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Threshold and saturation properties of the field-induced twist cell with two parameters weak anchoring boundaries*

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On the basis of two-parameter formula of weak surface coupling anchoring energy of nematic liquid crystals proposed by Zhao *et al* recently, the general torque equilibrium equation and boundary conditions of the director have been obtained and the threshold field as well as the saturation field of the field-induced twist cell have been analysed for three kinds of configurations, i.e. homogeneous, splay and Pi cells formed by different rubbing conditions and pretilt angles. The results indicated that the polar anchoring has no effect on the threshold field. It is determined only by the twist anchoring and pretilt angle. But, the polar anchoring and twist anchoring are coupling with each other and have a great influence on the saturation field. The formulae for calculating the threshold field and saturation field are given. These results will be very useful in understanding surface physics and the design of liquid crystal cells.

Keywords: threshold field, saturation field, two-parameter formula, nonzero pretilt angle
PACC: 6130, 6470M, 6130G

1. Introduction

Surface anchoring effects have been extensively studied for nematic liquid crystal (NLC) during the last thirty years.[1−4] The threshold field and saturation field of NLC cell were studied under different surface anchoring forms.[5−9] In the early stage, one-dimensional weak anchoring interfacial energy was described by Rapini and Papoular (RP) equation:[5] $g_{\rm s} = A\sin^2\Phi_0$, where Φ_0 is the angle between easy direction $\boldsymbol{e}$ and the director $\boldsymbol{n}$ at the NLC surface, and the constant A is termed anchoring strength or anchoring energy. After that, many attempts were made to generalize the RP equation in order to describe the two-dimensional case of the planar and tilt anchoring.[6−8] Sugimura *et al* proposed an interfacial energy formula with a single anchoring parameter.[7] But their arguments were criticized because of the incapability on distinguishing the difference between azimuthal and polar couplings. Becker *et al* have made a meaningful improvement of the results in Ref.[6] and proposed a new formula for anchoring energy, in which two anchoring strength parameters appear.[6] But their expression is still incomplete and gives wrong predictions for the homeotropic anchoring case. More recently, Zhao *et al* [8] have improved the RP formula further by introducing two anchoring parameters and one easy direction, which can be expressed as

$$g_{\rm s} = W_\xi\,(\boldsymbol{n}\cdot\boldsymbol{\xi})^2 + W_\eta\,(\boldsymbol{n}\cdot\boldsymbol{\eta})^2\,, \qquad (1)$$

where $g_{\rm s}$ is the anchoring energy, W_ξ and W_η are the polar and azimuthal anchoring parameters respectively. The unit vectors $\boldsymbol{\xi}, \boldsymbol{\eta}$, together with easy axis $\boldsymbol{e}$ constitute the stationary directions of an orthonormal triplet. Equation (1) is successful in studying the twist and homeotropic to twist-planar cells, and is accepted by most researches.

On the other hand, recently Derfel and Gajewska reported that there have two thresholds in the field-induced twist cell with weak anchoring boundaries.[10] A field-induced twist LC cell means an in-plane

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switching (IPS) mode of liquid crystal display (LCD) when the magnetic field is replaced by an electric field, which is applied to the plane of the substrate. In their experiments, the director remains undeformed until the lower threshold is reached. While above another higher threshold, the director is uniformly aligned along the field. The two thresholds imply that there are the threshold and the saturation fields in a twisted chiral nematic liquid crystal cell.[5] In previous works,[11−13] the pretilt angle of director was always assumed to be zero in analysing the two threshold fields of IPS cell with symmetrical or asymmetrical weak boundaries. Actually, the pretilt angle is not zero and need consider it carefully.

In this paper, we study the threshold and saturation fields of the field-induced twist cell with nonzero pretilt angle based on the Zhao's expression of anchoring energy proposed recently. For simplification, a magnetic field is applied. In fact, the electric and magnetic fields are equivalent if we ignore the streaky distribution of the field. As shown in Fig.1, three kinds of configurations are discussed. One is homogeneous configuration (parallel rubbing, Fig.1(a)), the second is splay configuration (anti-parallel rubbing, small pretilt angle, Fig.1(b)), and the third is Pi-cell configuration (anti-parallel rubbing, large pretilt angle, Fig.1(c)). Each configuration makes the same pretilt angle with upper and lower substrates (see Fig.1). The general torque equilibrium equations and boundary conditions of the director are deduced. Based on the equation and the boundary conditions the formulae for calculating threshold field and saturation field are obtained. Under a zero pretilt angle and strong anchoring, the formulae reduce to those obtained by Yoneya *et al.*[11] The results also show that the polar anchoring has no effect on the threshold field. However, the twist and polar anchoring are coupling with each other and have great influence on the saturation field.

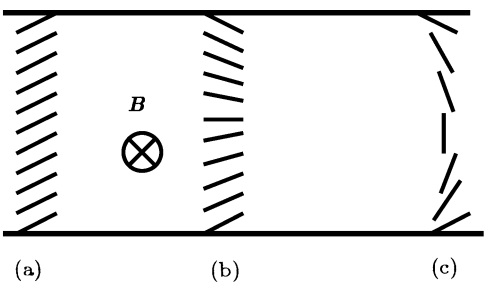


Fig.1. The three configurations for the cell with same pretilt angle. (a) homogeneous, (b) splay, (c) Pi-cell configuration.

2. General torque equilibrium equations

We now investigate a field-induced twisted NLC cell with nonzero pretilt angle using anchoring energy formula (1). We consider the sandwiched cell of thickness d located between $z = 0$ and $z = d$ in a Cartesian coordinate system. The external magnetic field is applied to the cell and it is expressed as $\boldsymbol{B} = (0, B, 0)$. The director $\boldsymbol{n}$ may be expressed as $\boldsymbol{n} = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$, where θ and φ are the tilt and azithmual angles, respectively and are functions of z.

