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Repulsive effects in steady axisymmetric fields*

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In this paper the repulsive effects in the Kerr and Kerr–Newman fields are discussed. The contributions made by all parameters of the fields and test particles to the repulsive effects are also discussed, and the accretive effect on interstellar dust, i.e. the distribution of dust is calculated. The discussion is also carried out on the slow rotation of the Kerr field in which the effect is related to the positions and velocities of the particles and the orientations of their trajectories as well.

Keywords: gravitational effect, general relativity, steady axisymmetric field
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1. Introduction

In general relativity, the gravitational effect is essentially different from that in Newtonian gravitation theory in which the acceleration of a test particle is determined by the equation of motion and the test particle is always attracted by a force produced by the mass source. The right-hand side of the Einstein equation is the energy-momentum tensor of all fields except the gravitational field. So the parameters in the gravitational potential $g_{\mu\nu}$ must be more than those in the Newtonian potential, and they may produce some new effects such as the repulsive effect that is not present in the Newtonian theory. The repulsive effect in a spherical symmetric field was fully studied.[1−3] Ernst *et al* presented a theorem and gave a new solution about the axisymmetric field,[4−6] which is very interesting because it is very important for the study of some astrophysical phenomena. So the study of repulsive effect in this field is also very important. In this paper, we investigate the repulsive effects in the Kerr and Kerr–Newman fields.

2. Expression of acceleration

In general relativity, establishing the relation between the gravitation and the acceleration of a test particle moving in a gravitational field, and interpreting it are much more complex than in the Newtonian gravitation theory. Take a neutral particle moving along the geodesics in the 4-dimensional space–time for example.[7] When we research it in a Lorentz local space section, the 3-dimensional acceleration in the direction of x^i is expressed as[8]

$$g^i = -\Gamma^i_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = \frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}s}\frac{v^i}{\sqrt{1-\beta^2}}, \tag{1}$$

where the Greek letters μ, ν take the values of 0, 1, 2, 3; the Latin letters i, j, k take the values of 1, 2, 3; $\beta = v/c$; and v^i is related to the velocity in the same direction. The 3-dimensional velocity v can be obtained by the pure space metric

$$v = \sqrt{\gamma_{ij}v^i v^j},$$

where

$$\gamma_{ij} = \frac{g_{0i}g_{0j}}{g_{00}} - g_{ij}.$$

According to Eq.(1),the acceleration can also be expressed as

$$\begin{aligned} g^i =& \frac{1}{1-\beta^2}\left[-\frac{\Gamma^i_{00}}{g_{00}} - \frac{2}{\sqrt{g_{00}}}\left(\Gamma^i_{0k} - \frac{g_{0k}}{g_{00}}\Gamma^i_{00}\right)\frac{v^k}{c}\right. \\ & - \left(\Gamma^i_{kj} - \frac{g_{0j}}{g_{00}}\Gamma^i_{0k} - \frac{g_{0k}}{g_{00}}\Gamma^i_{0j}\right. \\ & \left.\left. + \frac{g_{0j}g_{0k}}{g^2_{00}}\Gamma^i_{00}\right)\frac{v^j v^k}{cc}\right]. \end{aligned} \tag{2}$$

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And the 3-dimensional acceleration g can also be obtained by the pure space metric

$$g = \sqrt{\gamma_{ij} g^i g^j}. \tag{3}$$

If the test particle is static in relation to the coordinates we choose (comoving coordinates), where v^i=0, the acceleration is simplified into the following form:

$$g^i(v^i = 0) = -\Gamma^i_{00}/g_{00}. \tag{4}$$

By introducing a special local Lorentz space section, the acceleration of a test particle moving in the gravitational field can be given, and it is related to the parameters of the source field and test particle. According to Eq.(2), some contributions made by the parameters in general relativity are to produce an attractive effect and the others are to produce a repulsive effect, which is different from the case in the Newtonian theory.

3. Repulsive effect in Kerr field

The Kerr metric is expressed by the line elements as[9]

$$\begin{aligned} \mathrm{d}s^2 =& \left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right)\mathrm{d}t^2 - \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2 - 2mr}\mathrm{d}r^2 \\ & - (r^2 + a^2\cos^2\theta)\mathrm{d}\theta^2 \\ & - \left[(r^2 + a^2)\sin^2\theta + \frac{2mra^2\sin^4\theta}{r^2 + a^2\cos^2\theta}\right]\mathrm{d}\phi^2 \\ & + \frac{4mra\sin^2\theta}{r^2 + a^2\cos^2\theta}\mathrm{d}t\mathrm{d}\phi, \end{aligned} \tag{5}$$

where $a = \dfrac{J}{m}$ represents the angular momentum per unit mass.

