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First integrals and stability of second-order differential equations*

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The stability of second-order differential equations is studied by using their integrals. A system of second-order differential equations can be considered as a mechanical system with holonomic constraints. A conserved quantity of the mechanical system or a part of the system is obtained by using the Noether theory. It is possible that the conserved quantity becomes a Liapunov function and the stability of the system is proved by the Liapunov theorem.

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1. Introduction

The Liapunov stability of differential equations can be studied by using the Liapunov's direct method.[1−3] 'The essential difficulty in utilizing Liapunov's direct method is that of constructing a Liapunov function. Sometimes, it is possible to find first integrals which are not in themselves definite in sign, but which can be combined so as to give a Liapunov function. Such a case is quoted by Malkin and is due to Chetaev. It concerns the stability of motion of a spinning projectile.'[4] This paper is devoted to the study of the stability of second-order differential equations by using the integrals of a mechanical system. A system of second-order differential equations is expressed as a holonomic mechanical system. The first integrals of the mechanical system are deduced by using the Noether theory. If a first integral or a combination of a few integrals can be regarded as a Liapunov function, the stability will be determined by the Liapunov theorem. If there is no integral in the system or it is difficult to find one, we will discuss a part of the system that may possess the integrals, and then a Liapunov function will be constructed by these integrals.

2. Fundamental theory

A system of second-order differential equations is

$$\ddot{q}_s = f_s(\boldsymbol{q}, \dot{\boldsymbol{q}}) \quad (s = 1, \ldots, n). \tag{1}$$

Supposing that there exists an isolated equilibrium position $q_s = q_s^0 = \text{const.}$ $(s = 1, \ldots, n)$ in the system, we have

$$f_s(\boldsymbol{q}^0, 0) = 0, \quad (s = 1, \ldots, n). \tag{2}$$

In order to study the stability of the equilibrium position, we express the system (1) as a holonomic mechanical system where Lagrangian and generalized forces are, respectively,

$$L = \frac{1}{2}\dot{q}_s\dot{q}_s + U(\boldsymbol{q}),$$

and

$$Q_s = f_s - \frac{\partial U}{\partial q_s}, \quad (s = 1, \ldots, n). \tag{3}$$

According to the Noether theory, the Noether identity of the system (3) has the form[5−7]

$$L\dot{\xi}_0 + \dot{q}_s(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) + \frac{\partial U}{\partial q_s}\xi_s + Q_s(\xi_s - \dot{q}_s\xi_0) + \dot{G}_{\rm N} = 0. \tag{4}$$

If we find the infinitesimal generators $\xi_0 = \xi_0(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, $\xi_s = \xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ and the gauge function $G_{\rm N} =$

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$G_{\rm N}(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ satisfying the relation (4), we shall obtain the Noether conserved quantity

$$I_{\rm N} = L\xi_0 + \dot{q}_s(\xi_s - \dot{q}_s\xi_0) + G_{\rm N} = \text{const.} \qquad (5)$$

which is a first integral of the system (1). If one integral (5) or a combination of a few integrals (5) can become a Liapunov function V, using the Liapunov theorem, we can identify the stability of the equilibrium position.

If it is difficult to find any integral of the system (1), the relation (1) will be written in the form

$$\ddot{q}_s = g_s(\boldsymbol{q},\dot{\boldsymbol{q}}) + h_s(\boldsymbol{q},\dot{\boldsymbol{q}}), \quad (s=1,\ldots,n), \qquad (6)$$

so that the auxiliary system

$$\ddot{q}_s = g_s(\boldsymbol{q},\dot{\boldsymbol{q}}) \quad (s=1,\ldots,n) \qquad (7)$$

possesses integrals. We choose one integral or a combination of a few integrals of the system (7) as a Liapunov function V and differentiate it with respect to time t along the trajectory of the system (6). Subsequently, the stability will be identified by using the Liapunov theorem.

3. Application

We give two examples in order to illustrate the above result.