The total bulk free energy density of the system may be expressed as[14]

$$\begin{aligned} g_b =& \frac{1}{2}\left\{k_{11}\left(\nabla\cdot\boldsymbol{n}\right)^2 + k_{22}\left(\boldsymbol{n}\cdot\nabla\times\boldsymbol{n}\right)^2 + k_{33}\left[\boldsymbol{n}\times\left(\nabla\times\boldsymbol{n}\right)\right]^2\right\} - \frac{1}{2}\Delta\chi(\boldsymbol{n}\cdot\boldsymbol{B})^2 \\ =& \frac{1}{2}\left\{f\left(\theta\right)\left(\frac{\partial\theta}{\partial z}\right)^2 + h\left(\theta\right)\left(\frac{\partial\phi}{\partial z}\right)^2 - \Delta\chi B^2\cos^2\theta\sin^2\phi\right\}, \end{aligned} \tag{2}$$

where k_{11}, k_{22}, k_{33} and $\Delta\chi$ are the splay, twist, bend elastic constants and the diamagnetic anisotropy of liquid crystal medium, respectively. $\gamma_1 = \left(k_{33} - k_{11}\right)/k_{33}$, $\gamma_2 = \left(k_{33} - k_{22}\right)/k_{33}$ and $f\left(\theta\right) = k_{33}\left(1 - \gamma_1\cos^2\theta\right)$, $h\left(\theta\right) = k_{33}\cos^2\theta\left(1 - \gamma_2\cos^2\theta\right)$.

Before writing out the surface anchoring energies, two points must be explained. One is that the surface anchoring energy of the substrate $g_{\rm s}^-$ at z=0 and $g_{\rm s}^+$ at $z = d$ are equivalent according to the symmetry of the system. The other is that the orthonormal vector $(\boldsymbol{\xi}, \boldsymbol{\eta}, \boldsymbol{e})$ may be described with the elementary coordinates $(\hat{x}, \hat{y}, \hat{z})$ as follows: $\boldsymbol{\xi} = \sin\theta_0\hat{x} - \cos\theta_0\hat{z}$, $\boldsymbol{\eta} = \hat{y}$ and $\boldsymbol{e} = \cos\theta_0\hat{x} + \sin\theta_0\hat{z}$ on the surface of z =0 (where θ_0 is the pretilt angle and $0 < \theta_0 < \pi/2$). In

the form of Eq.(1), the surface anchoring energy can be expressed as

$$\begin{aligned} g_{\mathrm{s}}^{+} =& g_{\mathrm{s}}^{-} \\ =& W_{\xi}\left(\cos\theta\cos\phi\sin\theta_0-\sin\theta\cos\theta_0\right)^2 \\ &+W_{\eta}\left(\cos\theta\sin\phi\right)^2 . \end{aligned} \tag{3}$$

The total free energy F is the sum of the bulk energy and the surface free energy

$$F = S\int_0^d g_{\mathrm{b}}\mathrm{d}z + 2Sg_{\mathrm{s}}^{-}, \tag{4}$$

where S is the area of the substrate. Because the free energy F is the minimal in the stable state of director distribution, we can obtain the torque equilibrium equations and boundary conditions of θ and φ from Euler–Lagrange equations[7]

$$\frac{\partial g_{\mathrm{b}}}{\partial\theta}-\frac{\partial}{\partial z}\frac{\partial g_{\mathrm{b}}}{\partial\theta_z}=0, \tag{5a}$$

$$\frac{\partial g_{\mathrm{b}}}{\partial\phi}-\frac{\partial}{\partial z}\frac{\partial g_{\mathrm{b}}}{\partial\phi_z}=0, \tag{5b}$$

$$\left.\frac{\partial g_{\mathrm{b}}}{\partial\theta_z}\right|_{z=0}-\frac{\partial}{\partial z}\frac{\partial g_{\mathrm{s}}}{\partial\theta_0}=0, \tag{6a}$$

$$\left.\frac{\partial g_{\mathrm{b}}}{\partial\phi_z}\right|_{z=0}-\frac{\partial}{\partial z}\frac{\partial g_{\mathrm{s}}}{\partial\phi_0}=0, \tag{6b}$$

where $\theta_z=\partial\theta/\partial z$ and $\phi_z=\partial\phi/\partial z$.

Substituting Eq.(2) into Eq.(5), we obtain the torque equilibrium equations

$$\begin{aligned} 0 =& \Delta\chi B^2\sin\theta\cos\theta\sin^2\phi+\frac{1}{2}\frac{\partial h}{\partial\theta}\left(\frac{\partial\phi}{\partial z}\right)^2 \\ &-f(\theta)\frac{\partial^2\theta}{\partial z^2}-\frac{1}{2}\frac{\partial f(\theta)}{\partial\theta}\left(\frac{\partial\theta}{\partial z}\right)^2, \end{aligned} \tag{7a}$$

$$\begin{aligned} 0 =& \Delta\chi B^2\cos^2\theta\sin\phi\cos\phi \\ &+\frac{\partial h(\theta)}{\partial\theta}\frac{\partial\phi}{\partial z}\frac{\partial\theta}{\partial z}+h(\theta)\frac{\partial^2\phi}{\partial z^2}. \end{aligned} \tag{7b}$$

Equation (7a) multiplied by $\mathrm{d}\phi/\mathrm{d}z$ is subtracted from Eq.(7b) multiplied by $\mathrm{d}\theta/\mathrm{d}z$, then the torque equilibrium equation also can be expressed as follows:

$$0=\Delta\chi B^2\left(\cos^2\theta\sin\phi\cos\phi\frac{\mathrm{d}\phi}{\mathrm{d}z}-\sin\theta\cos\theta\sin^2\phi\frac{\mathrm{d}\theta}{\mathrm{d}z}\right)+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left[f(\theta)\left(\frac{\partial\theta}{\partial z}\right)^2\right]+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}z}\left[h(\theta)\left(\frac{\partial\phi}{\partial z}\right)^2\right]. \tag{7c}$$

Substituting Eqs.(2) and (3) into Eq.(6), we obtain the boundary conditions

$$\begin{aligned} & f(\theta(0))\frac{\partial\theta}{\partial z}|_{z=0} = \frac{\partial g_{\mathrm{s}}}{\partial\theta} \\ =& 2W_{\xi}\left[\sin\theta(0)\cos\theta(0)\cos^2\theta_0-\sin^2\theta_0\sin\theta(0)\cos\theta(0)\cos^2\phi(0)\right. \\ &\left.-\sin\theta_0\cos\theta_0\cos\phi(0)\left(\cos^2\theta(0)-\sin^2\theta(0)\right)\right]-2W_{\eta}\sin\theta(0)\cos\theta(0)\sin^2\phi(0), \end{aligned} \tag{8a}$$

$$\begin{aligned} & h(\theta(0))\frac{\partial\phi}{\partial z}|_{z=0} = \frac{\partial g_{\mathrm{s}}}{\partial\phi} \\ =& 2W_{\xi}\left[\sin\theta_0\cos\theta_0\sin\theta(0)\cos\theta(0)\sin\phi(0)-\sin^2\theta_0\cos^2\theta(0)\sin\phi(0)\cos\phi(0)\right] \\ &+2W_{\eta}\cos^2\theta(0)\sin\phi(0)\cos\phi(0). \end{aligned} \tag{8b}$$

