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The properties and geodesics related to the NUT–Taub-like spacetime*

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Some properties related to the NUT–Taub-like spacetime, such as the surface of infinite red-shift, horizon, singularity and the area of the NUT–Taub-like black hole are discussed. Furthermore, the geodesics in the NUT–Taub-like spacetime are obtained in some special cases. Specifically, the circular orbits for a massive particle are derived, which can reduce to the cases of the Schwarzschild spacetime and the NUT–Taub spacetime when $m^* = 0$ and $m^* \ll M$, respectively.

Keywords: NUT–Taub spacetime, Schwarzschild metric, geodesics
PACC: 0420, 0240, 0470

1. Introduction

The Ernst equation[1] is a basic equation for finding stationary axisymmetric vacuum solutions of Einstein's equations. The symmetric groups and generation of new solutions of Ernst's equation have been extensively studied.[2−12] Especially, Reina and Treves[13] obtained NUT–Taub(NT) and Kerr–NUT (KN) solutions[14,15] by using a unitary transformation on Schwarzschild solution $\xi_{\rm S}$ and Kerr solution $\xi_{\rm K}$ respectively. In the latest paper,[16] we found two solutions by applying the Ehlers transformation[17] to $\xi_{\rm S}$ and $\xi_{\rm K}$. Furthermore, by means of the mappings and coordinate transformations, we have obtained two NUT-like solutions, i.e. the so-called NUT–Taub-like (NT-like) and Kerr–NUT-like (KN-like) solutions, which can reduce to the NT metric and the KN metric respectively in some special cases. However, so far researchers have not deeply studied properties of the NT-like solutions and geodesics in the NT-like spacetime.

In this paper we investigate some properties on the NT-like spacetime such as surface of infinite red-shift, horizons, and singularities, etc. Furthermore we derive and solve the geodesic equations for a massive particle in the NT-like spacetime.

2. Some properties on the NT-like spacetime

The NT-like metric[16] is

$$\begin{aligned} {\rm d}s^2 =& \Delta^* U \left({\rm d}t - 2\frac{S^*}{\Delta^*}\cos\theta {\rm d}\phi \right)^2 \\ & - \Delta^{*-1} U^{-1} {\rm d}r^2 - \Delta^{*-1}\left(r^2 + S^{*2}\right) \\ & \times \left({\rm d}\theta^2 + \sin^2\theta {\rm d}\phi^2\right), \end{aligned} \tag{1}$$

where

$$\begin{aligned} \Delta^* &= \sqrt{\frac{M^2 - m^{*2}}{M^2}}, \\ S^* &= m^* \sqrt{\frac{M^2}{M^2 - m^{*2}}}, \\ U &= \frac{r^2 - 2Mr - S^{*2}}{r^2 + S^{*2}}, \end{aligned} \tag{2}$$

and r, θ and ϕ are the spherical coordinates, the parameter M is mass of the source and m^* is the NUT parameter interpreted as the gravomagnetic monopole,[18] and its value is in a range of $-M < m^* < M$, $|m^*| \neq M$. However, when $|m^*| \ll M$, Eq.(1) can reduce to the NT metric.[14]

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If we make the transformation

$$t = t^{'} + 2\frac{S^{*}\phi}{\Delta^{*}}, \tag{3}$$

the NT-like metric can be written as

$$\begin{aligned} \mathrm{d}s^2 =& \Delta^{*}U\left(\mathrm{d}t + 4\frac{S^{*}}{\Delta^{*}}\sin^2\frac{\theta}{2}\mathrm{d}\phi\right)^2 \\ & - \Delta^{*-1}U^{-1}\mathrm{d}r^2 - \frac{r^2+S^{*2}}{\Delta^{*}}\mathrm{d}\theta^2 \\ & - \frac{1}{\Delta^{*}}\left(r^2+S^{*2}\right)\sin^2\theta\mathrm{d}\phi^2, \end{aligned} \tag{4}$$

where we have replaced t' with t.

For a general stationary spacetime, the surface of infinite red-shift is determined by $g_{00}=0$, which in this case is

$$\Delta^{*}\frac{r^2-2Mr-S^{*2}}{r^2+S^{*2}} = 0. \tag{5}$$

Since Δ^{*} and $\left(r^2+S^{*2}\right)$ must not be zero, we obtain the solutions of Eq.(5) as follows:

$$r_{+}^{\mathrm{s}} = M + \sqrt{M^2+S^{*2}}, \tag{6}$$

$$r_{-} = M - \sqrt{M^2+S^{*2}}. \tag{7}$$

Because $r_{-}(<0)$ is meaningless in physics, r_{+}^{s} is only the surface of infinite red-shift in the NT-like spacetime.

