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Berry phase of coupled two arbitrary spins in a time-varying magnetic field*

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In this paper, we investigate the Berry phase of two coupled arbitrary spins driven by a time-varying magnetic field where the Hamiltonian is explicitly time-dependent. Using a technique of time-dependent gauge transform the Berry phase and time-evolution operator are found explicitly in the adiabatic approximation. The general solutions for arbitrary spins are applied to the spin-1/2 system as an example of explanation.

Keywords: Berry phase, time-dependent gauge transformation, coupled two arbitrary spins
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The Berry phase, which describes a time-dependent quantum system undergoing a cyclic and adiabatic evolution, was first discovered by Berry.[1] Simon gave a simple geometric interpretation of the Berry phase.[2] In order to remove the adiabatic condition in any cyclic evolution, Aharonov and Anandan proposed a more general type of the Berry phase called Aharonov–Anandan (AA) phase.[3] Now, the Berry phase has been comprehensively discussed in many fields of modern physics both theoretically and experimentally[4−7] and generalized to local spin moments in itinerant magnets,[8] entangled systems,[9] mixed states,[10] open systems,[11,12] two-species Bose–Einstein condensates,[13] any two states including orthogonal states,[14] spin-1/2 systems,[15,16] composite systems,[17,18] dephasing systems,[19] and non-isolated system[20] etc. Furthermore, it has been recognized that the Berry and AA phases are an important way to implement quantum computation.[21−24] It should be addressed that the geometric quantum computing is a way of manipulating quantum states using quantum gates based on the Berry phase shifts and holds the merit of some built-in fault tolerant features. However, the discussion about the Berry phase in a composite system has been completed only in a spin 1/2 quantum system.[17,18]

In this paper, we mainly investigate the Berry phase of a two-interaction-spin system with an arbitrary spin quantum number. If this quantum system with an isotropic exchange coupling is driven by a time-varying magnetic field,[25] under the adiabatic condition the Berry phase can be obtained explicitly by means of the time-dependent gauge transform.

In a time-dependent rotational magnetic field, the Hamiltonian in our model reads

$$\hat{H} = 2J\widehat{\boldsymbol{S}}_1 \cdot \widehat{\boldsymbol{S}}_2 + g\mu_{\rm B}\boldsymbol{B}(t)\cdot(\widehat{\boldsymbol{S}}_1 + \widehat{\boldsymbol{S}}_2), \qquad (1)$$

where $J > 0$ describes the antiferromagnetic coupling and $J < 0$ the ferromagnetic coupling, and $\boldsymbol{B}(t)$ denotes the magnetic field. If $\boldsymbol{B}(t)$ is chosen as $\boldsymbol{B}(t) = B(\sin\theta\cos\varphi(t),\ \sin\theta\sin\varphi(t),\ \cos\theta)$, Hamiltonian (1) can be rewritten as

$$\begin{aligned}\hat{H} =& J(S^2 - S_1^2 - S_2^2) + g\mu_{\rm B}B\widehat{S}_z\cos\theta \\ &+ \frac{1}{2}g\mu_{\rm B}B(\widehat{S}_+ {\rm e}^{-{\rm i}\varphi(t)} + \widehat{S}_- {\rm e}^{{\rm i}\varphi(t)}). \qquad (2)\end{aligned}$$

For simplicity the spin quantum numbers of two particles are taken to be identical, that is, $S_1 = S_2$, from which it follows that $\boldsymbol{S} = \boldsymbol{S}_1 + \boldsymbol{S}_2$, $S_z = S_{1z} + S_{2z}$ and $S_\pm = S_{1\pm} + S_{2\pm}$. In the eigenvector $|S, m\rangle = |S_1, S_2, S, m\rangle$ of $(\hat{S}^2, S_1^2, S_2^2, \hat{S}_z)$, we have $S^2|S,m\rangle = S(S+1)|S,m\rangle$, $S_z|S,m\rangle = m|S,m\rangle$ and $S_i^2|S,m\rangle = S_i(S_i+1)|S,m\rangle$.

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In the generalized time-dependent gauge transform[13,26,27] defined by

$$\hat{R}(t) = \mathrm{e}^{\lambda/2(\hat{S}_- \mathrm{e}^{\mathrm{i}\varphi(t)} - \hat{S}_+ \mathrm{e}^{-\mathrm{i}\varphi(t)})} \tag{3}$$

the time-dependent Schrödinger equation is a covariant such that ($\hbar = 1$)

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi'(t)\rangle = \hat{H}'\,|\psi'(t)\rangle\,, \tag{4}$$

where

$$\hat{H}' = \hat{R}\hat{H}\hat{R}^\dagger - \mathrm{i}\hat{R}\frac{\partial}{\partial t}\hat{R}^\dagger, \quad |\psi'(t)\rangle = \hat{R}\,|\psi(t)\rangle\,, \tag{5}$$

and the state $|\psi(t)\rangle$ is the solution of the original Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\,|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle\,. \tag{6}$$

