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Restrictions on negative energy density for the Dirac field in flat spacetime*

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This paper investigates the quantum Dirac field in $n+1$-dimensional flat spacetime and derives a lower bound in the form of quantum inequality on the energy density averaged against spacetime sampling functions. The state-independent quantum inequality derived in the present paper is similar to the temporal quantum energy inequality and it is stronger for massive field than for massless one. It also presents the concrete results of the quantum inequality in 2 and 4-dimensional spacetimes.

Keywords: negative energy density, quantum inequality, spacetime averaged
PACC: 0420, 0370, 0490

1. Introduction

In classical general relativity, the stress tensor is required to satisfy the classical energy conditions so that the solution to the Einstein field equation is physically reasonable. For instance, the stress tensor obeys the weak energy condition (WEC): $T_{\mu\nu}u^\mu u^\nu \geq 0$ for all timelike observers, where u^μ is the observer's four-velocity. WEC suggests that all observers must see positive energy density. The classical energy conditions are obeyed by all observed forms of classical matter and energy and play important roles in proving singularity theorems[1,2] and positive mass theorems.[3,4]

However, the classical energy conditions are doomed to fail in quantum field theory. It has been demonstrated that the energy density can become negative for certain states in quantum fields.[5] Furthermore, the Casimir effect[6,7] and the squeezed state of light[8] studied experimentally have given indirect support to the existence of negative energy densities. At the same time, if no restrictions were placed on negative energy, then many exotic effects including the violation of the second law of thermodynamics[9] or of cosmic censorship,[10,11] and superluminal travel by 'warp drive'[12] or time machines[13] could occur.

Certainly, the quantum field theory by itself contains mechanisms to prevent the bizarre results from emergence. Now people have found severe constraints on negative energy.[14−31] The constraints pioneered by Ford have been developed into the quantum energy inequality (QEI) which is a bound on the average of energy density weighted by a sampling function. The QEIs have been found by various authors for different quantum fields.[20,21,23−25,27,31] As for the Dirac field, Vollick[27] obtained an optimal QEI

$$\langle\rho\rangle_T = \int \mathrm{d}t \,\langle :T_{00}: \rangle f(t) \geq -\frac{1}{24\pi} \int_{-\infty}^{\infty} \mathrm{d}t \,\frac{f'(t)^2}{f(t)}, \tag{1}$$

which is state-independent in two-dimensional flat spacetime. Here the normal ordered energy density is measured by an inertial observer whose proper time is t and $f(t)$ is a smooth and positive sampling function whose integral is unity. Fewster and Mistry[23] derived a QEI in four-dimensional flat spacetime

$$\langle\rho\rangle_T \geq -\frac{1}{12\pi^3} \int_m^{\infty} \mathrm{d}u \,|\widehat{f^{1/2}}(u)|^2 \, u^4 \, Q_3^D(u/m), \tag{2}$$

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where

$$Q_3^D(x) = \left(1 - \frac{1}{x^2}\right)^{1/2} \left(1 - \frac{5}{2x^2}\right) + \frac{3}{2x^4} \log[x + (x^2-1)^{1/2}] \qquad (3)$$

is a monotonically increasing function for x in $(1, \infty)$ with the maximum value of one. The QEIs constrain the magnitude and duration of negative energy density and have been used to show that the above mentioned exotic phenomena are impossible in reality.[17,32]

In these QEIs the energy densities are all averaged along timelike worldlines. It is natural to enquire whether there exists any QEI with the energy density weighted along a spacelike curve or over a spacetime volume.[33] In this regard, Vollick,[27] and Fewster and Hollands[34] found that space averaged QEI exists for the scalar field and the conformal field respectively in two dimensions, but Ford and colleagues demonstrated that similar conclusions do not apply to the four-dimensional case.[35] It should be noted that the bound on the spacetime-averaged energy density, called spacetime quantum energy inequality (STQEI), might hold as argued by Helfer.[36] In two-dimensional flat spacetime, Borde *et al* established a STQEI for the massless scalar field[37] and Fewster and Hollands found one for the conformal field.[34] So far, there has not been much work to search for STQEI in other fields or in higher dimensions.