Equations (7) and (8) are just the general torque equilibrium equation and boundary condition which the director satisfies. Obviously, $\theta(z)=0$, $\phi(z)=\pi/2$, and $\theta(z)$ is the steadiness, $\phi(z)=0$ are the solutions of Eqs.(7) and (8). It implies that there exist the threshold field and the saturation field for the three configurations of LC cell.

When the magnetic field tends to the threshold field or the saturation field, we can simplify the above torque equilibrium equations and boundary conditions as follows:

(1) The torque equilibrium equations and boundary conditions for the threshold field B_{F}.

As the external magnetic field B tends to the threshold field B_{F}, we have $\theta(0)\to\theta_0$, $\phi(0)\to 0$. Using $\cos\phi(0)\approx 1-\frac{1}{2}\sin^2\phi(0)$ and ignoring the infinitesimal quantity $\sin^2\phi(0)$, we can transform boundary conditions (8) to

$$f(\theta(0))\frac{\partial\theta}{\partial z}|_{z=0}=0, \tag{9a}$$

$$h(\theta(0))\frac{\partial\phi}{\partial z}|_{z=0}=2W_{\eta}\cos^2\theta_0\sin\phi(0). \tag{9b}$$

On the other hand, from general torque equilibrium equation (7c), one can see that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{\Delta\chi B^2\cos^2\theta\sin^2\phi+h(\theta)\left(\frac{\partial\phi}{\partial z}\right)^2\right\}\Big|_{\theta(z)}+\frac{\mathrm{d}}{\mathrm{d}z}\left\{\Delta\chi B^2\cos^2\theta\sin^2\phi+f(\theta)\left(\frac{\partial\theta}{\partial z}\right)^2\right\}\Big|_{\phi(z)\equiv0}=0.$$

Then

$$\left\{\Delta\chi B^2\cos^2\theta\sin^2\phi+h(\theta)\left(\frac{\partial\phi}{\partial z}\right)^2\right\}_{\theta(z)}+\left\{\Delta\chi B^2\cos^2\theta\sin^2\phi+f(\theta)\left(\frac{\partial\theta}{\partial z}\right)^2\right\}_{\phi(z)\equiv0}=C_1,$$

where C_1 is a constant. Denoting $\theta(d/2)=\theta_{\mathrm{M}}$, $\phi(d/2)=\phi_{\mathrm{M}}$, in the physics view, we have

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=d/2}=0,$$

then the constant C_1 can be expressed as

$$C_1=\left(\Delta\chi B^2\cos^2\theta_{\mathrm{M}}\sin^2\phi_{\mathrm{M}}\right)_{\theta(z)}+f(\theta_{\mathrm{M}})\left(\frac{\partial\theta_{\mathrm{M}}}{\partial z}\right)^2_{\phi\equiv0}.$$

We then obtain

$$\left(\Delta\chi B^2\cos^2\theta_{\mathrm{M}}\sin^2\phi_{\mathrm{M}}\right)_{\theta(z)}+f(\theta_{\mathrm{M}})\left(\frac{\partial\theta_{\mathrm{M}}}{\partial z}\right)^2_{\phi(z)\equiv0}=\left\{\Delta\chi B^2\cos^2\theta\sin^2\phi+h(\theta)\left(\frac{\partial\phi}{\partial z}\right)^2\right\}_{\theta(z)}+\left\{f(\theta)\left(\frac{\partial\theta}{\partial z}\right)^2\right\}_{\phi(z)\equiv0}.$$

When the external magnetic field B tends to the threshold field B_{F}, we have also $\theta_{\mathrm{M}}\to\theta(0)\to\theta_0$, $\phi_{\mathrm{M}}\to\phi(0)\to0$. So we can rewrite the above torque equilibrium equation as two equivalent equations

$$0=f(\theta)\left(\frac{\partial\theta}{\partial z}\right)^2-f(\theta_{\mathrm{M}})\left(\frac{\partial\theta_{\mathrm{M}}}{\partial z}\right)^2\quad(\text{for }\ \phi(z)\equiv0),\tag{10a}$$

$$0=\Delta\chi B^2\cos^2\theta_0\left(\sin^2\phi-\sin^2\phi_{\mathrm{M}}\right)+h(\theta)\left(\frac{\partial\phi}{\partial z}\right)^2\quad(\text{for constant }\ \theta(z)).\tag{10b}$$

(2) The torque equilibrium equations and boundary conditions for the saturation field B_{s}

As the external magnetic field B tends to the saturation field B_{s}, we have $\theta(0)\to0$, $\phi(0)\to\pi/2$. Using $\sin\phi(0)\approx1-\frac{1}{2}\cos^2\phi(0)$, $\cos\theta(0)\approx1-\frac{1}{2}\sin^2\theta(0)$ and ignoring infinitesimal quantities $\sin^2\phi(0)$, $\cos^2\phi(0)$ we can obtain boundary conditions from Eq.(8),

$$f(\theta(0))\frac{\partial\theta}{\partial z}|_{z=0}=2\left(W_\xi\cos^2\theta_0-W_\eta\right)\sin\theta(0)-2W_\xi\sin\theta_0\cos\theta_0\cos\phi(0),\tag{11a}$$

$$h(\theta(0))\frac{\partial\phi}{\partial z}|_{z=0}=2\left(W_\eta-W_\xi\sin^2\theta_0\right)\cos\phi(0)+2W_\xi\sin\theta_0\cos\theta_0\sin\theta(0).\tag{11b}$$

On the other hand, from general torque equilibrium equation (7c), one also can see that

