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Interaction of a submerged horseshoe vortex with a free surface*

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In this paper a submerged horseshoe vortex under a free surface is discussed and the algebraic expression of the wave elevation is obtained. From this expression, some characteristics of the ship wave are described. There exists a smooth region near $\theta = 0°$, but when the uniform stream passes the other singularities (source, sink, doublet, etc.) there is no smooth region. The mechanism of synthetic aperture radar (SAR) images of the narrow ship wakes is also explained.

Keywords: free surface, horseshoe vortex, ship wakes, SAR
PACC: 0340G, 0340K

1. Introduction

As is well known, when a ship moves on the water, there are steady V-shaped waves behind it. The wake with an aperture angle of $38°56'$ is usually called the Kelvin ship wave. The Kelvin wave system is composed of diverging waves and transverse waves. Diverging waves and transverse waves intersect each other at two Kelvin arms, where the undulation is especially obvious. Synthetic aperture radar (SAR) images of the sea surface can show the Kelvin wake which has been investigated widely.[1−6] Many scholars have studied ship waves regarding the water as an inviscid fluid[7−11] or as a viscous fluid.[12−13] The singularity of ship waves exists at the two sides ($\theta = 19°28'$, see Fig.1) and the centreline ($\theta = 0°$). We resolved the former in Ref.[14] and address the latter in this paper.

However, there exists a turbulent wake near the centreline region of a narrow angle, and more SAR images of narrow V-shaped ship waves than that of the Kelvin wakes are obtained. Moreover, they are longer than the Kelvin wakes. In good weather the far wake as a region of smooth water as far as 100 km behind the ship can be seen with conventional high altitude photographs. Reed[2] thought that it comes from Bragg scattering by short diverging Kelvin waves near the centreline of the ship wake. But he also thought there is controversy as to where these narrow V-images come from. As is well known, the water will flow around the bottom of the ship when a ship moves, and the two parallel vortex lines are formed. In addition, a pair of propellers can make the two parallel vortex lines. So a ship with an unpropelled hull generally generates two pairs of longitudinal counter-rotating vortices, referred to as the bilge vortex and the side vortex pairs.[15] For the propeller hull the rotating direction will decide whether the swirl from the propeller will add to or be subtracted from the rotation of the bilge vortex.[16] The vortex wakes are little studied in theory. Results drawn from the analysis of many images are that wake manifestations seem to cover the entire range of half-angles between $1°$ and $19°28'$ (full Kelvin-wave system in classical deep water). [2] In this paper the effect of vortex wakes on the free surface will be studied.

This paper makes reference to the lifting theory of the submerged hydrofoil of finite span and regards the above two vortex lines as the two free vortex lines of the hydrofoil. Although there may not exist a bound vortex at the bottom of the ship we assume that the free vortex is connected with the bound vortex for the

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satisfaction of Helmholtz's vortex theory. In this way the problem we consider is the interaction between a submerged horseshoe vortex and a free surface far from the ship. So the assumption of the bound vortex does not greatly affect the problem we consider.

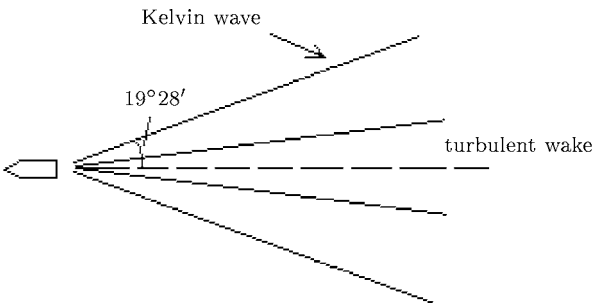


Fig.1. The sketch map of the ship waves.

The problem of the submerged hydrofoil of finite span was studied by Nishiyama[17] and Wu.[18] Nishiyama obtained the expression of the velocity potential of the submerged horseshoe vortex and computed the resistance of the hydrofoil. But we think the boundary condition in his paper may be in question. So we re-deduce the boundary condition and the expression of the velocity potential we obtain is slightly different from that of Nishiyama.[16] What we obtain is the theoretic result. Wu[18] also studied the submerged hydrofoil of finite span but his expression of wave elevation did not include the variable in the direction of span. In fact, what he obtained is the result of an infinite straight line vortex. We derive the theoretic expression of wave elevation of the submerged horseshoe vortex for the first time as far as we know by employing the Lighthill's two-step method in this paper. Furthermore, the graph of the wave elevation is given and the two main results are obtained which are of military significance and practical application.