In this paper, we will generalize the methods developed by Fewster and Mistry in deriving QEI in the flat spacetime of four dimensions[23] to $n+1$ dimensions and obtain a lower bound on the energy density averaged against spacetime sampling functions for the Dirac field. The bound has the form of quantum inequality and amounts to the QEI. In the end we will evaluate our bound explicitly for the 2 and 4-dimensional spacetime.

We use the unit system where $c = \hbar = 1$ and the metric sign convention $(+ - \cdots -)$.

2. Derivation of the quantum inequality for the Dirac field

Let us begin with some basics of quantum Dirac field theory. For the Dirac field the Lagrangian density is

$$L = \frac{\mathrm{i}}{2}\overline{\psi}\gamma^\mu(-\overleftarrow{\partial^\mu} + \overrightarrow{\partial^\mu})\psi - m\overline{\psi}\psi, \qquad (4)$$

from which the symmetrized stress tensor[38] can be calculated:

$$T_{\mu\nu} = \frac{\mathrm{i}}{4}\left[\overline{\psi}\gamma^\mu(-\overleftarrow{\partial^\nu} + \overrightarrow{\partial^\nu})\psi + \overline{\psi}\gamma^\nu(-\overleftarrow{\partial^\mu} + \overrightarrow{\partial^\mu})\psi\right]. \qquad (5)$$

The field operator can be expanded as

$$\psi(x) = \sum_k \sum_{\alpha=1,2} [b_\alpha(\boldsymbol{k})u^\alpha(\boldsymbol{k})\mathrm{e}^{\mathrm{i}k\cdot x} + d_\alpha^\dagger(\boldsymbol{k})v^\alpha(\boldsymbol{k})\mathrm{e}^{-\mathrm{i}k\cdot x}], \qquad (6)$$

where $b_\alpha(\boldsymbol{k})$ and $b_\alpha^\dagger(\boldsymbol{k})$ are the annihilation and creation operators for the electron respectively, while $d_\alpha(\boldsymbol{k})$ and $d_\alpha^\dagger(\boldsymbol{k})$ are the respective annihilation and creation operators for the positron. The four operators anticommute except in the cases

$$\{b_\alpha(\boldsymbol{k}), b_{\alpha'}^\dagger(\boldsymbol{k}')\} = \{d_\alpha(\boldsymbol{k}), d_{\alpha'}^\dagger(\boldsymbol{k}')\} = \delta_{\alpha,\alpha'}\delta_{k,k'}I, \qquad (7)$$

where I is the unit matrix. The mode functions are taken to be

$$u^\alpha(\boldsymbol{k}) = \begin{pmatrix} \sqrt{\dfrac{\omega_k + m}{2\omega_k V}}\phi^\alpha \\ \dfrac{\sigma\boldsymbol{k}}{\sqrt{2\omega_k(\omega_k+m)V}}\phi^\alpha \end{pmatrix}, \qquad (8)$$

$$v^\alpha(\boldsymbol{k}) = \begin{pmatrix} \dfrac{\sigma\boldsymbol{k}}{\sqrt{2\omega_k(\omega_k+m)V}}\phi^\alpha \\ \sqrt{\dfrac{\omega_k + m}{2\omega_k V}}\phi^\alpha \end{pmatrix}, \qquad (9)$$

and $\phi^{1\dagger} = (1, 0)$, $\phi^{2\dagger} = (0, 1)$. The expectation value of the normal ordered energy density in an arbitrary quantum state is