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{-\Delta\chi B^2\cos^2\theta\cos^2\phi+h(\theta)\left(\frac{\partial\phi}{\partial z}\right)^2\right\}\Big|_{\theta\equiv0}+\frac{\mathrm{d}}{\mathrm{d}z}\left\{-\Delta\chi B^2\sin^2\theta\sin^2\phi+f(\theta)\left(\frac{\partial\theta}{\partial z}\right)^2\right\}\Big|_{\phi(z)\equiv\pi/2}=0.$$

Then

$$\left\{-\Delta\chi B^2\cos^2\theta\cos^2\phi+h(\theta)\left(\frac{\partial\phi}{\partial z}\right)^2\right\}_{\theta\equiv0}+\left\{-\Delta\chi B^2\sin^2\theta\sin^2\phi+f(\theta)\left(\frac{\partial\theta}{\partial z}\right)^2\right\}_{\phi\equiv\pi/2}=C_2,$$

where C_2 is a constant. In the physical view

$$\left.\frac{\partial\theta}{\partial z}\right|_{z=d/2}=0,\quad\left.\frac{\partial\phi}{\partial z}\right|_{z=d/2}=0,$$

then the constant C_2 can be expressed as

$$C_2=-\Delta\chi B^2\cos^2\theta_{\mathrm{M}}\cos^2\phi_{\mathrm{M}}-\Delta\chi B^2\sin^2\theta_{\mathrm{M}}\sin^2\phi_{\mathrm{M}}.$$

We then obtain

$$0=\left\{\Delta\chi B^2\left(\cos^2\theta_{\rm M}\cos^2\phi_{\rm M}-\cos^2\phi\cos^2\phi\right)+h\left(\theta\right)\left(\frac{\partial\phi}{\partial z}\right)^2\right\}_{\theta\equiv 0}$$
$$+\left\{\Delta\chi B^2\left(\sin^2\theta_{\rm M}\sin^2\phi_{\rm M}-\sin^2\theta\sin^2\phi\right)+f\left(\theta\right)\left(\frac{\partial\theta}{\partial z}\right)^2\right\}_{\phi\equiv\pi/2}.$$

When the external magnetic field B tends to the saturation field $B_{\rm s}$, we have also $\theta(z)\to\theta_{\rm M}\to 0$, $\phi(z)\to\phi_{\rm M}\to\pi/2$. So we can rewrite the above torque equilibrium equation as two equivalent equations

$$0=\Delta\chi B^2\left(\sin^2\theta_{\rm M}-\sin^2\theta\right)+f\left(\theta\right)\left(\frac{\partial\theta}{\partial z}\right)^2 \quad (\text{for } \phi\equiv\pi/2), \tag{12a}$$

$$0=\Delta\chi B^2\left(\cos^2\phi_{\rm M}-\cos^2\phi\right)+h\left(\theta\right)\left(\frac{\partial\phi}{\partial z}\right)^2 \quad (\text{for } \theta\equiv 0). \tag{12b}$$

By this time, the general torque equilibrium equations (7) and boundary conditions (8) are obtained. Based on them, the torque equilibrium equations (10), (12) and the boundary conditions (9), (11) of the threshold field and saturation field are also obtained. Above equations are the basis for discussions.

3. Freedericksz transition

3.1. Homogeneous configuration

To derive the threshold magnetic field $B_{\rm F}$ of the Fréedericksz transition, we suppose that the original alignment of liquid crystal molecules is described by $\phi(0)=\phi_{\rm M}=0$ and $\theta(0)=\theta_{\rm M}=\theta_0$. When the external magnetic field B reaches the threshold field $B_{\rm F}$, we have $\phi(z)\to\phi_{\rm M}\to 0$, $\theta(z)\to\theta_{\rm M}\to\theta_0$, $h(\theta)\to k_{33}\cos^2\theta_0\left(1-\gamma_2\cos^2\theta_0\right)$, $f(\theta)\to k_{33}\left(1-\gamma_1\cos^2\theta_0\right)$. Substituting the above approximations into torque equilibrium equations (12b) of the threshold field, we have

$$\frac{\partial\phi}{\partial z}=\left[\frac{\Delta\chi B^2\left(\sin^2\phi_{\rm M}-\sin^2\phi\right)}{k_{33}\left(1-\gamma_2\cos^2\theta_0\right)}\right]^{\frac{1}{2}}. \tag{13}$$

We introduce a new variable α under the condition of $\phi\le\phi_{\rm M}$ as follows,

$$\sin\phi=\sin\phi_{\rm M}\sin\alpha \quad \left(\sin\phi\left(0\right)=\sin\phi_{\rm M}\sin\alpha^0\right).$$

From Eq.(13) we have

$$\frac{d}{2}=\int_{\alpha^0}^{\pi/2}\left[\frac{k_{33}\left(1-\gamma_2\cos^2\theta_0\right)}{\Delta\chi B^2}\right]^{\frac{1}{2}}\times\frac{{\rm d}\alpha}{\sqrt{1-\sin^2\phi_{\rm M}\sin^2\alpha}}. \tag{14}$$

Denoting $B_0=\pi/d\sqrt{k_{22}/\Delta\chi}$ which is the threshold magnetic field of a field-induced twist cell with pretilt angle $\theta_0=0$ and strong boundary coupling, after integration of Eq.(14), we obtain

$$\frac{\pi}{2}\frac{B_{\rm F}}{B_0}\left(\frac{k_{22}}{k_{33}\left(1-\gamma_2\cos^2\theta_0\right)}\right)^{\frac{1}{2}}=\lim_{\phi_{\rm M}\to 0}\left[\frac{\pi}{2}-\sin^{-1}\frac{\sin\phi\left(0\right)}{\sin\phi_{\rm M}}\right]. \tag{15}$$

On the other hand, boundary condition (9b) of the threshold field gives

$$\left[k_{33}\left(1-\gamma_2\cos^2\theta_0\right)\right]^{\frac{1}{2}}\sqrt{\Delta\chi B^2}\times\left(\sin^2\phi_{\rm M}-\sin^2\phi\left(0\right)\right)^{\frac{1}{2}}=2W_\eta\sin\phi\left(0\right).$$

When $\phi(0)\to 0$, we have $\phi_{\rm M}\to 0$. The Above equation can be expressed as

$$\left[k_{33}\left(1-\gamma_2\cos^2\theta_0\right)\right]^{\frac{1}{2}}\sqrt{\Delta\chi B_{\rm F}^2}=2W_\eta\lim_{\phi_{\rm M}\to 0}\frac{\sin\phi\left(0\right)}{\left(\sin^2\phi_{\rm M}-\sin^2\phi\left(0\right)\right)^{\frac{1}{2}}}=2W_\eta\cot\left\{\lim_{\phi_{\rm M}\to 0}\left[\frac{\pi}{2}-\sin^{-1}\frac{\sin\phi(0)}{\sin\phi_{\rm M}}\right]\right\}. \tag{16}$$

We can see that the threshold field is related to the azimuthal anchoring only.