Substituting Eq.(5) into Eq.(4), and supposing that the test particle moves in the comoving coordinates ($v^i = 0$), we obtain the expressions of g^i as

$$g^1 = \frac{(r^2 - 2m + a^2)(-mr^2 + ma^2\cos^2\theta)}{(r^2 + a^2 - 2mr)(r^2 + a^2\cos^2\theta)^2}, \tag{6}$$

$$g^2 = \frac{mra^2\sin(2\theta)}{(r^2 + a^2\cos^2\theta - 2mr)(r^2 + a^2\cos^2\theta)^2}, \tag{7}$$

$$g^3 = 0. \tag{8}$$

Considering the pure space metric in the Kerr field

$$\begin{aligned} &\gamma_{11} = -g^{11}, \quad \gamma_{22} = -g^{22}, \\ &\gamma_{12} = \gamma_{21} = 0, \end{aligned} \tag{9}$$

we can easily obtain the 3-dimensional acceleration of a neutral particle moving in the comoving coordinates

$$\begin{aligned} g_{m,a} =& \frac{m}{(r^2 + a^2\cos^2\theta)^{3/2}(r^2 + a^2\cos^2\theta - 2m)} \\ & \times [(r^2 - 2mr + a^2)(2r^2 - r^2 - a^2\cos^2\theta)^2 \\ & + r^2a^4\sin^2 2\theta]^{1/2}, \end{aligned} \tag{10}$$

where subscripts m and a in $g_{m,a}$ mean that the acceleration is related to the gravitational mass and angular momentum per unit mass. According to Eqs.(7), (8) and (9),we find that there is non-vanishing component g^2 of g^i besides g^1. g^1, g^2 and g are related to the parameters r and θ, which reveals that the acceleration of the test particle is dependent on its position.

According to Eq.(6), it is obvious that the radial acceleration of the test particle vanishes in the infinite red-shift surface ($r = m \pm \sqrt{m^2 - a^2}$) and the surface of $r = |a\cos\theta|$. According to Eq.(7), the acceleration is possessed of the lateral component g^2 because of the rotation of the source field, and it vanishes around the equator $\left(\theta = \dfrac{\pi}{2}\right)$ and the two poles ($\theta = 0, \pi$). When $\theta = \dfrac{\pi}{4}$, the lateral acceleration reach a maximum. That is to say, the particle has a tendency to drop down to the equator and two poles.

Therefore, we can find a balanced location of the test particle in the Kerr field. Here we only discuss two particular cases: around the equator $\left(\theta = \frac{\pi}{2}\right)$ and around the two poles ($\theta = 0, \pi$).

When $\theta = 0$ or $\theta = \pi$, we have

$$g(\theta = 0, \pi) \approx \frac{m}{r^2}\left(1 - \frac{3a^2}{r}\right), \tag{11}$$

which indicates that around the two poles the contribution made by the angular momentum of the gravitational field is to produce a repulsive force and then to retard the Newtonian acceleration, and when $r > 3a^2$, the attractive force is stronger than the repulsive force, so the particle moves along the direction toward the centre of the source field. When $r = r_c = 3a^2$, we have g=0, that is to say, when a particle moving through the surface of r_c, the force acting on it changes from attractive to repulsive. Therefore, the particle cannot reach the centre of the source field, and only stays in the surface of r_c.

When $\theta = \frac{\pi}{2}$, we have

$$g\left(\theta = \frac{\pi}{2}\right) \approx \frac{m}{r^2}\left(1 + \frac{a^2}{r}\right), \tag{12}$$

that is to say, the contribution made by the angular momentum is to produce an attractive force and then to strengthen the Newtonian acceleration around the equator, so the particle can reach the centre of the source field.

If we discuss arbitrary cases with Eqs.(6), (7) and (10), the accretive effect on interstellar dust, i.e. the distribution of dust can be calculated then.