$$\begin{aligned}\langle :T_{00}: \rangle =& \frac{1}{2}\sum_{k,k'}\sum_{\alpha,\alpha'}(\omega_k + \omega_{k'}) \\ &\times [\langle b_\alpha^\dagger(\boldsymbol{k})b_{\alpha'}(\boldsymbol{k}')\rangle u^{\dagger\alpha}(\boldsymbol{k})u^{\alpha'}(\boldsymbol{k}')\mathrm{e}^{-\mathrm{i}(k-k')\cdot x} \\ &+ \langle d_{\alpha'}^\dagger(\boldsymbol{k}')d_\alpha(\boldsymbol{k})\rangle v^{\dagger\alpha}(\boldsymbol{k})v^{\alpha'}(\boldsymbol{k}')\mathrm{e}^{\mathrm{i}(k-k')\cdot x}] \\ &+ \frac{1}{2}\sum_{k,k'}\sum_{\alpha,\alpha'}(\omega_{k'} - \omega_k) \\ &\times [\langle d_\alpha(\boldsymbol{k})b_{\alpha'}(\boldsymbol{k}')\rangle v^{\dagger\alpha}(\boldsymbol{k})u^{\alpha'}(\boldsymbol{k}')\mathrm{e}^{\mathrm{i}(k+k')\cdot x} \\ &- \langle b_\alpha^\dagger(\boldsymbol{k})d_{\alpha'}^\dagger(\boldsymbol{k}')\rangle u^{\dagger\alpha}(\boldsymbol{k})v^{\alpha'}(\boldsymbol{k}')\mathrm{e}^{-\mathrm{i}(k+k')\cdot x}]. \end{aligned} \qquad (10)$$

Now we define the spacetime-averaged energy density measured by a stationary observer as

$$\begin{aligned}\langle\rho\rangle_{\mathrm{ST}} =& \int_{-\infty}^{\infty}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}x_1\cdots \\ &\times\int_{-\infty}^{\infty}\mathrm{d}x_n\langle :T_{00}:\rangle f_0(t)f_1(x_1)\cdots f_n(x_n), \end{aligned} \qquad (11)$$

where $f_0, \cdots, f_n$ are assumed to be smooth, nonnegative and compactly supported spacetime sampling functions whose integrals about the corresponding coordinates are all unity. Substituting Eq.(10) into the above equation we obtain

$$\begin{aligned}\langle\rho\rangle_{\mathrm{ST}} =&\frac{1}{2}\sum_{k,k'}\sum_{\alpha,\alpha'}(\omega_k+\omega_{k'})\\ &\times[\langle b^{\dagger}_{\alpha}(\boldsymbol{k})b_{\alpha'}(\boldsymbol{k}')\rangle u^{\dagger\alpha}(\boldsymbol{k})u^{\alpha'}(\boldsymbol{k}')\widehat{f}_0(\omega_{k'}-\omega_k)\widehat{f}_1(k_1-k_1^{'})\cdots\widehat{f}_n(k_n-k_n^{'})\\ &+\langle d^{\dagger}_{\alpha'}(\boldsymbol{k}')d_{\alpha}(\boldsymbol{k})\rangle v^{\dagger\alpha}(\boldsymbol{k})v^{\alpha'}(\boldsymbol{k}')\widehat{f}_0(\omega_k-\omega_{k'})\widehat{f}_1(k_1^{'}-k_1)\cdots\widehat{f}_n(k_n^{'}-k_n)]\\ &+\frac{1}{2}\sum_{k,k'}\sum_{\alpha,\alpha'}(\omega_{k'}-\omega_k)\\ &\times[\langle d_{\alpha}(\boldsymbol{k})b_{\alpha'}(\boldsymbol{k}')\rangle v^{\dagger\alpha}(\boldsymbol{k})u^{\alpha'}(\boldsymbol{k}')\widehat{f}_0(\omega_k+\omega_{k'})\widehat{f}_1(-k_1-k_1^{'})\cdots\widehat{f}_n(-k_n-k_n^{'})\\ &-\langle b^{\dagger}_{\alpha}(\boldsymbol{k})d^{\dagger}_{\alpha'}(\boldsymbol{k}')\rangle u^{\dagger\alpha}(\boldsymbol{k})v^{\alpha'}(\boldsymbol{k}')\widehat{f}_0(-\omega_k-\omega_{k'})\widehat{f}_1(k_1+k_1^{'})\cdots\widehat{f}_n(k_n+k_n^{'})].\end{aligned}\tag{12}$$