2. Submerged horseshoe vortex model

We take the origin of coordinate system o on the ship with the ox-axis in the reverse direction of advance, and oz-axis vertically upward, with $z = 0$ representing the undisturbed free surface, as shown in Fig.2. Then in this system the flow picture would appear to be stationary with a uniform stream velocity V approaching the ship. The bound vortex lies between $(0, -b/2, -f)$ and $(0, b/2, -f)$ and the free vortex lines along the ox-axis. We can assume that the horseshoe vortex is rigid and the circulation strength is Γ. It should be pointed out that Γ is a complicated function of y due to the existence of the free surface. But for the sake of qualitative analysis and simplification of the problem, we can assume Γ is a constant.

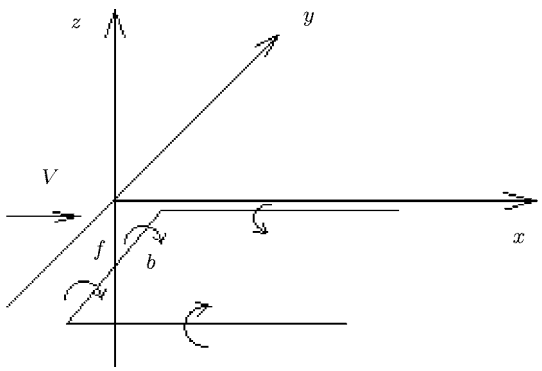


Fig.2. The coordinate system.

Let the water be inviscid. The perturbation velocity potential is Φ and the perturbation velocity $(\frac{\partial \Phi}{\partial x}, \frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial z})$. The total velocity field is

$$u = V + \frac{\partial \Phi}{\partial x}, \quad v = \frac{\partial \Phi}{\partial y}, \quad w = \frac{\partial \Phi}{\partial z}. \tag{1}$$

We introduce an artificial viscous term into the Euler equation. Then the formula can be expressed as follows:

$$\frac{D\boldsymbol{v}}{Dt} = -\frac{1}{\rho}\nabla p - g\nabla y - \mu\nabla\Phi = \nabla\left(-\frac{p}{\rho} - gy - \mu\Phi\right). \tag{2}$$

From Eq.(2) the modified Lagrange formula can be obtained

$$\frac{\partial \Phi}{\partial t} + \frac{\boldsymbol{v}\nabla\boldsymbol{v}}{2} + \frac{p}{\rho} + gy + \mu\Phi = c(t). \tag{3}$$

Linearizing Eq.(3), we have

$$\frac{\partial \Phi}{\partial t} + V\frac{\partial \Phi}{\partial x} + \frac{p}{\rho} + gy + \mu\Phi = c(t). \tag{4}$$

Considering the steady case with $p = 0$ at the free surface we obtain

$$V\frac{\partial \Phi}{\partial x} + g\zeta + \mu\Phi = 0. \tag{5}$$

That is

$$\zeta = -\frac{1}{g}\left(V\frac{\partial \Phi}{\partial x} + \mu\Phi\right). \tag{6}$$

Combining Eq.(6) with the kinematical condition at the free surface

$$-\frac{\partial \Phi}{\partial z} + V\frac{\partial \zeta}{\partial x} = 0, \tag{7}$$

we obtain

$$-\frac{\partial \Phi}{\partial z} = \frac{V}{g}\left(V\frac{\partial^2 \Phi}{\partial x^2} + \mu\frac{\partial \Phi}{\partial x}\right). \tag{8}$$