When the angular momentum per unit mass of the source field $a \to m$, the non-linear effect is extremely complex. We limit ourselves to the case of slow rotation ($a \ll m$), in which the Kerr metric can be linearized at the origin of $r \geq r_+ = m + \sqrt{m^2 - a^2}$ and it can be expressed by the line elements as

$$\begin{aligned} \mathrm{d}s^2 =&(1 - 2m/r)c^2\mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - 2m/r} - r^2\mathrm{d}\theta^2 \\ &- r^2\sin^2\theta\mathrm{d}\varphi^2 + \frac{4ma\sin^2\theta}{r}\mathrm{d}t\mathrm{d}\phi. \end{aligned} \tag{13}$$

Substituting Eq.(13) into Eq.(4), and assuming the particle to have only the lateral component of v^i (say $v^3 \neq 0$, and $v^1 = v^2 = 0$), we obtain the expression of g^i as follows:

$$\begin{aligned} g^1 =&-\frac{1}{1-\beta^2}\left[-\frac{m}{r^2} + \frac{2ma\sin^2\theta}{r^2}\left(1 - \frac{2m}{r}\right)^{1/2}\frac{v^3}{c}\right. \\ &\left. + r\sin^2\theta\left(1 - \frac{2m}{r}\right)\left(\frac{v^3}{c}\right)^2\right], \end{aligned} \tag{14}$$

$$\begin{aligned} g^2 =&\frac{1}{1-\beta^2}\left[-\frac{2ma\sin 2\theta}{r^3\sqrt{1-2m/r}}\frac{v^3}{c}\right. \\ &\left. + \frac{\sin(2\theta)}{2}\left(\frac{v^3}{c}\right)^2\right], \end{aligned} \tag{15}$$

$$g^3 = 0. \tag{16}$$

According to Eqs.(7), (8) and (9), we find that the acceleration of the particle is dependent on its velocity besides its position. Assume that the velocity is low enough compared with the velocity of light, then we will be able to cancel the second order of g^i and obtain its non-vanishing components of g^i as follows:

$$\begin{aligned} g^1 =&-\frac{1}{1-\beta^2}\frac{m}{r^2}\left[-1 + 2a\sin^2\theta\left(1 - \frac{2m}{r}\right)^{1/2}\frac{v^3}{c}\right], \\ g^2 =&\frac{1}{1-\beta^2}\left[-\frac{2ma\sin 2\theta}{r^3\sqrt{1-2m/r}}\frac{v^3}{c}\right]. \end{aligned} \tag{17}$$

It can be deduced from Eq.(17) that the acceleration of a particle falling at a classical velocity is related to the orientation of the trajectory. When the particle moves around the equator $\left(\theta = \frac{\pi}{2}\right)$, the lateral acceleration vanishes. Thus we obtain

$$g^1 = -\frac{1}{1-\beta^2}\frac{m}{r^2}\left[-1 + 2a\left(1 - \frac{2m}{r}\right)^{1/2}\frac{v^3}{c}\right]. \tag{18}$$

The 3-dimensional acceleration of the particle is given by

$$\begin{aligned} g_{m,a} =&-\frac{1}{1-\beta^2}\frac{m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1/2} \\ &\times\left[1 - 2a\left(1 - \frac{2m}{r}\right)\frac{v^3}{c}\right]. \end{aligned} \tag{19}$$

It is obvious that when a and v^3 have the same directions, the contribution of a is to produce a repulsive force; when a and v^3 have the opposite directions, the contribution of a is to produce an attractive force. Especially, when the trajectory through the rotation axis, the acceleration vanishes.

4. Repulsive effect in Kerr–Newman field

The Kerr–Newman metric is expressed by the line elements as[10−15]

$$\begin{aligned} \mathrm{d}s^2 =&\left(1 - \frac{2mr - q^2}{\rho^2}\right)\mathrm{d}t^2 - \frac{\rho^2}{\Delta}\mathrm{d}r^2 - \rho^2\mathrm{d}\theta^2 \\ &- \left[\frac{(2mr - q^2)a^2\sin^2\theta}{\rho^2} + (r^2 + a^2)\right] \\ &\times\sin^2\theta\mathrm{d}\phi^2 + 2\frac{a\sin^2\theta}{\rho^2}(2mr - q)\mathrm{d}t\mathrm{d}\phi, \end{aligned} \tag{20}$$

where

$$\begin{aligned} \Delta &= r^2 - 2mr + a^2 + q^2, \\ \rho^2 &= r^2 + a^2\cos^2\theta, \end{aligned} \tag{21}$$

q and a represent the electric charge and angular momentum per unit mass, respectively.