Defining $2W_\eta=\dfrac{\pi k_{22}}{\lambda d}$, $u'=B_{\rm F}/B_0$, from Eq.(16) the reduced threshold magnetic field u' is obtained finally

$$u'=\frac{1}{\lambda}\left(\frac{k_{22}}{k_{33}\left(1-\gamma_2\cos^2\theta_0\right)}\right)^{\frac{1}{2}}\times\cot\left\{\left(\frac{k_{22}}{k_{33}\left(1-\gamma_2\cos^2\theta_0\right)}\right)^{\frac{1}{2}}\frac{\pi}{2}u'\right\}. \tag{17}$$

It is instructive to compare the present paper with previous studies. In the case of $\theta_0 = 0$ and $\theta_0 = \pi/2$, $k_{22} = k_{33}$, which represent the second Freedericksz transition and the third Freedericksz transition, respectively, Eq.(17) reduces to

$$u' = \frac{1}{\lambda} \cot\left(\frac{\pi}{2} u'\right).$$

This result is consistent with that in Ref.[11]. Here, we can use one equation to denote the two Freedericksz transitions.

3.2. Splay configuration

In this case, we should make the zero point of z-coordinate at the centre of LC layer because of the asymmetric surface conditions. With the similar analysis in Section 2, the boundary conditions (9) for the bottom substrate becomes

$$f\left(\theta\left(-d/2\right)\right) \frac{\partial \theta}{\partial z}\Big|_{z=-\frac{d}{2}} = 0,$$
$$h\left(\theta\left(-d/2\right)\right) \frac{\partial \phi}{\partial z}\Big|_{z=-\frac{d}{2}} = 2W_\eta \cos^2\theta_0 \sin\phi\left(-d/2\right).$$

The torque equilibrium equations (10) becomes

$$0 = f(\theta)\left(\frac{\partial \theta}{\partial z}\right)^2 \quad (\text{for } \phi \equiv 0),$$
$$0 = \Delta\chi B^2 \cos^2\theta\left(\sin^2\phi - \sin^2\phi_{\mathrm{M}}\right) + h(\theta)\left(\frac{\partial \phi}{\partial z}\right)^2$$
$$(\text{for constant } \theta(z)).$$

In this configuration, we know $\phi\left(-d/2\right) = \phi_{\mathrm{M}} = 0$, $\theta\left(-d/2\right) = \theta_0$ and $\theta\left(d/2\right) = -\theta\left(-d/2\right)$ for $B < B_{\mathrm{F}}$, so we have $\theta(0) \equiv 0$. With the same analytical procedure as above, we have

$$\frac{\partial \phi}{\partial z} = \pm\left[\frac{\Delta\chi B^2\left(\sin^2\phi_{\mathrm{M}} - \sin^2\phi\right)}{k_{33}\left(1 - \gamma_2 \cos^2\theta_0\right)}\right]^{\frac{1}{2}},$$

where "+" is for $-d/2 < z < 0$ and "–" is for $0 < z < d/2$. We introduce a new variable β under the condition of $\phi \le \phi_{\mathrm{M}}$ as follows

$$\sin\phi = \sin\phi_{\mathrm{M}} \sin\beta \quad \left(\sin\phi\left(-d/2\right) = \sin\phi_{\mathrm{M}} \sin\beta^0\right).$$

For $-d/2 < z < 0$, the integration of the above equation yields

$$\int_{-d/2}^{0}\left[\frac{\Delta\chi B^2}{k_{33}\left(1 - \gamma_2\cos^2\theta\right)}\right]^{\frac{1}{2}} \mathrm{d}z$$
$$= \int_{\beta^0}^{\pi/2} \frac{\mathrm{d}\alpha}{\sqrt{1 - \sin^2\phi_{\mathrm{M}} \sin^2\beta}},$$
$$\frac{d}{2}\sqrt{\frac{\Delta\chi B^2}{k_{22}}}\left[\frac{k_{22}}{k_{33}\left(1 - \gamma_2\cos^2\theta_0\right)}\right]^{\frac{1}{2}}$$
$$= \lim_{\phi_{\mathrm{M}} \to 0}\left[\frac{\pi}{2} - \sin^{-1}\frac{\sin\phi\left(-d/2\right)}{\sin\phi_{\mathrm{M}}}\right].$$

For $0 < z < d/2$, the same result as the above can be obtained. Denoting $B_0 = \pi/\mathrm{d}\sqrt{k_{22}/\Delta\chi}$, which is the threshold magnetic field for a field-induced twist cell with $\theta_0 = 0$ and strong boundary coupling, we can obtain

$$\frac{\pi}{2}\frac{B_{\mathrm{F}}}{B_0}\left[\frac{k_{22}}{k_{33}\left(1 - \gamma_2\cos^2\theta_0\right)}\right]^{\frac{1}{2}}$$
$$= \lim_{\phi_{\mathrm{M}} \to 0}\left[\frac{\pi}{2} - \sin^{-1}\frac{\sin\phi\left(-d/2\right)}{\sin\phi_{\mathrm{M}}}\right].$$

The surface boundary condition (9b) gives

$$\left[k_{33}\left(1 - \gamma_2\cos^2\theta_0\right)\right]^{\frac{1}{2}}\sqrt{\Delta\chi B^2}$$
$$\times\left(\sin^2\phi_{\mathrm{M}} - \sin^2\phi\left(-d/2\right)\right)^{\frac{1}{2}}$$
$$= 2W_\eta \sin\phi\left(-d/2\right).$$

