + \alpha_1 s_1 + \beta_1 s_1^{q_1/p_1}, \\ &\qquad \vdots \\ &s_{n-1} = \dot{s}_{n-2} + \alpha_{n-2} s_{n-2} + \beta_{n-2} s_{n-2}^{q_{n-2}/p_{n-2}}, \end{aligned} \tag{8}$$

where α_i, β_i, and $i = 0, 1, \ldots, n-2$, are positive constants, p_i, q_i, and $i = 0, 1, \ldots, n-2$ are positive odd integers, with $p_i > q_i$. It follows from Eq.(6) that $s_{i+1} = \dot{s}_i + \alpha_i s_i + \beta_i s_i^{q_i/p_i}$, and s_i will reach zero in a finite time when s_{i+1} reaches zero according to Lemma 1. Therefore, s_0 will reach zero in a finite time.

Lemma 2 The motion of the sliding mode (8) is asymptotically stable, if the following condition is satisfied:

$$s_{n-1}(t)\dot{s}_{n-1}(t) < 0. \tag{9}$$

Obviously, if one chooses

$$V(s_{n-1}(t)) = \frac{1}{2}s_{n-1}^2(t), \tag{10}$$

then

$$\dot{V}(s_{n-1}(t)) = \dot{s}_{n-1}(t)s_{n-1}(t) < 0. \tag{11}$$

Based on the Layout stability theory, $s_{n-1}(t)$ is globally exponentially stable about zero.

To satisfy the reaching condition indicated in Lemma 1, a scheme is proposed as follows:

$$u(t) = -\gamma\eta\text{sign}(s_{n-1}(t)), \qquad \gamma > \frac{1}{k_1}, \tag{12}$$

where

$$\eta = |f(X(t))| + |\dot{x}_{\text{dn}}(t)| + \sum_{j=0}^{n-2}\alpha_j\left|s_j^{(n-1)}\right| + \sum_{j=0}^{n-2}\beta_j\left|\frac{\text{d}^{n-j-1}}{\text{d}t^{n-j-1}}s_j^{q_j/p_j}\right|.$$

In the following, the proposed scheme (12) will be proved to be able to drive the chaotic system (1) with nonlinear input onto the sliding mode $s_{n-1}(t) = 0$.

Theorem 1 Consider a nonlinear system in the form of expression (1). The motion of the sliding mode (8) is asymptotically stable, if the control $u(t)$ is designed via expression (12).

Proof It follows from expression (8) that

$$\dot{s}_{n-1} = s_0^{(n)} + \sum_{j=0}^{n-2}\alpha_j s_j^{(n-1)} + \sum_{j=0}^{n-2}\beta_j\frac{\text{d}^{n-j-1}}{\text{d}t^{n-j-1}}s_j^{q_j/p_j}. \tag{13}$$

Using expression (13), one has

$$\begin{aligned} s_{n-1}(t)\dot{s}_{n-1}(t) =& s_{n-1}\left(s_0^{(n)} + \sum_{j=0}^{n-2}\alpha_j s_j^{(n-1)} + \sum_{j=0}^{n-2}\beta_j\frac{\text{d}^{n-j-1}}{\text{d}t^{n-j-1}}s_j^{q_j/p_j}\right) \\ =& s_{n-1}(t)\left(e_1^{(n)} + \sum_{j=0}^{n-2}\alpha_j s_j^{(n-1)} + \sum_{j=0}^{n-2}\beta_j\frac{\text{d}^{n-j-1}}{\text{d}t^{n-j-1}}s_j^{q_j/p_j}\right). \end{aligned} \tag{14}$$

Substituting expression (5) into expression (14) gives

$$\begin{aligned} s_{n-1}(t)\dot{s}_{n-1}(t) =& s_{n-1}(t)\left(f(X(t)) + \phi(u(t)) - \dot{x}_{\text{dn}}(t) + \sum_{j=0}^{n-2}\alpha_j s_j^{(n-1)} + \sum_{j=0}^{n-2}\beta_j\frac{\text{d}^{n-j-1}}{\text{d}t^{n-j-1}}s_j^{q_j/p_j}\right) \\ \le& |s_{n-1}(t)|\left(|f(X(t))| + |\dot{x}_{\text{dn}}(t)| + \sum_{j=0}^{n-2}\alpha_j\left|s_j^{(n-1)}\right| + \sum_{j=0}^{n-2}\beta_j\left|\frac{\text{d}^{n-j-1}}{\text{d}t^{n-j-1}}s_j^{q_j/p_j}\right|\right) \\ &+ |\phi(u(t))|\,|s_{n-1}(t)| \\ =& |s_{n-1}(t)|\,\eta + |\phi(u(t))|\,|s_{n-1}(t)|. \end{aligned} \tag{15}$$