Substituting Eq.(20) into Eq.(4), and assuming the test particle to move in the comoving coordinates ($v^i = 0$), we obtain the expressions of g^i as follows:

$$\begin{aligned} g^1 =&\frac{\Delta}{(\rho^2)^2}(ma^2\cos^2\theta + rq^2 - mr^2) \\ &\times\left(1 - \frac{2mr - q^2}{\rho^2}\right)^{-1}, \end{aligned} \tag{22}$$

$$g^2 = \frac{1}{2}\frac{2mr - q^2}{(\rho^2)^2}a^2\sin(2\theta)\left(1 - \frac{2mr - q^2}{\rho^2}\right)^{-1}. \tag{23}$$

Considering the pure space metric in the Kerr–Newman Kerr field

$$\gamma_{11} = -g^{11}, \quad \gamma_{22} = -g^{22},$$
$$\gamma_{12} = \gamma_{21} = 0.$$

We can obtain the 3-dimensional acceleration of a neutral particle moving in the comoving coordinates

$$g = \frac{1}{(\rho^2)^{3/2}}[\rho^2 - (2mr - q^2)]^{-1} \times \{\Delta(2mr - q^2)r - m\rho^2]^2 + \frac{1}{4}(2mr - q^2)^2 a^4 \sin^2(2\theta)\}^{1/2}. \quad (24)$$

According to Eqs.(22), (23) and (24), we find that there is non-vanishing component g^2 of g^i besides $g^1(g^2 \neq 0)$. g^1, g^2 and g are all related to the parameters r and θ. It reveals that the acceleration of the test particle is dependent on its position. Thus, at different positions, the acceleration is different.

When $\theta = \frac{\pi}{2}$, we have

$$g = \frac{1}{r^3}(r^2 - 2mr + q^2)^{-1}(r^2 - 2mr + q^2 + a^2)^{1/2} \times (mr - q^2), \quad (25)$$

from which we can find that around the equator the contribution made by the electric charge in the gravitational field is to produce a repulsive force and then to retard the acceleration that is related to gravitational mass and angular momentum per unit mass. When $r > \frac{q^2}{m}$, the attractive force is stronger than the repulsive force, so the particle moves along the direction toward the centre of the source field. When $r = r'_{\rm c} = \frac{q^2}{m}$, we have g=0, that is to say, when a particle moves through the surface of $r'_{\rm c}$, the force acting on it changes from attractive to repulsive. Therefore, the particle cannot reach the centre of the source field, and only stays in the surface of $r'_{\rm c}$.

When $\theta = 0$ or $\theta = \pi$, we have

$$g(\theta = 0, \pi) = \frac{1}{(r^2 + a^2)^{3/2}}\left(1 - \frac{3a^2}{r}\right) \times (r^2 - 2mr + q^2 + a^2)^{-1/2} \times (mr^2 - q^2 r - ma^2), \quad (26)$$

which indicates that around the two poles the contributions made by electric charge and angular momentum of the gravitational field are to produce a repulsive force and then to retard the Newtonian acceleration. When $r < \frac{q^2 + \sqrt{q^4 + 4m^2a^2}}{2m}$, the repulsive force is stronger than the attractive force. The particle cannot reach the centre of the source field, either, and only stays in the surface of $r = \frac{q^2 + \sqrt{q^4 + 4m^2a^2}}{2m}$.

It is similar to the case of the Kerr field that if we discuss the arbitrary cases with Eqs.(22), (23) and (24), the accretive effect on the interstellar dust, i.e. the distribution of dust can be calculated. But we find that the distribution of dust is different from that in the Kerr field because of the electric charge.

5. Conclusions

In this paper, we have discussed the repulsive effects in the Kerr and the Kerr–Newman fields. The particles cannot arrive at the centre of the source field from around the two poles but can reach it from the equatorial plane in the Kerr field, and the particles cannot reach the centre of the source field from these two directions because of the repulsive force produced by the electric charge in the Kerr–Newman field. It can be deduced that the dust distributions are different in the two fields. The discussion is also carried out on the slow rotation of the Kerr field in the case $a \ll m$. In this case, the acceleration is related to the position and velocity of the particle, and the orientation of the trajectory as well.

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