When $\phi(0) \to 0$, we have

$$\left[k_{33}\left(1 - \gamma_2\cos^2\theta_0\right)\right]^{\frac{1}{2}}\sqrt{\Delta\chi B_{\mathrm{F}}^2}$$
$$= 2W_\eta \lim_{\phi_{\mathrm{M}} \to 0}\frac{\sin\phi\left(-d/2\right)}{\left(\sin^2\phi_{\mathrm{M}} - \sin^2\phi\left(-d/2\right)\right)^{\frac{1}{2}}}$$
$$= 2W_\eta \cot\left\{\lim_{\phi_{\mathrm{M}} \to 0}\left[\frac{\pi}{2} - \sin^{-1}\frac{\sin\phi(-d/2)}{\sin\phi_{\mathrm{M}}}\right]\right\}.$$

We see that, in this configuration, the bulk and surface equation are the same as those of the homogeneous configuration, so the reduced threshold magnetic field is also given by

$$u' = \frac{1}{\lambda}\left(\frac{k_{22}}{k_{33}\left(1 - \gamma_2\cos^2\theta_0\right)}\right)^{\frac{1}{2}}$$
$$\times \cot\left\{\left[\frac{k_{22}}{k_{33}\left(1 - \gamma_2\cos^2\theta\right)}\right]^{\frac{1}{2}}\frac{\pi}{2}u'\right\}. \quad (18)$$

3.3. Pi-cell configuration

With the same coordinate transformation as that made for the splay configuration, we have, in this case, $\phi\left(-d/2\right) = \phi_{\mathrm{M}} = 0$, $\theta\left(-d/2\right) = \theta_0$ and $\theta\left(d/2\right) =$

$\pi-\theta(-d/2)$ for $B<B_{\mathrm{F}}$, so $\theta(0)=\pi/2$ is obtained under the threshold field. With the identical analytical procedure as that for splay configuration, the reduced threshold magnetic field can be obtained

$$u'=\frac{1}{\lambda}\left(\frac{k_{22}}{k_{33}\left(1-\gamma_2\cos^2\theta_0\right)}\right)^{\frac{1}{2}} \times\cot\left\{\left[\frac{k_{22}}{k_{33}\left(1-\gamma_2\cos^2\theta\right)}\right]^{\frac{1}{2}}\frac{\pi}{2}u'\right\}. \quad (19)$$

In this section, we have deduced and proved that the formula of the threshold field for the three kinds of configurations, namely homogeneous, splay and Pi cells, has the identical expression. From Eq.(17) or (18) or (19), we find that the threshold magnetic field is determined not only by elastic constants k_{ii}, surface azimuthal anchoring $W_\eta(W_\eta=\pi k_{22}/2\lambda d)$ but also by pretilt angle θ_0. However, the polar anchoring W_ξ has no effect on the threshold field. The threshold field is indeed dependent only on the azimuthal anchoring W_η. Thus W_η can be determined if the threshold field has been given. Figure 2 shows the dependence of threshold field on anchoring coefficients (λ) under different pretilt angles. The threshold field decreases as the anchoring coefficient decreases.

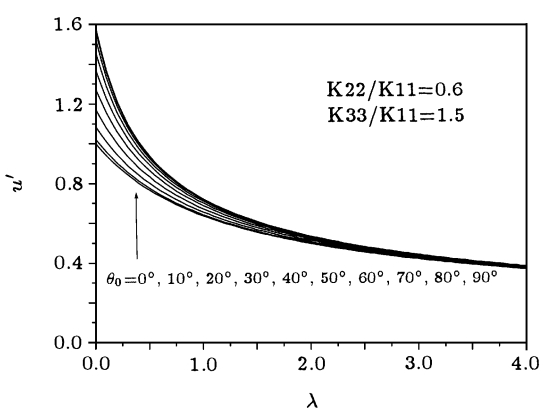


Fig.2. The pretilt angles dependence of the reduced threshold field for different anchoring coefficients.

4. Saturation field

4.1. Homogeneous configuration

When the field reaches the saturation field B_{s}, the director is aligned along the field completely. To derive the saturation field B_{s}, we need to suppose $\theta(0)\to\theta_{\mathrm{M}}\to 0$ and $\phi(0)\to\phi_{\mathrm{M}}\to\pi/2$ when $B\to B_{\mathrm{s}}$,[15] i.e., once $\phi\equiv\pi/2$, we have $\theta\equiv 0$. The saturation field can be analytically solved through solving Eqs.(11) and (12) in the limit $\theta\to 0$, $\phi\to\pi/2$, $f(\theta)\to k_{11}$, $h(\theta)\to k_{22}$ when $B\to B_{\mathrm{s}}$. It is convenient to introduce the following two variables, α and β, for $0<z<d/2$.

$$\sin\theta=\frac{\sin\theta_{\mathrm{M}}}{\cos\alpha}\qquad\left(\sin\theta(0)=\frac{\sin\theta_{\mathrm{M}}}{\cos\alpha(0)}\right),$$

$$\cos\phi=\frac{\cos\phi_{\mathrm{M}}}{\cos\beta}\qquad\left(\cos\phi(0)=\frac{\cos\phi_{\mathrm{M}}}{\cos\beta(0)}\right).$$

$$\mathrm{d}\theta=\frac{\sin\theta_{\mathrm{M}}}{\sqrt{1-\dfrac{\sin^2\theta_{\mathrm{M}}}{\cos^2\alpha}}}\frac{\sin\alpha}{\cos^2\alpha}\mathrm{d}\alpha,$$

$$\mathrm{d}\phi=-\frac{\cos\phi_{\mathrm{M}}}{\sqrt{1-\dfrac{\cos^2\phi_{\mathrm{M}}}{\cos^2\beta}}}\frac{\sin\beta}{\cos^2\beta}\mathrm{d}\beta.$$

Integration of Eq.(12) can be performed analytically to give

$$\begin{cases}\dfrac{d}{2}=\displaystyle\int_{\alpha(0)}^{0}\sqrt{\frac{k_{11}}{\Delta\chi B^2}}\frac{1}{\sqrt{\cos^2\alpha-\sin^2\theta_{\mathrm{M}}}}\mathrm{d}\alpha,\\ \dfrac{d}{2}=\displaystyle\int_{0}^{\beta(0)}\sqrt{\frac{k_{22}}{\Delta\chi B^2}}\frac{1}{\sqrt{\cos^2\beta-\cos^2\phi_{\mathrm{M}}}}\mathrm{d}\beta.\end{cases}$$