Example 1 Differential equations are

$$\ddot{q}_1 = -q_1 - 2q_2^2\dot{q}_1,$$

and

$$\ddot{q}_2 = -q_2 - 3q_2^2 - 2q_1^2\dot{q}_2. \qquad (8)$$

We study the stability of the equilibrium position

$$q_1^0 = q_2^0 = 0. \qquad (9)$$

We express Eq.(8) as a holonomic mechanical system whose Lagrangian and generalized forces respectively are

$$L = \frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2) - \frac{1}{2}(q_1^2+q_2^2) - q_2^3,$$

$$Q_1 = -2q_2^2\dot{q}_1, \quad \text{and} \quad Q_2 = -2q_1^2\dot{q}_2. \qquad (10)$$

The Noether identity (4) gives

$$\begin{aligned} &L\dot{\xi}_0 + \dot{q}_1(\dot{\xi}_1 - \dot{q}_1\dot{\xi}_0) + \dot{q}_2(\dot{\xi}_2 - \dot{q}_2\dot{\xi}_0) \\ &- q_1\xi_1 - q_2\xi_2 - 3q_2^2\xi_2 \\ &+ Q_1(\xi_1 - \dot{q}_1\xi_0) + Q_2(\xi_2 - \dot{q}_2\xi_0) + \dot{G}_{\rm N} = 0. \end{aligned} \qquad (11)$$

It is very difficult to find the generators ξ_0, ξ_1, ξ_2 and the gauge function $G_{\rm N}$ satisfying the relation (11). Assume

$$Q_1 = Q_2 = 0, \qquad (12)$$

then the auxiliary system (7) will become

$$\ddot{q}_1 = -q_1, \quad \text{and} \quad \ddot{q}_2 = -q_2 - 3q_2^2. \qquad (13)$$

In this case, we can find the solution of the relation (11) as

$$\xi_0 = -1, \quad \xi_1 = \xi_2 = 0, \quad G_{\rm N} = 0. \qquad (14)$$

The conserved quantity (5) gives

$$I_{\rm N} = \frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2) + \frac{1}{2}(q_1^2+q_2^2) + q_2^3 = \text{const.} \qquad (15)$$

We choose it as a Liapunov function

$$V = I_{\rm N} = \frac{1}{2}(\dot{q}_1^2+\dot{q}_2^2) + \frac{1}{2}(q_1^2+q_2^2) + q_2^3, \qquad (16)$$

which is a positive definite function in the neighbourhood of the equilibrium position. Differentiating V with respect to time t along the trajectory of Eq.(8), we have

$$\dot{V}\big|_{(8)} = -2q_1^2\dot{q}_2^2 - 2q_2^2\dot{q}_1^2 \le 0. \qquad (17)$$

According to the Liapunov theorem, the equilibrium position (9) is stable.

Example 2 Study the stability of the equilibrium position $q^0 = 0$ of the system

$$2(t+2)\ddot{q} + 2\dot{q} = -q. \qquad (18)$$

Equation (18) can be rewritten in the form

$$\ddot{q} + \frac{q+\dot{q}}{2(t+2)} = -\frac{\dot{q}}{2(t+2)}, \qquad (19)$$

and its auxiliary system chosen as

$$\ddot{q} + \frac{q+\dot{q}}{2(t+2)} = 0 \qquad (20)$$

is a holonomic mechanical system with a Lagrangian and a generalized force expressed respectively by

$$L = \frac{1}{2}\dot{q}^2, \quad \text{and} \quad Q = -\frac{q+\dot{q}}{2(t+2)}. \qquad (21)$$

The Noether identity of the system (20) is

$$L\dot{\xi}_0 + \dot{q}(\dot{\xi} - \dot{q}\dot{\xi}_0) - \frac{q+\dot{q}}{2(t+2)}(\xi - \dot{q}\xi_0) + \dot{G}_{\rm N} = 0, \qquad (22)$$

which has a solution as

$$\xi_0 = 0, \quad \xi = 2(t+2)\dot{q}, \quad G_{\rm N} = \frac{1}{2}q^2 - (t+2)\dot{q}^2, \qquad (23)$$

and the Noether conserved quantity (5) gives

$$I_{\mathrm{N}} = (t+2)\dot{q}^2 + \frac{1}{2}q^2. \tag{24}$$

Let the relation (24) be a Liapunov function, i.e. $V = I_{\mathrm{N}}$, and we have

$$\dot{V}\,\big|_{(18)} = -\dot{q}^2 \leq 0. \tag{25}$$

By the Liapunov theorem, we know that the equilibrium position is stable.

4. Conclusions

The relation between the integrals and the stability of second-order differential equations is discussed in this paper. Two given examples show how the stability of the equilibrium position for a system of second-order differential equations is identified by using the first integrals of the holonomic mechanical system deduced from the differential equations, which is a useful method. The $f_s(\boldsymbol{q}, \dot{\boldsymbol{q}})$ in Eq.(1) is not directly related to time t, it is easy to find an isolated equilibrium position. Example 2 indicates that the method can be applied to a special case while a useful Liapunov function exists.

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