Here $\widehat{f}$ is the Fourier transform of f:

$$\widehat{f}(A)=\int_{-\infty}^{\infty}\mathrm{d}x\, f(x)\mathrm{e}^{-\mathrm{i}Ax}.\tag{13}$$

Now we prove an identity which is crucial for obtaining the lower bound on $\langle\rho\rangle_{\mathrm{ST}}$. We first define the function g as

$$g(A)=\frac{\widehat{f^{1/2}}(A)}{\sqrt{2\pi}},\tag{14}$$

and g obeys

$$g\star g=\widehat{f},\tag{15}$$

where the convolution $\star$ is defined as

$$(h_1\star h_2)(A)=\int_{-\infty}^{\infty}\mathrm{d}A'\,h_1(A-A')h_2(A').\tag{16}$$

Following Fewster and Mistry's method[23] we define a family of operators on the Fock space as

$$\begin{aligned}\mathcal{O}_{\mu}=&\sum_{\alpha,k}[\overline{g_0(\mu_0-\omega_k)}g_1(\mu_1-k_1)\cdots g_n(\mu_n-k_n)b_{\alpha}(\boldsymbol{k})u^{\alpha}(\boldsymbol{k})\\ &+\overline{g_0(\mu_0+\omega_k)}g_1(\mu_1+k_1)\cdots g_n(\mu_n+k_n)d^{\dagger}_{\alpha}(\boldsymbol{k})v^{\alpha}(\boldsymbol{k})].\end{aligned}\tag{17}$$

Here the subscript μ indicates a group of variables $\{\mu_0,\mu_1,\cdots,\mu_n\}$. Considering the integral symmetry about $\boldsymbol{k}$ and $\boldsymbol{k}'$ and using Eq.(7), we derive (no summation over μ)

$$\begin{aligned}\mathcal{O}^{\dagger}_{\mu}\mathcal{O}_{\mu}=&S_{\mu}I+\sum_{\alpha,\alpha'}\sum_{k,k'}\Big[g_0^{-}\overline{g_1^{-}\cdots g_n^{-}g_0'^{-}}g_1'^{-}\cdots g_n'^{-}b^{\dagger}_{\alpha}(\boldsymbol{k})b_{\alpha'}(\boldsymbol{k}')u^{\alpha\dagger}(\boldsymbol{k})u^{\alpha'}(\boldsymbol{k}')\\ &+g_0^{-}\overline{g_1^{-}\cdots g_n^{-}g_0'^{+}}g_1'^{+}\cdots g_n'^{+}b^{\dagger}_{\alpha}(\boldsymbol{k})d^{\dagger}_{\alpha'}(\boldsymbol{k}')u^{\alpha\dagger}(\boldsymbol{k})v^{\alpha'}(\boldsymbol{k}')\\ &+g_0^{+}\overline{g_1^{+}\cdots g_n^{+}g_0'^{-}}g_1'^{-}\cdots g_n'^{-}d_{\alpha}(\boldsymbol{k})b_{\alpha'}(\boldsymbol{k}')v^{\alpha\dagger}(\boldsymbol{k})u^{\alpha'}(\boldsymbol{k}')\\ &-g_0^{+}\overline{g_1^{+}\cdots g_n^{+}g_0'^{+}}g_1'^{+}\cdots g_n'^{+}d^{\dagger}_{\alpha'}(\boldsymbol{k}')d_{\alpha}(\boldsymbol{k})v^{\alpha\dagger}(\boldsymbol{k})v^{\alpha'}(\boldsymbol{k}')\Big],\end{aligned}\tag{18}$$