Then, from the inequality (2) and expression (12), one obtains

$$\begin{aligned} u(t)\phi(u(t)) =& -\gamma\eta\text{sign}(s_{n-1}(t))\phi(u(t)) \\ \ge& k_1\gamma^2\eta^2(\text{sign}(s_{n-1}(t))^2. \end{aligned}$$

Since $s_{n-1}^2(t) \ge 0$, one has

$$\begin{aligned} &-\gamma\eta\text{sign}(s_{n-1}(t))\phi(u(t))s_{n-1}^2(t) \\ \ge& k_1\gamma^2\eta^2(\text{sign}(s_{n-1}(t))^2 s_{n-1}^2(t), \end{aligned}$$

$$-\gamma\eta s_{n-1}(t)\,|s_{n-1}(t)|\,\phi(u(t)) \ge k_1\gamma^2\eta^2 s_{n-1}^2(t),$$

$$s_{n-1}(t)\phi(u(t)) \le -\gamma\eta k_1\,|s_{n-1}((t)|\,. \tag{16}$$

By substituting expression (16) into Eq.(15), one can obtain

$$s_{n-1}(t)\dot{s}_{n-1}(t) \le (1 - k_1\gamma)\eta\,|s_{n-1}(t)|\,. \tag{17}$$

Since $\gamma > \dfrac{1}{k_1}$ has been selected in expression (12), the reaching condition $s_{n-1}(t)\dot{s}_{n-1}(t) < 0$ is always satisfied. Thus, the proof is completed.

Note that in order to avoid the singularity problem in the sliding modes, the conditions

$$\frac{q_k}{p_k} > \frac{n-k-1}{n-k}, \qquad k = n-2, \ldots, 0 \tag{18}$$

must be satisfied.[15]

Remark In fact, the controller proposed is robust as can be seen from the following discussion: If the system has an external noise perturbation in the form

$$\dot{x}_i(t) = x_{i+1}(t), \qquad i = 1, 2, \cdots, n-1,$$

$$\dot{x}_n(t) = f(X(t)) + \delta(t) + \phi(u(t)), \tag{19}$$

where $\delta(t)$ denotes the external bounded noise perturbation, that is, there exists a positive δ_1 such that

$$|\delta(t)| \leq \delta_1, \tag{20}$$

then, one can choose

$$\eta = |f(X(t))| + |\dot{x}_{dn}(t)| + \sum_{j=0}^{n-2} \alpha_j \left| s_j^{(n-1)} \right| + \sum_{j=0}^{n-2} \beta_j \left| \frac{\mathrm{d}^{n-j-1}}{\mathrm{d}t^{n-j-1}} s_j^{q_j/p_j} \right| + \delta_1. \tag{21}$$

In the same way, one can prove that the controller (12) is also a finite-time controller that can drive the system, with the external perturbation in the form of the expression (19), to track a target smooth signal in a finite time.

4. Control Simulation and Results

Example 1 The representative chaotic Düffing–Holmes oscillator is described by

$$\dot{x}_1(t) = x_2(t),$$

$$\dot{x}_2(t) = -p_1 x_1(t) - p_2 x_2(t) - x_1^3(t) + q\cos(w_1 t). \tag{22}$$

With parameter values of $p_1 = -1, p_2 = 0.25$, $q = 0.3$ and $w_1 = 1.0$, this system displays chaotic behaviour as shown in Fig. 1.

The control objective is to drive the system (22) to track the target signal $x_d(t) = \sin(t)$. Suppose that the nonlinear input is

$$\phi(u(t)) = (0.8 + 0.2\sin(u(t)))u(t). \tag{23}$$

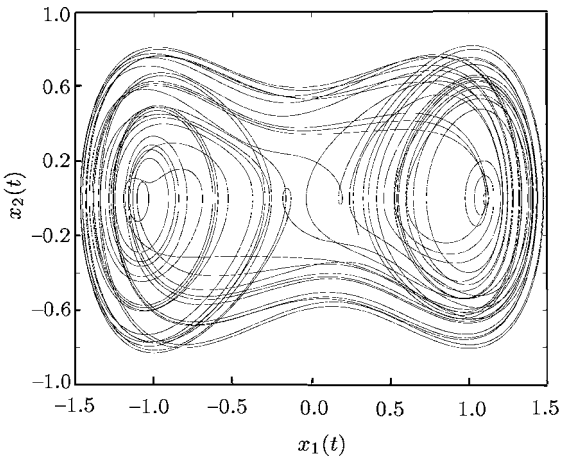


Fig.1. Chaotic attractor of the system (25).