That is

$$\frac{\partial^2 \Phi}{\partial x^2} + k_0 \frac{\partial \Phi}{\partial z} + \frac{\mu}{V}\frac{\partial \Phi}{\partial x} = 0\,, \tag{9}$$

where $k_0 = g/V^2$. This formula is slightly different from that of Nishiyama[16] whose sign of the viscous term is negative. But the result is equal because we have $\mu \to 0$ finally. According to Eq.(9) and using Nishiyama's method, we can obtain

$$\begin{aligned}\Phi = -\frac{\mathrm{i}}{8\pi^2}\Big[&\int_{-b/2}^{b/2} \Gamma \mathrm{d}\,\eta \int_{-\pi}^{\pi} \sec\beta \mathrm{d}\,\beta \int_0^{\infty} \mathrm{e}^{-k(z+f)+\mathrm{i}k\omega}\mathrm{d}\,k \\ &- \int_{-b/2}^{b/2} \Gamma \mathrm{d}\,\eta \int_{-\pi}^{\pi} \sec\beta \mathrm{d}\,\beta \int_0^{\infty} \mathrm{e}^{-k(-z+f)+\mathrm{i}k\omega}\mathrm{d}\,k \\ &- 2k_0 \int_{-b/2}^{b/2} \Gamma \mathrm{d}\,\eta \int_{-\pi}^{\pi} \sec^3\beta \mathrm{d}\,\beta \int_0^{\infty} \frac{\mathrm{e}^{-k(-z+f)+\mathrm{i}k\omega}}{k - k_0\sec^2\beta - \mathrm{i}\mu\sec\beta/v}\mathrm{d}\,k \\ &- 2\pi\mathrm{i} \int_{-b/2}^{b/2} \Gamma \mathrm{d}\,\eta \int_0^{\infty} \mathrm{e}^{-k(z+f)+\mathrm{i}k\omega}\cos(k(y-\eta))\mathrm{d}\,k \\ &+ 2\pi\mathrm{i} \int_{-b/2}^{b/2} \Gamma \mathrm{d}\,\eta \int_0^{\infty} \mathrm{e}^{-k(-z+f)+\mathrm{i}k\omega}\cos(k(y-\eta))\mathrm{d}\,k\Big]\,,\end{aligned} \tag{10}$$

where $\omega = x\cos\theta + (y-\eta)\sin\theta$. So the wave elevation is

$$\zeta = -\frac{v}{g}\frac{\partial \Phi}{\partial x}\bigg|_{z=0} = \frac{vk_0}{4\pi^2 g}\int_{-b/2}^{b/2} \Gamma \mathrm{d}\,\eta \int_{-\pi}^{\pi} \sec^2\beta \mathrm{d}\,\beta \int_0^{\infty} \frac{k\mathrm{e}^{-kf+\mathrm{i}k\omega}}{k - k_0\sec^2\beta - \mathrm{i}\mu\sec\beta/v}\mathrm{d}\,k\,. \tag{11}$$

3. Solution of the submerged horseshoe vortex

Assuming $k\cos\beta = k_1$, $k\sin\beta = k_2$, we have

$$\zeta = \frac{vk_0}{4\pi^2 g}\int_{-b/2}^{b/2} \Gamma \mathrm{d}\,\eta \int_{-\infty}^{\infty} \mathrm{d}\,k_1 \int_{-\infty}^{\infty} \frac{\sqrt{k_1^2+k_2^2}\mathrm{e}^{-\sqrt{k_1^2+k_2^2}f+\mathrm{i}k\omega}}{k_1^2 - k_0\sqrt{k_1^2+k_2^2} - \mathrm{i}\varepsilon k_1}\mathrm{d}\,k_2\,, \tag{12}$$

where $\varepsilon = \mu/V$. Then Lighthill's two-stage scheme can be used, in which the first step is to employ the residue theorem to calculate the integral of k_1 and the second is to employ the stationary phase method to calculate the integral of k_2. Finally the algebraic expression of the wave elevation can be obtained.