where

$$S_{\mu}=\frac{2}{V}\sum_{k}|g_0(\mu_0+\omega_k)|^2|g_1(\mu_1+k_1)|^2\cdots|g_n(\mu_n+k_n)|^2\tag{19}$$

and we have taken the following notations for convenience:

$$g_0^{\pm}=g_0(\mu_0\pm\omega_k),g_i^{\pm}=g_i(\mu_i\pm k_i),g_0'^{\pm}=g(\mu_0\pm\omega_{k'}),g_i'^{\pm}=g_i(\mu_i\pm k_i^{'}),\ (i=1,\cdots,n).\tag{20}$$

Then we can prove

$$\langle\rho\rangle_{\mathrm{ST}}=\int_{-\infty}^{\infty}\mathrm{d}\mu_0\int_{-\infty}^{\infty}\mathrm{d}\mu_1\cdots\int_{-\infty}^{\infty}\mathrm{d}\mu_n\mu_0\langle\mathcal{O}^{\dagger}_{\mu}\mathcal{O}_{\mu}-S_{\mu}I\rangle,\tag{21}$$

where we have used Eq.(15) and the lemma[23]

$$\frac{1}{2}(\omega+\omega')\widehat{f}(\omega-\omega')=\int_{-\infty}^{\infty}\mathrm{d}\mu\mu g(\omega-\mu)\overline{g(\omega'-\mu)}.\tag{22}$$

Upon calculating

$$\begin{aligned}\left\{\mathcal{O}_\mu^\dagger, \mathcal{O}_\mu\right\} =& \frac{2}{V}\sum_{\alpha,k}|g_0(\mu_0-\omega_k)|^2|g_1(\mu_1-k_1)|^2\cdots|g_n(\mu_n-k_n)|^2 I \\ &+\frac{2}{V}\sum_{\alpha,k}|g_0(\mu_0+\omega_k)|^2|g_1(\mu_1+k_1)|^2\cdots|g_n(\mu_n+k_n)|^2 I \\ \equiv& \left(S_{-\mu}+S_\mu\right) I, \end{aligned} \tag{23}$$

we find that

$$\begin{aligned}\langle\rho\rangle_{\mathrm{ST}} =& \int_0^\infty \mathrm{d}\mu_0\int_{-\infty}^\infty \mathrm{d}\mu_1\cdots\int_{-\infty}^\infty \mathrm{d}\mu_n\mu_0\langle\mathcal{O}_\mu^\dagger\mathcal{O}_\mu - S_\mu I\rangle \\ &+\int_{-\infty}^0 \mathrm{d}\mu_0\int_{-\infty}^\infty \mathrm{d}\mu_1\cdots\int_{-\infty}^\infty \mathrm{d}\mu_n\mu_0\langle -\mathcal{O}_\mu\mathcal{O}_\mu^\dagger + S_{-\mu} I\rangle. \end{aligned} \tag{24}$$

Considering $\mu_0\langle\mathcal{O}_\mu^\dagger\mathcal{O}_\mu\rangle \geq 0$ for any $\mu_0 \geq 0$ and $-\mu_0\langle\mathcal{O}_\mu\mathcal{O}_\mu^\dagger\rangle \geq 0$ for any $\mu_0 \leq 0$ in all quantum states, we have

$$\langle\rho\rangle_{\mathrm{ST}} \geq -\int_0^\infty \mathrm{d}\mu_0\int_{-\infty}^\infty \mathrm{d}\mu_1\cdots\int_{-\infty}^\infty \mathrm{d}\mu_n\mu_0 S_\mu + \int_{-\infty}^0 \mathrm{d}\mu_0\int_{-\infty}^\infty \mathrm{d}\mu_1\cdots\int_{-\infty}^\infty \mathrm{d}\mu_n\mu_0 S_{-\mu}. \tag{25}$$

