Vol 15 No 5, May 2006
1009-1963/2006/15(05)/0903-04
Chinese Physics
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Unified symmetry of nonholonomic mechanical systems with variable mass

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(Received 17 November 2005; revised manuscript received 12 January 2006)

Based on the total time derivative along the trajectory of the system the definition and the criterion for a unified symmetry of nonholonomic mechanical system with variable mass are presented in this paper. A new conserved quantity, as well as the Noether conserved quantity and the Hojman conserved quantity, deduced from the unified symmetry, are also obtained. An example is given to illustrate the application of the results.

Keywords: variable mass, nonholonomic mechanical system, unified symmetry, conserved quantity
PACC: 0320

1. Introduction

It is significant to study the symmetries and conserved quantities of dynamic systems not only in mathematics but also in physics. There exist three main kinds of symmetries: the Noether symmetry,[1−4] the Lie symmetry[5−8] and the form invariance,[9−15] and three main kinds of conserved quantities: the Noether conserved quantity,[1−4] the Hojman conserved quantity[16] and a new conserved quantity.[17]

Previous studies on symmetries and conserved quantities are all carried out from a certain kind of symmetry to seek a type of conserved quantity. Recently, Mei gave a Noether–Lie symmetry[18] of Lagrange system which could lead to the Noether conserved quantity and the Hojman conserved quantity. Wu firstly defined the Lie-form invariance[19] of holonomic mechanical system which could lead to the Hojman conserved quantity and a new conserved quantity. Mei *et al* also gave a unified symmetry[20] of a Lagrangian system which could lead to the three kinds of conserved quantities mentioned above. Xu *et al* generalized Mei's results to a holonomic mechanical system.[21] On the basis of the total time derivative along the trajectory of the system, this paper generalizes Mei's results to a nonholonomic mechanical system with variable mass and obtained three kinds of conserved quantities mentioned above. Since the holonomic mechanical system with constant mass is a special case of the nonholonomic mechanical system with variable mass, therefore the results of this paper have more universal significance.

2. Definition and criterion of unified symmetry

We consider a mechanical system composed of N particles, and its configuration is determined by the n generalized coordinates $q_s(s = 1, \cdots, n)$. Let its motion be subjected to the g ideal nonlinear nonholonomic constraints of Chetaev type

$$\varphi_\beta(t, q_s, \dot{q}_s) = 0, \quad (\beta = 1, \cdots, g), \tag{1}$$

$$\frac{\partial \varphi_\beta}{\partial \dot{q}_s}\delta q_s = 0. \tag{2}$$

The mass of a representative point i of the system is

$$m_i = m_i(t, q_s, \dot{q}_s), \quad (i = 1, \cdots, N). \tag{3}$$

Then its motion can be described by the differential equations as follows:

$$E_s(L) = Q_s + P_s + \lambda_\beta \frac{\partial \varphi_\beta}{\partial \dot{q}_s},$$
$$(s = 1, \cdots, n; \beta = 1, \cdots, g), \tag{4}$$

where $L = L(t, q_s, \dot{q}_s)$ is the Lagrangian, $Q_s = Q_s(t, q_s, \dot{q}_s)$ are the nonpotential generalized forces,

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$P_s = P_s(t, q_s, \dot{q}_s, \ddot{q}_s)$ are the generalized counter thrusts, and we have

$$P_s = (\dot{m}_i \boldsymbol{\mu}_i + \dot{m}_i \dot{\boldsymbol{r}}) \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_s} - \frac{1}{2} \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i \frac{\partial m_i}{\partial q_s} + \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \dot{\boldsymbol{r}}_i \cdot \dot{\boldsymbol{r}}_i \frac{\partial m_i}{\partial \dot{q}_s} \right), \quad (s = 1, \cdots, n; i = 1, \cdots, N), \tag{5}$$

where $\boldsymbol{r}_i$ and $\dot{\boldsymbol{r}}_i$ are the vector radius and the velocity of the particle i respectively. $\boldsymbol{\mu}_i$ is the relative velocity of the small particle corresponding to the particle i, E_s are the Euler operators

$$E_s = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}. \tag{6}$$

In general, it is supposed that Eqs.(4) is nonsingular, before integrating them, we can find λ_β as functions of t, q_s and $\dot{q}_s$, thus we can get the equations of motion of the holonomic system corresponding to the nonholonomic system expressed by Eqs.(1) and Eqs.(4):

$$E_s(L) = Q_s + P_s + \Lambda_s, \quad (s = 1, \cdots, n), \tag{7}$$

where $\Lambda_s = \Lambda_s(t, q_s, \dot{q}_s) = \lambda_\beta \dfrac{\partial \varphi_\beta}{\partial \dot{q}_s}$ are the generalized nonholonomic reactive constraint forces.

From Eqs.(7), we can find all the generalized accelerations as

$$\ddot{q}_s = \alpha_s(t, q_s, \dot{q}_s), \quad (s = 1, \cdots, n). \tag{8}$$

Definition[20] If a symmetry is a Noether one as well as a Lie one and is also a form invariant at one time, the symmetry is called a unified symmetry.