Considering

$$k_1^2 - k_0\sqrt{k_1^2+k_2^2} - \mathrm{i}\varepsilon k_1 = 0\,, \tag{13}$$

and assuming $k_1 = k_{1,j1} + \mathrm{i}\varepsilon k_{1,j2}$, from Eq.(13) we obtain

$$k_{1,j1} = (-1)^j k_0\sqrt{\frac{1+m}{2}}\,, \qquad k_{1,j2} = \frac{1+m}{2m} > 0\,, \tag{14}$$

where $m = \sqrt{1+4k_2^2/k_0^2}$. Employing the residue theorem,[12] we have

$$(k_{1,j}^2 + k_2^2)\lim_{k_1 \to k_{1,j}} \frac{k_1 - k_{1,j}}{k_1^2 - k_0\sqrt{k_1^2+k_2^2} - \mathrm{i}\varepsilon k_1} = (-1)^j \frac{1}{m}\Big(\frac{m+1}{2}\Big)^{3/2} k_0\,. \tag{15}$$

That is

$$\zeta = \sum_{j=1,2} 2\pi\mathrm{i} \times \frac{vk_0^2}{4\pi^2 g}\int_{-b/2}^{b/2} \Gamma(\eta)\mathrm{d}\,\eta \int_{-\infty}^{\infty} (-1)^j \frac{1}{m}\Big(\frac{m+1}{2}\Big)^{3/2} \mathrm{e}^{-\frac{m+1}{2}k_0 f+\mathrm{i}k\omega}\mathrm{d}\,k_2\,. \tag{16}$$

Assuming $(1+m)/2 = 1+u^2$, so $m = 2u^2+1$, that is $\sqrt{1+4k_2^2/k_0^2} = 2u^2+1$, therefore

$$\mathrm{d}\,k_2 = k_0\frac{1+2u^2}{\sqrt{1+u^2}}\mathrm{d}\,u\,. \tag{17}$$

Let $x = R\cos\theta$, $y = R\sin\theta$ and rewrite the integral in Eq.(16) as follows:

$$\sum_{j=1,2} \mathrm{i} \times \int_{-b/2}^{b/2} \Gamma(\eta)\mathrm{d}\,\eta \int_{-\infty}^{\infty} (-1)^j \frac{1}{m}\Big(\frac{m+1}{2}\Big)^{3/2} \mathrm{e}^{-\frac{m+1}{2}k_0 f + \mathrm{i}k\omega}\mathrm{d}\,k_2$$

$$= -4\Gamma \times \int_{-\infty}^{\infty} \sin\Big(\frac{bk_0 u\sqrt{1+u^2}}{2}\Big)\frac{(1+u^2)^{1/2}}{u}\mathrm{e}^{-k_0(1+u^2)f}[\sin A\cos B + \mathrm{i}\sin A\sin B]\mathrm{d}\,u\,, \tag{18}$$

where $A = Rk_0\sqrt{1+u^2}\cos\theta$, $B = Rk_0 u\sqrt{1+u^2}\sin\theta$. The term including $\sin A\sin B$ is an odd function of u, so the integral equals zero. Then

$$\zeta = -\frac{2vk_0^2\Gamma}{\pi g}\int_{-\infty}^{\infty} \sin\Big(\frac{bk_0 u\sqrt{1+u^2}}{2}\Big)\frac{(1+u^2)^{1/2}}{u}\mathrm{e}^{-k_0(1+u^2)f}\sin A\cos B\mathrm{d}\,u$$

$$= -\frac{vk_0^2\Gamma}{\pi g}\mathrm{Im}\Big\{\int_{-\infty}^{\infty} \sin\Big(\frac{bk_0 u\sqrt{1+u^2}}{2}\Big)\frac{(1+u^2)^{1/2}}{u}\mathrm{e}^{-k_0(1+u^2)f}(\mathrm{e}^{\mathrm{i}(A+B)} + \mathrm{e}^{\mathrm{i}(A-B)})\mathrm{d}\,u\Big\}\,. \tag{19}$$

Since the term including $\mathrm{e}^{\mathrm{i}(A+B)}$ has not any stationary point, it can be neglected. So

$$\zeta = -\frac{vk_0^2\Gamma}{\pi g}\mathrm{Im}\Big\{\int_{-\infty}^{\infty} \sin\Big(\frac{bk_0 u\sqrt{1+u^2}}{2}\Big)\frac{(1+u^2)^{1/2}}{u}\mathrm{e}^{-k_0(1+u^2)f}\mathrm{e}^{\mathrm{i}(A-B)}\mathrm{d}\,u\Big\}\,. \tag{20}$$