Inserting Eq.(19) into the above equation and taking the continuum limit $(1/V)\sum_k \longrightarrow \int \mathrm{d}^n\boldsymbol{k}/(2\pi)^n$, we obtain the inequality

$$\begin{aligned}\langle\rho\rangle_{\mathrm{ST}} \geq& -4\int_0^\infty \mathrm{d}\mu_0\int_{-\infty}^\infty \mathrm{d}\mu_1\cdots\int_{-\infty}^\infty \mathrm{d}\mu_n \\ &\times\int\frac{\mathrm{d}^n\boldsymbol{k}}{(2\pi)^n}|g_0(\mu_0+\omega_k)|^2|g_1(\mu_1+k_1)|^2\cdots|g_n(\mu_n+k_n)|^2\mu_0. \end{aligned} \tag{26}$$

For the multiple integral in the right-hand side of Eq.(26) we use the method of transforming integral variables to calculate the parts about μ_i and k_i for $i = 1, \cdots, n$, respectively. Finally we obtain

$$\langle\rho\rangle_{\mathrm{ST}} \geq -4C_n\int_0^\infty \mathrm{d}\mu_0\int_m^\infty \mathrm{d}\omega|g_0(\mu_0+\omega)|^2(\omega^2-m^2)^{n/2-1}\omega\mu_0, \tag{27}$$

where $\int_0^\infty \mathrm{d}u|g(u)|^2 = 1/2$ has been used and

$$C_n = \frac{1}{2^{n-1}\pi^{n/2}\Gamma(n/2)}. \tag{28}$$

Let $u_0 = \mu_0 + \omega, v = \omega$, and apply the way of changing the integral variables once again, then we can simplify Eq.(27) further:

$$\langle\rho\rangle_{\mathrm{ST}} \geq -\frac{2C_n}{\pi}\int_m^\infty \mathrm{d}u_0|\widehat{f_0^{1/2}}(u_0)|^2\int_m^{u_0}\mathrm{d}v(v^2-m^2)^{n/2-1}v(u_0-v). \tag{29}$$

We can also write the above inequality as

$$\langle\rho\rangle_{\mathrm{ST}} \geq -\frac{2C_n}{\pi n(n+1)}\int_m^\infty \mathrm{d}u_0|\widehat{f_0^{1/2}}(u_0)|^2 u_0^{n+1}Q_n^D\left(\frac{u_0}{m}\right), \tag{30}$$

where the bounded positive function $Q_n^D(x)$ is defined as

$$Q_n^D(x) = (n+1)\left(1-x^{-2}\right)^{n/2} - n(n+1)x^{-(n+1)}\int_1^x \mathrm{d}y\, y^2(y^2-1)^{n/2-1}. \tag{31}$$

The function $Q_n^D(x)$ is monotonically increasing for $x \geq 1$ and exhibits the maximum value of one as $x \to \infty$ corresponding to the massless case. In Eq.(30), the factor $|\widehat{f_0^{1/2}}(u_0)|^2$ decays faster than any inverse polynomial in u_0 as $u_0 \to \infty$ and therefore the above inequality is always finite for any smooth, positive and compactly supported spacetime sampling functions.

In the light of Eq.(30) we come to the conclusion that the lower bound for massive field is stronger than that for massless one because the integration interval and the function $Q_n^D(u_0/m)$ are smaller in the massive case than in the case $m = 0$.

3. Quantum inequalities in 2 and 4-dimensional spacetimes

In the following we evaluate the quantum inequalities in 2 and 4-dimensional Minkowski spacetimes and compare them with the corresponding results of QEIs derived previously to gain more insights into the quantum inequality bound.