Taking $\theta_{\mathrm{M}}=0$ and $\phi_{\mathrm{M}}=\pi/2$, eliminating $\sin\theta_{\mathrm{M}}$ and $\cos\phi_{\mathrm{M}}$ from the above equations, we have

$$\begin{cases}\dfrac{d}{2}\sqrt{\dfrac{\Delta\chi B_{\mathrm{s}}^2}{k_{11}}}=\displaystyle\int_0^{\alpha(0)}\frac{\mathrm{d}\alpha}{\cos\alpha},\\ \dfrac{d}{2}\sqrt{\dfrac{\Delta\chi B_{\mathrm{s}}^2}{k_{22}}}=\displaystyle\int_0^{\beta(0)}\frac{\mathrm{d}\beta}{\cos\beta}.\end{cases} \quad (20)$$

Using the integration formula

$$\int\frac{1}{b+c\cos(ax)}\mathrm{d}x =\frac{1}{a\sqrt{c^2-b^2}}\operatorname{arc\,tanh}\frac{\sqrt{c^2-b^2}\sin(ax)}{c+b\cos(ax)}\quad(|b|<|c|),$$

we can obtain the integral of Eq.(20) (where $b=0$, $c=a=1$)

$$\begin{cases}\sin\alpha(0)=\tanh\left(\dfrac{d}{2}\sqrt{\dfrac{\Delta\chi B_{\mathrm{s}}^2}{k_{11}}}\right),\\ \sin\beta(0)=\tanh\left(\dfrac{d}{2}\sqrt{\dfrac{\Delta\chi B_{\mathrm{s}}^2}{k_{22}}}\right).\end{cases} \quad (21)$$

In a similar process, in the limit of $\phi\to\pi/2$ and $\theta\to 0$, the integral of Eq.(11) leads to

$$\begin{cases} f(\theta(0))\dfrac{\partial\theta}{\partial z}|_{z=0} = 2\left(W_\xi\cos^2\theta_0 - W_\eta\right)\sin\theta(0) - 2W_\xi\sin\theta_0\cos\theta_0\cos\phi(0), \\ h(\theta(0))\dfrac{\partial\phi}{\partial z}|_{z=0} = 2\left(W_\eta - W_\xi\sin^2\theta_0\right)\cos\phi(0) + 2W_\xi\sin\theta_0\cos\theta_0\sin\theta(0); \end{cases}$$

$$\begin{cases} \sin\alpha(0) = \dfrac{2W_\eta - 2W_\xi\cos^2\theta_0 + [2W_\xi\sin\theta_0\cos\theta_0]\left(\dfrac{\cos\phi(0)}{\sin\theta(0)}\right)}{B_s\sqrt{k_{11}\Delta\chi}}, \\ \sin\beta(0) = \dfrac{2W_\eta - 2W_\xi\sin^2\theta_0 + [2W_\xi\sin\theta_0\cos\theta_0]\left(\dfrac{\sin\theta(0)}{\cos\phi(0)}\right)}{B_s\sqrt{k_{22}\Delta\chi}}. \end{cases} \tag{22}$$

Eliminating $\alpha(0)$ and $\beta(0)$ from Eqs.(21) and (22) yields

$$\begin{cases} \left[B_s\sqrt{k_{11}\Delta\chi}\tanh\left(\dfrac{d}{2}\sqrt{\dfrac{\Delta\chi B_s^2}{k_{11}}}\right) - 2W_\eta + 2W_\xi\cos^2\theta_0\right]\sin(\theta(0)) = [2W_\xi\sin\theta_0\cos\theta_0]\cos(\phi(0)), \\ \left[B_s\sqrt{k_{22}\Delta\chi}\tanh\left(\dfrac{d}{2}\sqrt{\dfrac{\Delta\chi B_s^2}{k_{22}}}\right) - 2W_\eta + 2W_\xi\sin^2\theta_0\right]\cos(\phi(0)) = [2W_\xi\sin\theta_0\cos\theta_0]\sin(\theta(0)). \end{cases} \tag{23}$$

Then, we obtain the analytical saturation field as follows

$$\frac{B_s\sqrt{k_{11}\Delta\chi}\tanh\left(\dfrac{d}{2}\sqrt{\dfrac{\Delta\chi B_s^2}{k_{11}}}\right) - 2W_\eta + 2W_\xi\cos^2\theta_0}{2W_\xi\sin\theta_0\cos\theta_0} = \frac{2W_\xi\sin\theta_0\cos\theta_0}{B_s\sqrt{k_{22}\Delta\chi}\tanh\left(\dfrac{d}{2}\sqrt{\dfrac{\Delta\chi B_s^2}{k_{22}}}\right) - 2W_\eta + 2W_\xi\sin^2\theta_0}. \tag{24}$$

We also can define $B_0 = \pi/\mathrm{d}\sqrt{k_{22}/\Delta\chi}$, $2W_\eta = \dfrac{\pi k_{22}}{\lambda d}$, $u'' = B_s/B_0$, $r = \sqrt{k_{22}/k_{11}}$ and $\gamma = W_\eta/W_\xi$, Eq.(24) can be rewritten as

$$\frac{\gamma\lambda u''\tanh\left(\dfrac{\pi}{2}u''r\right)/r - \gamma + \cos^2\theta_0}{\sin\theta_0\cos\theta_0} = \frac{\sin\theta_0\cos\theta_0}{\gamma\lambda u''\tanh\left(\dfrac{\pi}{2}u''\right) - \gamma + \sin^2\theta_0}, \tag{25}$$

$$\begin{aligned} &(\gamma-1) + \gamma\lambda^2 u''^2\tanh\left(\frac{\pi}{2}u''r\right)\tanh\left(\frac{\pi}{2}u''\right)/r - \left(\gamma - \sin^2\theta_0\right)\lambda u''\tanh\left(\frac{\pi}{2}u''r\right)/r \\ &- \left(\gamma - \cos^2\theta_0\right)\lambda u''\tanh\left(\frac{\pi}{2}u''\right) = 0. \end{aligned} \tag{26}$$

Equation (26) is just the final result of the saturation field. It is necessary to discuss two especial saturation fields from Eq.(26)