Then the stationary phase method can be used. Therefore we have $\partial(A-B)/\partial u = 0$, that is

$$\frac{\partial[Rk_0(\cos\theta - u\sin\theta)\sqrt{1+u^2}]}{\partial u} = 0\,. \tag{21}$$

We thus have

$$2u^2 - \cot\theta u + 1 = 0\,. \tag{22}$$

The two roots of Eq.(22) are

$$u_\pm = \frac{1}{4}\cot\theta\big(1 \pm \sqrt{1 - 8\tan^2\theta}\big)\,. \tag{23}$$

These are the stationary points. From Eq.(23) we notice that θ cannot be larger than $\arctan\sqrt{1/8}$, which means that the wake angle of the horseshoe vortex is the same as that of the classical Kelvin waves. Assuming

$$F(u) = \sin\Big(\frac{bk_0 u\sqrt{1+u^2}}{2}\Big)\frac{(1+u^2)^{1/2}}{u}\mathrm{e}^{-k_0(1+u^2)f}\,, \tag{24}$$

and

$$h(u) = k_0(\cos\theta - u\sin\theta)\sqrt{1+u^2}\,, \tag{25}$$

it is easy to derive

$$h''(u) = k_0\frac{\cos\theta - (2u^3+3u)\sin\theta}{(1+u^2)^{3/2}}\,, \tag{26}$$

According to the stationary phase method, we have

$$\zeta = -\frac{vk_0^2\Gamma}{\pi g}\Big\{F(u_+)\sqrt{\frac{-2\pi}{Rh''(u_+)}}\sin\Big(Rh(u_+) - \frac{\pi}{4}\Big)$$

$$+ F(u_-)\sqrt{\frac{-2\pi}{Rh''(u_-)}}\sin\Big(Rh(u_-) + \frac{\pi}{4}\Big)\Big\}\,. \tag{27}$$

4. Discussion

First let us analyse Eq.(27). When $h''(u)$ is equal to zero, the expression (27) approaches infinity if $F(u)$ has a finite value. It is easy to know that $h''(u_\pm) = 0$ when $u_+ = u_-$, and so we have $h'(u_+) = h'(u_-)$ from the structure of $h(u)$. Here

$$\theta = \arctan\sqrt{1/8}\,,$$

that is $19°28'$. This means the wake angle of the submerged horseshoe vortex on the free surface is the same as that of the classical Kelvin wave. Of course, the singularities at the two sides of the wake angle can be avoided by employing Ursell's method.[8]

Secondly, we consider the free surface of the region near $\theta = 0$. As shown in Fig.3, the dashed line denotes divergent waves (the second term on the right side of Eq.(27). The dot-dashed line denotes the transverse waves (the first term on the right side of Eq.(27)). The solid line denotes the total elevation. For the transverse waves, we have $u_- \to 0$ when $\theta \to 0$. It can be found that the change of the transverse waves is smooth as shown in Fig.3. For the diverging waves, we have $u_+ \to \infty$ when $\theta \to 0$. But for the existence of the extremely small term $\mathrm{e}^{-k_0(1+u_+^2)f}$ in Eq.(24) the amplitude of the diverging waves still approaches zero. It also can be found that the elevation of the diverging waves varies drastically only when they deviate by a certain distance from the axis (see Fig.3).

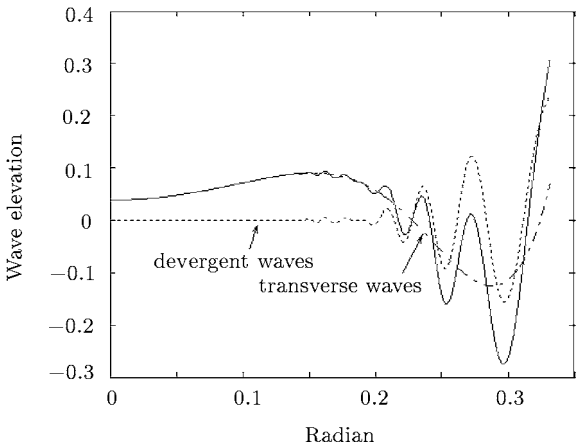


Fig.3. The wave elevation ($R = 500$ m, $V = 10$ m/s, $b = 10$ m, $\Gamma = 500$ m^2/s).