Generally, horizon function $f(x^{\mu})$ (exactly null-hypersurface) satisfies[6] $n_{\mu}n^{\mu} = g^{\mu\nu}\dfrac{\partial f}{\partial x^{\mu}}\dfrac{\partial f}{\partial x^{\nu}} = 0$, where $n^{\mu}\equiv\dfrac{\partial f}{\partial x^{\mu}}$ is the normal vector of hypersurface. For a stationary axisymmetric spacetime, this equation can reduce to[6]

$$\widehat{g}_{00}\left[g^{11}\left(\frac{\partial f}{\partial r}\right)^2 + g^{22}\left(\frac{\partial f}{\partial\theta}\right)^2\right] = 0, \tag{8}$$

where $\widehat{g}_{00}\equiv g_{00}-\dfrac{g_{03}^2}{g_{33}}$. Thus we obtain two equations:

$$g^{11}\left(\frac{\partial f}{\partial r}\right)^2 + g^{22}\left(\frac{\partial f}{\partial\theta}\right)^2 = 0 \quad \text{and} \quad \widehat{g}_{00}=0. \tag{9}$$

It is easy to identify these two equations with the following equation by using Eq.(4)

$$\Delta^{*}U - \frac{\left(4US^{*}\sin^2\frac{\theta}{2}\right)^2}{\frac{1}{\Delta^{*}}\left[16US^{*2}\sin^4\frac{\theta}{2} - \left(r^2+S^{*2}\right)\sin^2\theta\right]} = 0, \tag{10}$$

whose solutions are

$$r_{-} = M - \sqrt{M^2+S^{*2}}, \tag{11}$$

$$r_{+}^{\mathrm{h}} = M + \sqrt{M^2+S^{*2}}. \tag{12}$$

Similarly, because $r_{-}(<0)$ is meaningless in physics, r_{+}^{h} is only the event horizon in the NT-like spacetime. When $m^{*}=0$, we can obtain the Schwarzschild horizon $r_{\mathrm{S}}^{\mathrm{h}}=2M$.

According to Eq.(4), the covariant basis vectors are

$$\begin{aligned} \omega^0 &= \sqrt{\Delta^{*}U}\left(\mathrm{d}t + 4\frac{S^{*}}{\Delta^{*}}\sin^2\frac{\theta}{2}\mathrm{d}\phi\right), \\ \omega^1 &= \sqrt{\Delta^{*-1}U^{-1}}\mathrm{d}r, \\ \omega^2 &= \sqrt{\frac{r^2+S^{*2}}{\Delta^{*}}}\mathrm{d}\theta, \\ \omega^3 &= \sqrt{\frac{r^2+S^{*2}}{\Delta^{*}}}\sin\theta\mathrm{d}\phi. \end{aligned} \tag{13}$$

Thus we can obtain

$$\mathrm{d}t = \frac{1}{\sqrt{\Delta^{*}U}}\omega^0 - \frac{2S^{*}}{\sqrt{\Delta^{*}\left(r^2+S^{*2}\right)}}\tan\frac{\theta}{2}\omega^3. \tag{14}$$

Furthermore, we have

$$\left(\bigtriangledown t\right)^2 = \left(\Delta^{*}U\right)^{-1} + \frac{4S^{*2}}{\Delta^{*}\left(r^2+S^{*2}\right)}\tan^2\frac{\theta}{2}. \tag{15}$$

It follows that the coordinate time t has singularity when (a) $\Delta^{*}=0$, or (b) $U=0$, or (c) $\theta=\pi$. We can easily point out that the case (a) can be ruled out because of $|m^{*}|\neq M$ in Eq.(2), the case (b) gives the horizon (12) of the metric, and the case (c) leads to the singularity of the metric. From the determinant of the metric (4): $-\det g = \dfrac{\left(r^2+S^{*2}\right)^2}{\Delta^{*2}}\sin^2\theta$, we know that it equals zero when $\theta=0$ or π. So we can come to the conclusion that the metric has the singularity at $\theta=\pi$. In addition, according to the curvature tensors $R_{\mu\nu\lambda\rho}$ corresponding to the NT and NT-like metrics[16] we can see that the spacetime structure of the NT-like solution is different from that of the NT solution. The case of $|m^{*}|=M$ causes the curvature of NUT–like spacetime to diverge, which implies that the NT-like solution is "everywhere" singular and therefore meaningless when $|m^{*}|=M$.