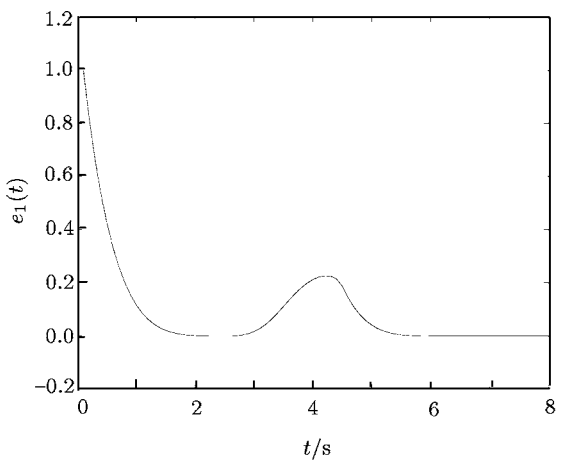


Fig.2. Time response of $e_1(t)$.

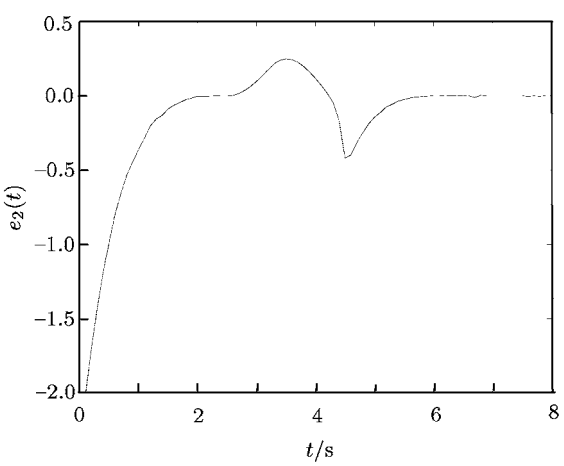


Fig.3. Time response of $e_2(t)$.

One can easily obtain $k_2 = 1$, and $k_1 = 0.6$. So, from the inequality (18), we can define a sliding surface as follows:

$$s_0 = e_1(t),$$

$$s_1 = \dot{s}_0 + s_0 + s_0^{3/5}. \tag{24}$$

From Theorem 1, one may select $\gamma = 2$, so the controller can be defined as

$$u(t) = -2\eta \text{sign}(s_1(t)), \tag{25}$$

with

$$\eta = |f(X(t))| + |\sin(t)| + |e_2(t)| + \frac{3}{5}\left|e_2(t)\frac{1}{\sqrt[5]{e_1(t)^2}}\right|. \tag{26}$$

The simulation results with initial values $x_1 = 1$ and $x_2 = -1$ are shown in Figs.2 and 3. Figure 2 shows the time response of $e_1(t)$, while Fig.3 shows the time response of $e_2(t)$. It can be seen that the system (21) tracks the target signal $x_d(t) = \sin(t)$ at the time 1.5s accurately.

Example 2 Where the chaotic Düffing–Holmes oscillator has an external bounded noise perturbation, such as

$$\dot{x}_1(t) = x_2(t),$$

$$\dot{x}_2(t) = -p_1x_1(t) - p_2x_2(t) - x_1^3(t) + q\cos(w_1t) + 1.2\sin(t). \tag{27}$$

We can define controller as the same as that before except that the equality (26) is replaced by

$$\eta = |f(X(t)| + |\sin(t)| + |e_2(t)| + \frac{3}{5}\left|e_2(t)\frac{1}{\sqrt[5/2]{e_1(t)}}\right| + 1.2 \tag{28}$$

where constant 1.2 is the biggest number of the perturbation. Time responses of $e_1(t)$ and $e_2(t)$ are shown in Figs.4 and 5.

Remark In fact, if an external perturbation of chaotic system is unknown but is bounded, the control is also effective. For example, if M is the upper bound of perturbation, then η in the example 2 can be given by

$$\eta = |f(X(t)| + |\sin(t)| + |e_2(t)| + \frac{3}{5}\left|e_2(t)\frac{1}{\sqrt[5/2]{e_1(t)}}\right| + M \tag{29}$$

and even if the parameters of chaotic systems are uncertain, as the states of chaotic system are bounded, without loss of generality, we assume that $|f(X(t))| \leq U$, then η in example 2 can be given by

$$\eta = U + |\sin(t)| + |e_2(t)| + \frac{3}{5}\left|e_2(t)\frac{1}{\sqrt[5/2]{e_1(t)}}\right| + M. \tag{30}$$

Figures 6 and 7 respectively show the time responses of $e_1(t)$ and $e_2(t)$ when the external perturbation is unknown and $M = 3$ under the assumption of chaotic system (27) tracking $x_d(t) = \sin(t)$.