(1) $\theta_0 = 0$. It corresponds to a parallel-aligned liquid crystal cell. Substituting $\theta_0 = 0$ into Eq.(26) yields

$$\left[(\gamma-1) - \gamma\lambda u''\tanh\left(\frac{\pi}{2}u''r\right)/r\right] \times \left[1 - \lambda u''\tanh\left(\frac{\pi}{2}u''\right)\right] = 0.$$

In this case, the polar angle is unchanged and W_ξ has no effect on B_s. Consequently, γ is an unmeaning quantity. So the saturation field is determined by

$$\lambda u''\tanh\left(\frac{\pi}{2}u''\right) - 1 = 0.$$

(2) $\theta_0 = \pi/2$. It corresponds to a perpendicularly aligned liquid crystal cell. Substituting $\theta_0 = \pi/2$ into Eq.(26) leads to

$$\left[1 - \lambda u''\tanh\left(\frac{\pi}{2}u''r\right)/r\right] \times \left[\gamma - 1 - \gamma\lambda u''\tanh\left(\frac{\pi}{2}u''\right)\right] = 0.$$

In this case, only polar anchoring W_ξ exists. So γ is also an unmeaning quantity. The saturation field

is determined by

$$\lambda u'' \tanh\left(\frac{\pi}{2}u''r\right)/r - 1 = 0.$$

We show the saturation field of the cell (homogeneous configuration) under different anchoring coefficients and pretilt angles in Fig.3. The saturation field increases as pretilt angle increases and anchoring coefficient decreases. The ratio γ of twist anchoring energy to polar anchoring energy is 0.1. In practice, γ is usually less than 0.1. Because the azimuthal anchoring is one or two orders weaker than the polar anchoring according to experimental results, we cannot separate them from each other in one figure.

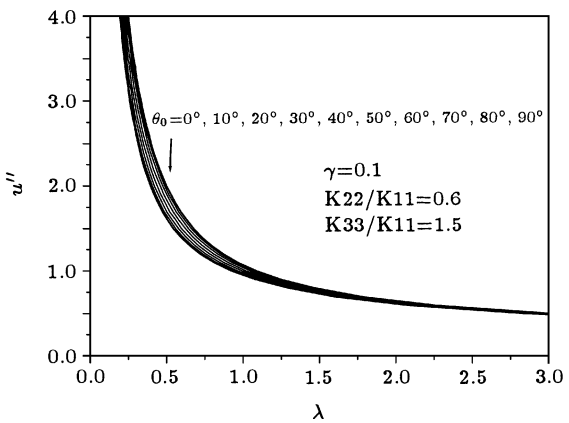


Fig.3. The anchoring coefficients dependence of the reduced saturation field for different pretilt angles.

We show the saturation field in Figs.4(a) and 4(b) for $\lambda = 0.2$ and $\lambda = 0.5$. From Fig.4, we can see that γ has no effect on the saturation field as the pretilt angle tends to zero or $\pi/2$. The largest difference appears at $\theta_0 = 45°$, corresponding to $B = 0.06B_0$ for $\lambda = 0.2$ and $B = 0.018B_0$ for $\lambda = 0.5$. As the threshold and saturation fields are known, the twist and polar anchoring energies can be solved using Eqs.(17) and (26). The results show that the LC cell with larger pretilt angle is needed to distinguish the difference of the saturation field under different γ's if we determine the twist and polar anchoring energies by this method.

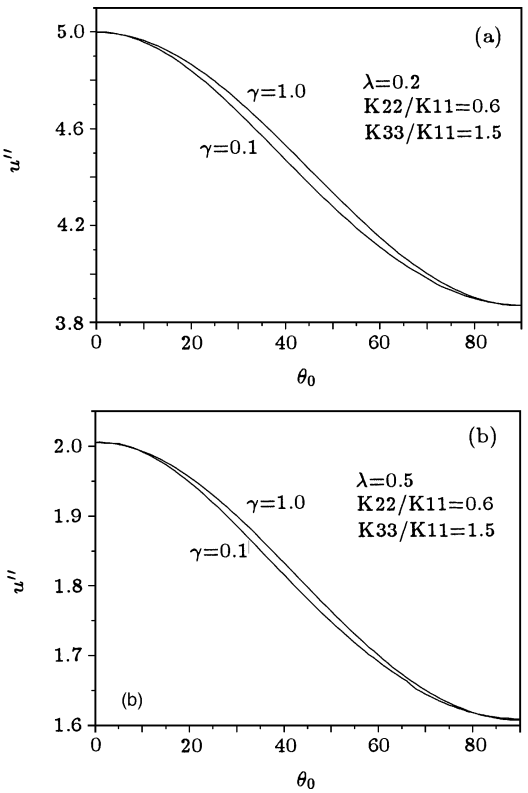


Fig.4. The saturation field for different γ's and the surface anchoring coefficients: (a) $\lambda = 0.2$, (b) $\lambda = 0.5$.

4.2. Splay and Pi-cell configuration

For the splay and Pi-cell configuration, the same expression for the saturation field can be deduced by the same analytical procedure as the homogeneous configuration. Because the three configurations have the same equation for surface boundary conditions on the bottom substrate, and the same bulk torque equilibrium equations near the saturation field even though they have different surface boundary conditions on the top substrate, the deduction of the analytical expressions for the other two configurations is omitted here.

5. Conclusion

We have analysed the threshold and saturation fields of a field-induced twist LC cell with nonzero pretilt angle and symmetric weak anchoring boundaries based on the two-parameter formula for describing surface anchoring energy proposed recently. The results show that both the second and the third Freedericksz transitions, and both the threshold and the saturation fields can be denoted by the same equation. The polar anchoring has no effect on the threshold field. The twist and polar anchorings are coupling with each other and have great influence on the saturation field. If the threshold field and pretilt angle are given, the twist anchoring energy can be obtained from Eq.(17). Also, if the saturation field is measured, the ratio γ can be solved from Eq.(26), then the polar anchoring energy can be obtained. If we use the

electric field to test the anchoring energy (this mode is the in-plane switching mode) larger electrodes gap is needed, because a uniform electric field is needed; and the threshold and saturation fields can be exactly measured when the pretilt angle tends to zero or $\pi/2$.

For an actual LC cell, the pretilt angle is usually not zero or $\pi/2$, so the analytical results of the present paper are useful for the field-induced twist LC cell and will be beneficial to the surface physics of liquid crystal displays.

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