In the case of time coordinate t =constant and radial coordinate $r=r_{+}$ (here the horizon radius r_{+}^{h} is replaced by r_{+}), the metric (4) can be written as

$$\begin{aligned} \mathrm{d}\sigma^2 =& -\frac{r^2+S^{*2}}{\Delta^{*}}\mathrm{d}\theta^2 + \frac{1}{\Delta^{*}}\left[16US^{*2}\sin^4\frac{\theta}{2}\right. \\ & \left. - \left(r^2+S^{*2}\right)\sin^2\theta\right]\mathrm{d}\phi^2. \end{aligned} \tag{16}$$

The determinant of the metric in the two-dimensional space mentioned above is

$$g = \begin{vmatrix} g_{22} & g_{23} \\ g_{32} & g_{33} \end{vmatrix} = \frac{\left(r_{+}^2+S^{*2}\right)^2}{\Delta^{*2}}\sin^2\theta.$$

So we can have the area of NT-like black hole, $A = \int \mathrm{d}A = \int \sqrt{g}\mathrm{d}\theta\mathrm{d}\phi = 4\pi\Delta^{*-1}\left(r_+^2 + S^{*2}\right)$.

When $m^* = 0$, $S^* = 0$ and $\Delta^* = 1$, the NT-like black hole reduces to the Schwarzschild black hole, whose area is $A_{\mathrm{Sch}} = 16\pi M^2$.

3. The geodesic equations of a massive particle for the NT-like metric

Now we consider the geodesic equations

$$\frac{\mathrm{d}}{\mathrm{d}s}(g_{\mu\nu}\dot{x}^\nu) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x^\mu}(g_{\tau\omega})\dot{x}^\tau\dot{x}^\omega = 0. \tag{17}$$

The dots above the variables denote derivatives with respect to some affine parameter. The Greek letters run from 0 to 3 and correspond to the (t, r, θ, ϕ) coordinates, respectively. Writing out the components of Eq.(17) for the NT-like metric in the expression (4), we obtain the following equations:

$$\mu = 0, \quad \frac{\mathrm{d}}{\mathrm{d}s}\left(\Delta^* U\dot{t} + 2US^*(1-\cos\theta)\dot{\phi}\right) = 0, \tag{18}$$

$$\mu = 1, \quad \ddot{r} - \frac{1}{2}U\frac{\mathrm{d}U^{-1}}{\mathrm{d}r}\dot{r}^2 - U\left[r\sin^2\theta - 2S^{*2}(1-\cos\theta)^2\frac{\mathrm{d}U}{\mathrm{d}r}\right]\dot{\phi}^2 - Ur\dot{\theta}^2 + 2\Delta^* S^* U(1-\cos\theta)\frac{\mathrm{d}U}{\mathrm{d}r}\dot{t}\dot{\phi} + \frac{1}{2}\Delta^{*2}U\frac{\mathrm{d}U}{\mathrm{d}r}\dot{t}^2 = 0, \tag{19}$$

$$\mu = 2, \quad \frac{\mathrm{d}}{\mathrm{d}s}[\left(r^2 + S^{*2}\right)\dot{\theta}] + [-(r^2 + S^{*2})\sin\theta\cos\theta + 4S^{*2}U\sin\theta(1-\cos\theta)]\dot{\phi}^2 + 2\Delta^* S^* U\sin\theta\dot{t}\dot{\phi} = 0, \tag{20}$$

$$\mu = 3, \quad \frac{\mathrm{d}}{\mathrm{d}s}\left\{2\Delta^* US^*(1-\cos\theta)\dot{t} + [4US^{*2}(1-\cos\theta)^2 - (r^2 + S^{*2})\sin^2\theta]\dot{\phi}\right\} = 0. \tag{21}$$

An additional equation follows from $u^\nu u_\nu = \epsilon$, i.e.