Because the oscillation of waves mainly lies on the diverging waves, we should consider the following term:

$$F(u_+)\sqrt{\frac{-2\pi}{Rh''(u_+)}}\,\sin\Big(Rh(u_+)-\frac{\pi}{4}\Big)\,. \tag{28}$$

We take $\sin(Rh(u_+)-\pi/4)$ as an oscillation term and $F(u_+)\sqrt{-2\pi/h''(u_+)}$ as a modulation term. Although there is the oscillation term in $F(u_+)$, the waves do not oscillate for the term $\mathrm{e}^{-k_0(1+u_+^2)f}$. The variation of the modulation term is small as shown in Fig.4 (the broken line). The appearance of zero point is determined by the term $\sin(bk_0u_+\sqrt{1+u_+^2}/2)$. At the critical line $\theta = 19°28'$, from Eq.(3) we see

$$u_+ = \sqrt{2}/2\,, \tag{29}$$

then

$$bk_0u_+\sqrt{1+u_+^2}/2 = \sqrt{3}bk_0/4\,. \tag{30}$$

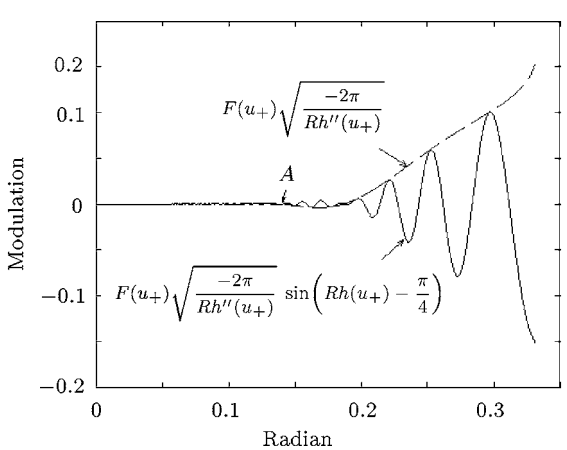


Fig.4. The wave modulation ($R = 500$ m, $V = 10$ m/s, $b = 10$ m, $\Gamma = 500$ m^2/s).

For the ordinary ship waves the value of $\sqrt{3}bk_0/4$ lies between 0 and π(for example, $b = 10$ m, $V = 10$ m/s). So the phase value of the point A in Fig.4 is 2π for $bk_0u_+\sqrt{1+u_+^2}/2$ is a decreasing function of θ as shown in Fig.5.

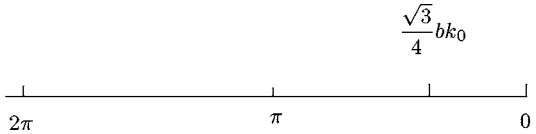


Fig.5. The zero points of $\sin(bk_0u_+\sqrt{1+u_+^2}/2)$.

From $bk_0u_+\sqrt{1+u_+^2}/2 = 2\pi$, we have

$$\begin{aligned} u_+ &= \sqrt{\sqrt{1/4+(4\pi/k_0b)^2}-1/2} \equiv a \\ &\approx \sqrt{4\pi/k_0b}\,. \end{aligned} \tag{31}$$

Let Eq.(31) be substituted into Eq.(22), the angle θ_A of the point A in Fig.4. can be obtained

$$\tan\theta_A = \frac{a}{2a^2+1} \approx \frac{2\sqrt{\pi k_0b}}{8\pi+k_0b}\,. \tag{32}$$

This means that a notable small swing oscillation begins when θ passes $\theta_c(= 19°28')$. Of course, it is an approximate estimation.