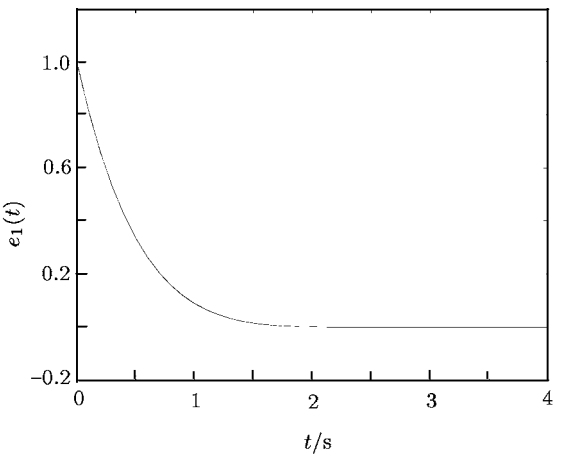


Fig.4. Time response of $e_1(t)$, with the perturbation added.

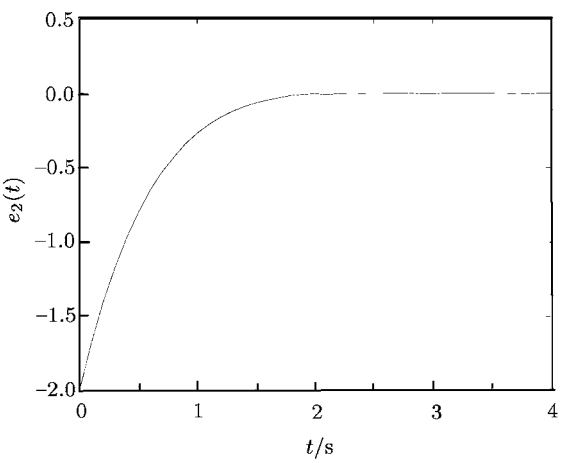


Fig.5. Time response of $e_2(t)$, with the perturbation added.

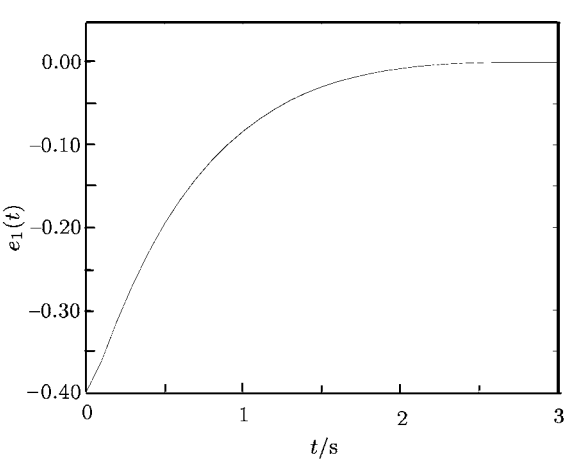


Fig.6. Time response of $e_1(t)$.

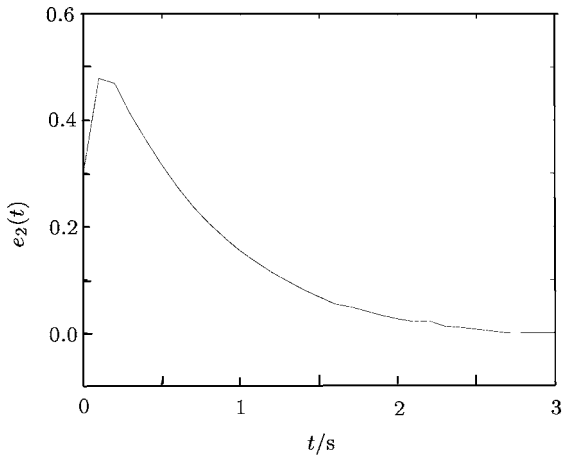


Fig.7. Time response of $e_2(t)$.

5. Conclusions

In this paper, a finite-time control design method has been proposed for a class of nonlinear systems with sector nonlinearities. This finite-time sliding-mode controller has been applied to the representative chaotic Düffing–Holmes oscillator for verification and demonstration. Based on the rigorous mathematical analysis and the excellent simulation results, it can be concluded that the proposed finite-time sliding-mode controller design approach has great potential for controlling chaotic systems even with input nonlinearities. A side benefit is that it also retains the same insensitivity to external disturbances as in the case of the system having no input nonlinearities.

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