$$\epsilon = \Delta^{*2}U\dot{t}^2 - U^{-1}\dot{r}^2 - H\dot{\theta}^2 + [4US^{*2}(1-\cos\theta)^2 - (r^2 + S^{*2})\sin^2\theta]\dot{\phi}^2 + 4\Delta^* US^*(1-\cos\theta)\dot{t}\dot{\phi}, \tag{22}$$

where u^ν is the tangent to the geodesic path, $\epsilon = 1$ for the time-like geodesic and $\epsilon = 0$ for the null geodesic. Two constants of motion follow immediately from the integrating of Eqs.(18) and (21), and they are

$$\Delta^* U\dot{t} + 2US^*(1-\cos\theta)\dot{\phi} = E, \tag{23}$$

$$2\Delta^* US^*(1-\cos\theta)\dot{t} + [4US^{*2}(1-\cos\theta)^2 - (r^2 + S^{*2})\sin^2\theta]\dot{\phi} = -J_z + 2S^* E. \tag{24}$$

Here, the constant E is defined as the energy of the orbit and J_z is chosen to agree with the z-component of the conserved total momentum $\boldsymbol{J}$.

We can eliminate $\dot{t}$ by substituting (23) into (19)–(22) as follows:

$$\ddot{r} - \frac{1}{2}\dot{r}U\frac{\mathrm{d}U^{-1}}{\mathrm{d}r} + \frac{E^2}{2U}\frac{\mathrm{d}U}{\mathrm{d}r} - Ur(\sin^2\theta\dot{\phi} + \dot{\theta}^2) = 0, \tag{25}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}[\left(r^2 + S^{*2}\right)\dot{\theta}] + 2ES^*\sin\theta\dot{\phi} + (r^2 + S^{*2})\sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{26}$$

$$\dot{\phi} = \frac{J_z - 2S^* E\cos\theta}{(r^2 + S^{*2})\sin^2\theta}, \tag{27}$$

$$\epsilon = E^2U^{-1} - U^{-1}\dot{r}^2 - (r^2 + S^{*2})(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2). \tag{28}$$

Using Eq.(27) to eliminate the $\dot{\phi}$ variable in Eq.(23), we obtain the following equation:

$$\dot{t} = \frac{E}{\Delta^* U} - \frac{2S(J_z - 2S^* E\cos\theta)}{\Delta^*(r^2 + S^{*2})(1+\cos\theta)}. \tag{29}$$

By means of the Killing symmetries[18] of the NT-like metric we can obtain the following relationships:

$$\begin{aligned}\boldsymbol{J} &= \boldsymbol{L} + \boldsymbol{S},\\ J^2 &= J_x^2 + J_y^2 + J_z^2\\ &= 4S^{*2}E^2 + L^2 = S^2 + L^2,\end{aligned} \tag{30}$$

where $\boldsymbol{L}$ and $\boldsymbol{S}$ are respectively the orbital angular momentum and field angular momentum (per unit mass) of a test particle in the NT-like spacetime. And L^2 satisfies

$$\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) = \frac{L^2}{\left(r^2 + S^{*2}\right)^2}. \tag{31}$$

Since J^2 and S^2 are constants, the magnitude of the orbital angular momentum $\boldsymbol{L}$ is also constant, but its direction is perpendicular to $\boldsymbol{S}$.

Thus, in the special case of $J_x = J_y = 0$, i.e. $\boldsymbol{J}$ is only along the z-axis, by virtual of Eqs.(31) and (28), we can obtain

$$\dot{r}^2 = (E^2 - 1) - V(r), \tag{32}$$

where

$$V(r) = \epsilon U - 1 + \frac{UL^2}{r^2 + S^{*2}}. \tag{33}$$

For the time-like geodesic, $\epsilon = 1$, thus we have

$$V(r) = -\frac{2Mr^3}{(r^2+S^{*2})^2} + \frac{(L^2 - 2S^{*2})r^2}{(r^2+S^{*2})^2} - \frac{(L^2+S^{*2})2Mr}{(r^2+S^{*2})^2} - \frac{(L^2+2S^{*2})S^{*2}}{(r^2+S^{*2})^2}. \tag{34}$$

The features of this potential are similar to those of the Schwarzschild potential. The first three terms on the right-hand side of the expression (34) are just the usual Schwarzschild terms with only a slight modification of S^*. The first term is just the classical Newtonian potential, the second term corresponds to the centrifugal force, and the third term is analogous to the usual general relativistic correction that occurs in the Schwarzschild potential. The main effect of S^* on these three terms is to modify the angular momentum barrier. The potential barrier appearing in the second term of the expression (34) is proportional to $(L^2 - 2S^{*2})$. The fourth term is unique to the NT-like metric and makes only a short-range contribution.