Firstly we discuss the 2-dimensional case. From Eq.(30) we obtain the quantum inequality for the massless Dirac field:

$$\langle\rho\rangle_{\rm ST} \geq -\frac{1}{\pi^2}\int_0^\infty {\rm d}u_0 |\widehat{f_0^{1/2}}(u_0)|^2 u_0^2. \qquad (32)$$

Equation (32) can also be transformed to t-space by using Parseval's theorem and can be rewritten as

$$\langle\rho\rangle_{\rm ST} \geq -\frac{1}{4\pi}\int_{-\infty}^\infty {\rm d}t \frac{f_0'(t)^2}{f_0(t)}. \qquad (33)$$

Comparing Eq.(33) with Eq.(1) we find that the quantum inequality is weaker than the optimal temporal QEI by a factor of 1/6. For the massive field, the spacetime inequality is

$$\langle\rho\rangle_{\rm ST} \geq -\frac{1}{\pi^2}\int_m^\infty {\rm d}u_0 |\widehat{f_0^{1/2}}(u_0)|^2 Q_1^D\left(\frac{u_0}{m}\right) u_0^2, \quad (34)$$

where

$$Q_1^D(x) = \left(1-\frac{1}{x^2}\right)^{1/2} - \frac{1}{x^2}\log[x+(x^2-1)^{1/2}]. \quad (35)$$

Because $Q_1^D(x) \leq 1$ the massive inequality is stronger than that for massless fields.

Now we deal with the case $n = 3$. In four-dimensional spacetime, the inequality for the massless field is worked out

$$\langle\rho\rangle_{\rm ST} \geq -\frac{1}{12\pi^3}\int_0^\infty {\rm d}u_0 u_0^4 |\widehat{f_0^{1/2}}(u_0)|^2. \qquad (36)$$

Equation (36) can also be written as

$$\langle\rho\rangle_{\rm ST} \geq -\frac{1}{12\pi^3}\int_{-\infty}^\infty {\rm d}t |f_0^{1/2''}(t)|^2. \qquad (37)$$

When the Lorentz function

$$f(t) = \frac{t_0}{\pi(t^2+t_0^2)} \qquad (38)$$

with characteristic width t_0 is adopted, the inequality can be easily calculated:

$$\langle\rho\rangle_{\rm ST} \geq -\frac{9}{512\pi^2 t_0^4}. \qquad (39)$$

If $t_0 \to 0$, the inequality is divergent. This fact is consistent with the conclusion that the spatial constraint on negative energy density does not exist in four dimensions.[35] For the massive field the quantum inequality bound is

$$\langle\rho\rangle_{\rm ST} \geq -\frac{1}{12\pi^3}\int_m^\infty {\rm d}u_0 |\widehat{f_0^{1/2}}(u_0)|^2 Q_3^D(\frac{u_0}{m}) u_0^4, \quad (40)$$

where $Q_3^D(x)$ is defined as Eq.(3). Comparing Eq.(30) with Eq.(2), we can see that the quantum inequality has the form of the temporal QEI.

4. Conclusion

We have derived new restrictions on the spacetime average of the energy density against spacetime sampling functions for the Dirac field in any dimensional flat spacetimes, and given the explicit results of the quantum inequality bound in 2 and 4-dimensions. The quantum inequality derived in this paper is similar to the temporal QEI in the same dimensions, but is not tighter than the latter. This indicates that the bound obtained by the current approach with the energy density averaged over the spacetime volume is not severer than that averaged only along the timelike curve. We also find that the quantum inequality for the massive field is stronger than that for massless case. This fact is natural and suggests that it is more difficult for the energy density to become negative in massive fields than in massless ones. Since the characteristic widths of space sampling functions do not emerge in the final result as clearly as that of the time sampling function, it is the direction for future efforts to search for sharper spacetime bound or the optimal one.

In conclusion the quantum inequality obtained in the present paper is a kind of constraint on negative energy density averaged over space and time, and we

may apply the method in this paper to other fields and expect to get similar restrictions.[39]

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