The time-dependent auxiliary parameter λ given in Eq.(3) is to be determined by requiring that the Hamiltonian operator $\hat{H}'$ is diagonal in the representation $|S,m\rangle$. Using the following relations:

$$\begin{aligned}\hat{R}\hat{S}_+\hat{R}^\dagger =&\hat{S}_+\cos^2\frac{\lambda}{2} - \hat{S}_-\mathrm{e}^{2\mathrm{i}\varphi(t)}\sin^2\frac{\lambda}{2}\\ &- \hat{S}_z\mathrm{e}^{\mathrm{i}\varphi(t)}\sin\lambda,\end{aligned} \tag{7}$$

$$\begin{aligned}\hat{R}\hat{S}_-\hat{R}^\dagger =&\hat{S}_-\cos^2\frac{\lambda}{2} - \hat{S}_+\mathrm{e}^{-2\mathrm{i}\varphi(t)}\sin^2\frac{\lambda}{2}\\ &- \hat{S}_z\mathrm{e}^{-\mathrm{i}\varphi(t)}\sin\lambda,\end{aligned} \tag{8}$$

$$\begin{aligned}\hat{R}\hat{S}_z\hat{R}^\dagger =&\hat{S}_z\cos\lambda + \frac{1}{2}(\widehat{S}_+\mathrm{e}^{-\mathrm{i}\varphi(t)}\\ &+ \widehat{S}_-\mathrm{e}^{\mathrm{i}\varphi(t)})\sin\lambda,\end{aligned} \tag{9}$$

$$\begin{aligned}\mathrm{i}\hat{R}\frac{\partial}{\partial t}\hat{R}^\dagger =& -2\frac{\mathrm{d}\varphi}{\mathrm{d}t}\sin^2\frac{\lambda}{2}\hat{S}_z\\ &+\frac{1}{2}\frac{\mathrm{d}\varphi}{\mathrm{d}t}\sin\lambda(\widehat{S}_+\mathrm{e}^{-\mathrm{i}\varphi(t)} + \widehat{S}_-\mathrm{e}^{\mathrm{i}\varphi(t)})\\ &+\frac{\mathrm{i}}{2}\frac{\mathrm{d}\lambda}{\mathrm{d}t}(\widehat{S}_+\mathrm{e}^{-\mathrm{i}\varphi(t)} - \widehat{S}_-\mathrm{e}^{\mathrm{i}\varphi(t)}),\end{aligned} \tag{10}$$

we obtain the Hamiltonian operator in the adiabatic approximation such that $\mathrm{d}\varphi/\mathrm{d}t \approx 0$ and $\mathrm{d}\lambda/\mathrm{d}t \approx 0$

$$\hat{H}' = J(S^2 - S_1^2 - S_2^2) + \Omega\hat{S}_z \tag{11}$$

and

$$\Omega = -g\mu_\mathrm{B}B\sin\lambda(\cos^2\theta + 1) \tag{12}$$

with the auxiliary parameter λ determined by

$$\cos\lambda = -\cos\theta\sin\lambda. \tag{13}$$

Therefore, it is easy to obtain the general solution of Eq.(6) as follows:

$$\begin{aligned}|\psi(t)\rangle =&\sum_{S,m} C_{S,m}\mathrm{e}^{-\mathrm{i}\alpha_m(t)}\,|S(t),m(t)\rangle\\ =&\sum_{S,m} C_{S,m}\mathrm{e}^{-\mathrm{i}\alpha_m(t)}\hat{R}^\dagger(t)\,|S,m\rangle\,.\end{aligned} \tag{14}$$

The total phase in Eq.(14) is seen to be

$$\begin{aligned}\alpha_{S,m}(t) =&\varepsilon_{S,m} + \gamma_{S,m}\\ =&\int_0^t \mathrm{d}t'\left\langle S,m\left|\mathrm{i}\hat{R}^+\frac{\partial}{\partial t'}\hat{R} - \hat{R}^+\hat{H}\hat{R}\right|S,m\right\rangle\\ =&[JS(S+1) - JS_1(S_1+1)\\ &- JS_2(S_2+1) + m\Omega]t,\end{aligned} \tag{15}$$

which consists of the usual dynamical phase given by

$$\varepsilon_{S,m} = \int_0^t \langle S,m|\,(\hat{R}\hat{H}\hat{R}^\dagger)\,|S,m\rangle\,\mathrm{d}t' \tag{16}$$

and the Berry phase written as

$$\gamma_{S,m}(t) = -\mathrm{i}\int_0^t \langle S,m|\,(\hat{R}\frac{\partial}{\partial t}\hat{R}^\dagger)\,|S,m\rangle\,\mathrm{d}t'. \tag{17}$$