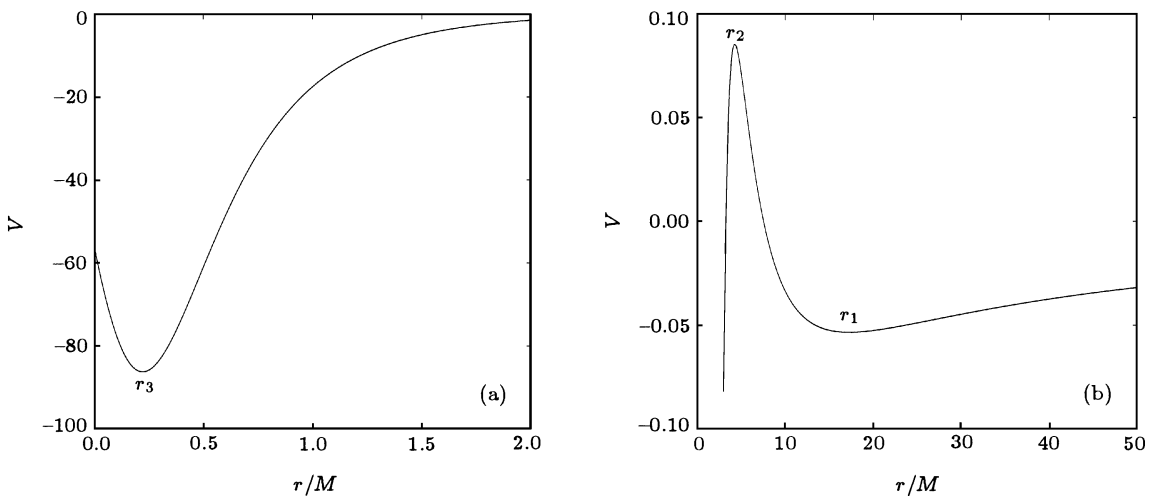


Fig.1. Qualitative features of the effective potential V varying with r/M, where $M = 1$, $L^2 = 22$ and $m^* = 0.54$.

The qualitative features of the potential for $L \neq 0$ in the expression (34) are shown in Fig.1. The shape of the potential is determined by the extreme points. Setting the derivative of the potential in (34) to be equal to zero, it follows that the extreme points are located at the roots of the equation,

$$Mr^4 + (2S^{*2} - L^2)r^3 + 3L^2Mr^2 + (2S^{*4} + 3L^2S^{*2})r - (S^{*2} + L^2)MS^{*2} = 0. \tag{35}$$

Thus, we can obtain the extreme position of the potential

$$r_1 = \frac{1}{2}\sqrt{\alpha + \frac{\beta}{M\eta} + \frac{\eta}{3\times 2^{1/3}M}} - \frac{2S^{*2} - L^2}{4M} + \frac{1}{2}\sqrt{2\alpha - \frac{\beta}{M\eta} - \frac{\eta}{3\times 2^{1/3}M} + \frac{\zeta}{4\sqrt{\alpha + \frac{\beta}{M\eta} + \frac{\eta}{3\times 2^{1/3}M}}}}, \tag{36}$$

$$r_2 = \frac{1}{2}\sqrt{\alpha + \frac{\beta}{M\eta} + \frac{\eta}{3\times 2^{1/3}M}} - \frac{2S^{*2}-L^2}{4M} - \frac{1}{2}\sqrt{2\alpha - \frac{\beta}{M\eta} - \frac{\eta}{3\times 2^{1/3}M} + \frac{\zeta}{4\sqrt{\alpha + \frac{\beta}{M\eta} + \frac{\eta}{3\times 2^{1/3}M}}}}, \tag{37}$$

$$r_3 = -\frac{1}{2}\sqrt{\alpha + \frac{\beta}{M\eta} + \frac{\eta}{3\times 2^{1/3}M}} - \frac{2S^{*2}-L^2}{4M} + \frac{1}{2}\sqrt{2\alpha - \frac{\beta}{M\eta} - \frac{\eta}{3\times 2^{1/3}M} - \frac{\zeta}{4\sqrt{\alpha + \frac{\beta}{M\eta} + \frac{\eta}{3\times 2^{1/3}M}}}}, \tag{38}$$