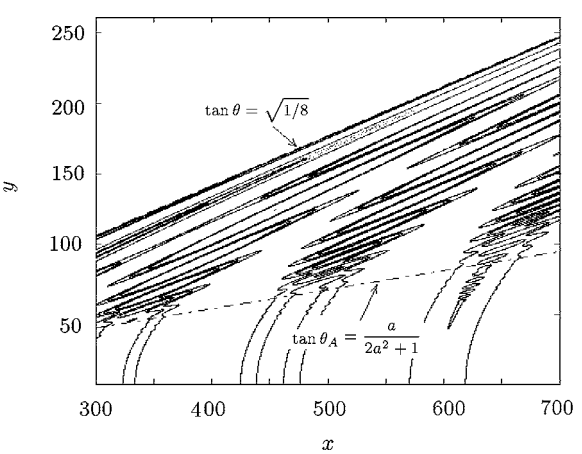


Fig.6. Contours of equal magnitude of the surface elevation ($V = 10$ m/s, $b = 10$ m, $\Gamma = 500$ m^2/s).

Assuming $V = 10$ m/s, $b = 10$ m, then $k_0 = 0.1$/m and $a \approx \sqrt{4\pi}$, we obtain

$$\tan\theta_A = \frac{2\sqrt{\pi}}{8\pi+1}\,. \tag{33}$$

θ_A is equal to about $7°42'$ and has nothing to do with vortex strength Γ. As shown in Fig.6, the line

$\theta = \theta_A$ lies just between the oscillation region and the smooth region. The contours of equal magnitude in the smooth region cannot be shown for there is almost no change of the elevation in this region as compared with that in the oscillation region. Moreover, we find from Eq.(32) that θ_A becomes smaller with the increase of V. When the uniform stream passing the other singularities (source, sink, doublet etc.) there is not any the smooth region near $\theta = 0$.

From the above analysis two results can be obtained:

1. When a submerged horseshoe vortex interacts with a free surface it is believed that the traditional Kelvin ship waves still exist. For the uniform stream problem the Kelvin waves can exist for any arbitrary disturbance form.

2. In the region with an aperture θ of just a few degrees the free surface is smooth. Then the fast oscillation begins when θ becomes larger. The region of small swing oscillation exists between the smooth region and the drastic oscillation region. It is believed that this region has an important effect on the SAR image of narrow V-shaped ship waves.

References

[1] Liu A K, Peng Y and Chang S Y 1997 *IEEE J. Oceanic Engin.* **22** 9

[2] Reed A M and Milgram J H, 2002 *Annu. Rev. Fluid Mech.* **34** 469

[3] Hennings I, Romeiser R, Alpers W and Viola A 1999 *Int. J. Remte Sensing* **20** 2519

[4] Lyden J D, Hammond R R *et al* 1988 *J. Geo-phys. Res.* **93**C 12293

[5] Milgram J H 1988 *J. Ship Res.* **32** 54

[6] Meng J M, Tao M D and Zhang J. 2001 *J. Hydrodynamics*, ser. B **13** (3) 122

[7] Wehausen J V and Laitone E V 1960 Surface Waves, In *Handbuch der Physik 8/3: Strömungsmechanik*, edited by Flügge S. (Berlin: Springer) 446

[8] Ursell F 1960 *J. Fluid Mech.* **8** 418

[9] Crapper G D 1964 *Proc. Roy. Soc.* Series A **282** 1388

[10] Yong K C and Jin S L 1991 *J. Ship Res.* **35** 3

[11] Timour R 2001 *J. Ship Res.* **45** 1

[12] Chan A T and Chwang A T 1996 *Phys. Fluid.* **8** 421

[13] Liu M J and Tao M D 2003 *Appl. Math. Mech.* **23** 10

[14] Wu Y G and Tao M D 2005 *Acta Phys. Sin.* **54** 4496 (in Chinese)

[15] Hoekstra M and Aalbers A 1997 *Twenty-First Symposium on Naval Hydrodynamics* (Marine Research Institute, The Netherlands) 278

[16] Reed A M, Beck R F, Griffin O M and Pletzer R D 1990 *SNAME Trans.* **98** 319

[17] Nishiyama T 1965 *J. Ship Res.* **19** 131

[18] Wu Y T 1954 *J. Math. Phys.* **33** 213