It should be noted that $|\psi(t)\rangle$ satisfies the Schrödinger equation, whereas

$$|S(t),m(t)\rangle = \hat{R}^\dagger(t)\,|S,m\rangle$$

does not. The appearance of the phase factor $\mathrm{e}^{-\mathrm{i}[\varepsilon_m(t)+\gamma_m(t)]}$ originates from the dynamical requirement that $|\psi(t)\rangle$ must satisfy the Schrödinger equation. For simplicity, we consider only a time-independent Ω . With the help of Eq.(10), the Berry phase is obtained by

$$\gamma_{S,m}(t) = 2m\sin^2\frac{\lambda}{2}\oint \mathrm{d}\varphi, \tag{18}$$

namely,

$$\gamma_{S,m}(T) = 4\pi m\sin^2\frac{\lambda}{2} = 2\pi m(1-\cos\lambda), \tag{19}$$

where T denotes the period. It is obvious that the Berry phase does not depend on an explicit form of the function $\varphi(t)$. For the initial state with $\theta = -\lambda$, the Berry phase then reduces to $\gamma_{S,m}(T) = 2\pi m$, which agrees with the recent result reported in Refs.[13,27]. In addition the Berry phases for the spin$-1/2$ system are obtained from Eq.(19) by

$$\gamma_{1,1}(T) = 2\pi(1-\cos\lambda), \text{ for } S=1, \quad m=1, \tag{20}$$

$$\gamma_{1,-1}(T) = -2\pi(1-\cos\lambda), \text{ for } S=1, \quad m=-1, \tag{21}$$

$$\gamma_{1,0}(t) = 0, \text{ for } S = 1, \quad m = 0, \tag{22}$$

$$\gamma_{0,0}(t) = 0, \text{ for } S = 0, \quad m = 0. \tag{23}$$

From Eq.(11), the Hamiltonian $\hat{H}'$ satisfies the following Schrödinger equation:

$$\hat{H}' \left| S, m \right\rangle = E'_m(t) \left| S, m \right\rangle, \tag{24}$$

and the initial state can be written in the basis of the eigenstate of the Hamiltonian $\hat{H}'$ as

$$\left| \psi'(0) \right\rangle = \sum_m C_{S,m} \left| S, m \right\rangle, \tag{25}$$

then the general state at time t, associated with the initial condition $\left| \psi'(0) \right\rangle$, can be found as

$$\begin{aligned} \left| \psi'(t) \right\rangle =& \sum_m \mathrm{e}^{-\mathrm{i} \int^t E'_m(t')\mathrm{d}t'} C_{S,m} \left| S, m \right\rangle \\ =& \mathrm{e}^{-\mathrm{i} \int^t \hat{H}'(t')\mathrm{d}t'} \left| \psi'(0) \right\rangle. \end{aligned} \tag{26}$$

Therefore, the solution of Eq.(6) is written as

$$\begin{aligned} \left| \psi(t) \right\rangle =& \hat{R}^\dagger \left| \psi'(t) \right\rangle \\ =& \hat{R}^\dagger \mathrm{e}^{-\mathrm{i} \int_0^t [\hat{R}\hat{H}\hat{R}^\dagger - \mathrm{i}\hbar \hat{R} \frac{\partial}{\partial t} \hat{R}^\dagger] \mathrm{d}t'} \hat{R}(0) \left| \psi(0) \right\rangle \\ =& \hat{U}(t,0) \left| \psi(0) \right\rangle, \end{aligned} \tag{27}$$

and the time-evolution operator becomes

$$\hat{U}(t,0) = \hat{R}^\dagger \mathrm{e}^{-\mathrm{i} \int_0^t [\hat{R}\hat{H}\hat{R}^\dagger - \mathrm{i}\hbar \hat{R} \frac{\partial}{\partial t} \hat{R}^\dagger] \mathrm{d}t'} \hat{R}(0). \tag{28}$$

The time evolution of arbitrary initial states can be evaluated with the time evolution operator given by Eq.(28), which is useful in quantum optics to investigate, for example, the time-dependent two-photon squeezing states, SU(2) squeezing states, and squeezing properties of the time-dependent SU(1,1) coherent states.[28]

In conclusion, we have obtained an exact solution of Schrödinger equation with two coupled arbitrary spins driven by a time-varying magnetic field. Using a technique of time-dependent gauge transform, the Berry phase is obtained explicitly if the quantum system undergoes a cyclic and adiabatic evolution. Compared with the dynamic phase, the Berry phase does not depend on time but is related to the area enclosed by the parameters over a cyclic evolution. It is, therefore, purely geometric in nature. One would, therefore, believe that quantum gates based on this geometric phase are possible to implement and the two coupled arbitrary spin system is a very good candidate for quantum computation.

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