$$r_4 = -\frac{1}{2}\sqrt{\alpha + \frac{\beta}{M\eta} + \frac{\eta}{3\times 2^{1/3}M}} - \frac{2S^{*2}-L^2}{4M} - \frac{1}{2}\sqrt{2\alpha - \frac{\beta}{M\eta} - \frac{\eta}{3\times 2^{1/3}M} - \frac{\zeta}{4\sqrt{\alpha + \frac{\beta}{M\eta} + \frac{\eta}{3\times 2^{1/3}M}}}}, \tag{39}$$

where

$$\alpha = -2L^2 + \frac{\left(2S^{*2}-L^2\right)^2}{4M^2},$$

$$\begin{aligned}
\beta =& 2^{1/3}\left(-4M^2S^{*4} - 4S^{*6} - 4L^2M^2S^{*2} - 4L^2S^{*4} + 3L^4M^2 + 3L^4S^{*2}\right),\\
\gamma =& 54L^6M^3 - 27L^2M\left(2S^{*2}-L^2\right)\left(2S^{*4}+3L^2S^{*2}\right) + 27M\left(2S^{*4}+3L^2S^{*2}\right)^2\\
& - 216L^2M^2\left(-MS^{*4} - L^2MS^{*2}\right) + 27\left(2S^{*2}-L^2\right)^2\left(-MS^{*4} - L^2MS^{*2}\right),\\
\kappa =& -4(-12M^2S^{*4} - 12S^{*6} - 12L^2M^2S^{*2} - 12L^2S^{*4} + 9L^4M^2 + 9L^4S^{*2})^3,\\
\zeta =& \frac{12L^2\left(2S^{*2}-L^2\right)}{M} - \frac{\left(2S^{*2}-L^2\right)^3}{M^3} - \frac{8\left(2S^{*4}+3L^2S^{*2}\right)}{M},\\
\eta =& \left(\gamma + \sqrt{\kappa+\gamma^2}\right)^{1/3}.
\end{aligned}$$

When $m^* = 0$ Eqs. (36) and (37) can reduce to the standard Schwarzschild results:

$$r_{S1} = (L^2 + \sqrt{L^4 - 12M^2L^2})/2M,$$

$$r_{S2} = (L^2 - \sqrt{L^4 - 12M^2L^2})/2M,$$

and $r_3 = r_4 = 0$.

It follows from the expression (39) that r_4 is always negative (meaningless in physics), so there are only three possible circular orbits, which are located at r_1, r_2 and r_3 respectively. The orbits located at r_1 and r_3 are stable, while the orbit at r_2 is unstable. The two circular orbits r_1 and r_2 lie outside the horizon and are analogous to the stable and unstable orbits in the Schwarzschild potential respectively. The circular orbit at r_3 is unique to the NT-like potential and is due to the short-ranged fourth term in the expression(34).

Figures 2 and 3 reflect the dependences of the orbits r_1, r_2 and r_3 on m^*. From Figs.2 and 3 we can easily find that r_1 increases but r_2 and r_3 decrease with $|m^*|$ decreasing.

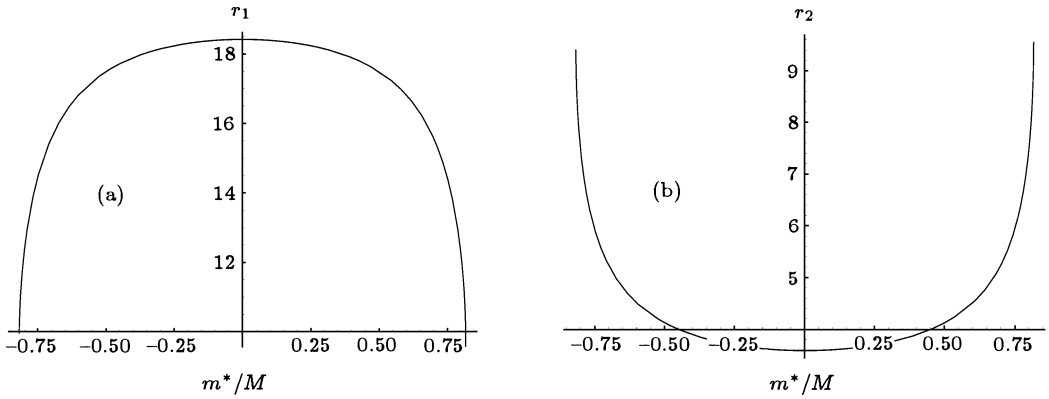


Fig.2. Quantitative features of two circular orbits r_1 and r_2 varying with the NUT parameter m^*, where $L^2 = 22$ and $M = 1$.

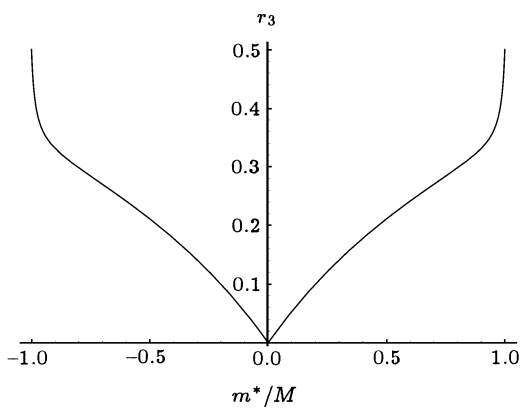


Fig.3. Quantitative features of the circular orbit r_3 varying with the NUT parameter m^*, where $L^2 = 22$ and $M = 1$.

If we let $m^* \ll M$, from Eqs.(36)–(38), we can obtain the extreme position r_1', r_2' and r_3' of the potential in the NUT–Taub spacetime, which have the same expressions as r_1, r_2 and r_3 except for S^* in Eqs.(36)–(38) being substituted by m^*.

Similar to r_1, r_2 and r_3 in the circular orbits of NT-like spacetime, the orbits r_1' and r_3' are stable, while the orbits r_2' is unstable. The two circular orbits r_1' and r_2' lie outside the horizon, while the circular orbit r_3' is always located inside the horizon.

Now we concretely solve the orbits for the explicit case of radial motion. From Eq.(31) it follows that the radial orbits have $L = 0$ in Eqs.(32)–(33), and we have $\dot{r}^2 = E^2 - U$. Taking the square root and integrating, we obtain

$$\tau = \int \frac{\left[r^2 + S^{*2}\right]^{1/2} \mathrm{d}r}{\left[(E^2-1)\, r^2 + 2Mr + (E^2+1)\, S^{*2}\right]^{1/2}}, \quad (40)$$

which is an expression for the proper time as a function of the radial coordinate. In a similar manner, we can have the coordinate time as a function of the radial position. By setting $\dot{\phi} = 0$ in Eq.(23) and using the resulting expression between the coordinate and the proper times to eliminate the proper time, we can have

$$t = \int \frac{E\left[r^2 + S^{*2}\right]^{3/2} \mathrm{d}r}{\Delta^* \left(r^2 - 2Mr - S^{*2}\right)\left[(E^2-1)\, r^2 + 2Mr + (E^2+1)\, S^{*2}\right]^{1/2}}. \quad (41)$$

From the expression (40) we can see that it only takes a finite amount of proper time for a moving test particle to arrive at the coordinate r_c=const (e.g. when $m^* = 0.9M$, $0.47M < r_c < 0.48M$). On the other hand, from Eq.(41) we can find that the coordinate time increases to infinity as a test particle approaches the horizon.

When $m^* \ll M$, t and τ reduce to the NUT–Taub results.[18] However, there is a distinct difference: in the NUT–Taub spacetime it takes a finite amount of proper time for a massive particle to be able to arrive at the origin $r = 0$,[15] while in the NT-like spacetime, it takes a finite amount of proper time for the particle to arrive at r_c=const.

4. Conclusions

So far, on the basis of the NUT–Taub-like solution, we have discussed some properties such as surface of infinite red-shift, event horizon, metric singularity and the area of the NUT–Taub-like black hole. We

have also derived the geodesic equations. Specifically, we have solved the orbits in the case of radial motion for a massive particle and obtained the circular orbits that can reduce to the Schwarzschild case for $m^* = 0$ and to NUT–Taub case for $m^* \ll M$. In fact, for a massless particle we can make similar discussion. However, in this paper we have not given out the physical interpretation of the NUT parameter m^*. We will investigate this subject in a forthcoming paper.

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