For nonholonomic system, the constrained Eqs.(1) retain the form invariance:

$$X^{(1)}(\varphi_\beta) = 0. \tag{9}$$

For nonholonomic system with variable mass (1) and (4), the Noether identity can be written in the Eqs.(2) and has the form

$$L \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_0 + X^{(1)}(L) + (Q_s + \Lambda_s + P_s)(\xi_s - \dot{q}_s \xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} G_{\mathrm{N}} = 0. \tag{10}$$

The determining equations of the Lie symmetry are Eqs.(9) and

$$X^{(2)} \left\{ \bar{E}_s(L) \right\} = X^{(1)}(Q_s) + X^{(2)}(P_s) + X^{(1)}(\Lambda_s). \tag{11}$$

The criterion equations of the form invariance are Eqs.(9) and

$$\bar{E}_s \left\{ X^{(1)}(L) \right\} = X^{(1)}(Q_s) + X^{(2)}(P_s) + X^{(1)}(\Lambda_s). \tag{12}$$

The following equations are obtained on the basis of the total time derivative along the trajectory of the system

$$\begin{cases} X^{(1)} = \xi_0 \dfrac{\partial}{\partial t} + \xi_s \dfrac{\partial}{\partial q_s} + \left(\dfrac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_s - \dot{q}_s \dfrac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_0 \right) \dfrac{\partial}{\partial \dot{q}_s}, \\ X^{(2)} = X^{(1)} + \left(\dfrac{\bar{\mathrm{d}}}{\mathrm{d}t} \dfrac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_s - 2\ddot{q}_s \dfrac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_0 - \dot{q}_s \dfrac{\bar{\mathrm{d}}}{\mathrm{d}t} \dfrac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_0 \right) \dfrac{\partial}{\partial \ddot{q}_s}, \\ \dfrac{\bar{\mathrm{d}}}{\mathrm{d}t} = \dfrac{\partial}{\partial t} + \dot{q}_s \dfrac{\partial}{\partial q_s} + \alpha_s \dfrac{\partial}{\partial \dot{q}_s}, \\ \bar{E}_s = \dfrac{\bar{\mathrm{d}}}{\mathrm{d}t} \dfrac{\partial}{\partial \dot{q}_s} - \dfrac{\partial}{\partial q_s}. \end{cases} \tag{13}$$

Using the definition and relations (9), (10), (11) and (12), we have:

Criterion if there exists such a gauge function $G_{\mathrm{N}} = G_{\mathrm{N}}(t, q_s, \dot{q}_s)$ that the infinitesimal generators ξ_0 and ξ_s satisfy Eqs.(2), (9) and

$$\left[L \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \xi_0 + X^{(1)}(L) + (Q_s + \Lambda_s + P_s)(\xi_s - \dot{q}_s \xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} G_{\mathrm{N}} \right]^2 + \sum_{s=1}^{n} \left\{ \bar{E}_s \left[X^{(1)}(L) \right] - X^{(1)}(Q_s) - X^{(2)}(P_s) - X^{(1)}(\Lambda_s) \right\}^2 + \sum_{s=1}^{n} \left\{ X^{(2)} \left[\bar{E}_s(L) \right] - X^{(1)}(Q_s) - X^{(2)}(P_s) - X^{(1)}(\Lambda_s) \right\}^2 = 0, \tag{14}$$

then the symmetry is a unified symmetry of the nonholonomic mechanical system with variable mass.

3. Conserved quantity deduced from unified symmetry

The unified symmetry of the nonholonomic mechanical system with variable mass can lead to all of the Noether conserved quantity, the Hojman conserved quantity and the new conserved quantity under certain conditions.

Proposition 1 For the nonholonomic mechanical system with variable mass, the unified symmetry will lead to the Noether conserved quantity

$$I_{\mathrm{N}}=L\xi_0+\frac{\partial L}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)+G_{\mathrm{N}}=\text{const.} \tag{15}$$

Proof Because the unified symmetry is certainly a Noether symmetry, there exists such a gauge function $G_{\mathrm{N}}=G_{\mathrm{N}}(t,q_s,\dot{q}_s)$ that the Noether identity (10) holds. According to Ref.[22], the system has the conserved quantity (15).

Proposition 2[23] For the nonholonomic mechanical system with variable mass, under the general infinitesimal transformations of groups in which time is variable, if there is a function $\mu=\mu(t,q_s,\dot{q}_s)$ such that

$$\frac{\partial\alpha_s}{\partial\dot{q}_s}+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu=0, \tag{16}$$

then the unified symmetry will lead to a Hojman conserved quantity of a general form

$$\begin{aligned}I_{\mathrm{H}}=&\frac{1}{\mu}\frac{\partial}{\partial t}(\mu\xi_0)\\&+\frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s)+\frac{1}{\mu}\frac{\partial}{\partial\dot{q}_s}\left[\mu\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s-\dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\right]\\&-\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0=\text{const.}\end{aligned} \tag{17}$$

Proof Because the unified symmetry is certainly a Lie symmetry, so we have the determining equations of Lie symmetry

$$\begin{aligned}&\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s-\dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\\&-2\alpha_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0-\frac{\partial\alpha_s}{\partial t}\xi_0-\frac{\partial\alpha_s}{\partial q_k}\xi_k\\&-\frac{\partial\alpha_s}{\partial\dot{q}_k}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k-\dot{q}_k\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)=0.\end{aligned} \tag{18}$$

From expressions (16) and (18), by using Hojman's proof method, we can prove proposition 2.

Proposition 3 For the nonholonomic mechanical system with variable mass, if there is a function $G_{\mathrm{F}}=G_{\mathrm{F}}(t,q_s,\dot{q}_s)$ such that

$$\begin{aligned}&X^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0+X^{(1)}\left[X^{(1)}(L)\right]\\&+\left[X^{(1)}(Q_s+\Lambda_s)+X^{(2)}(P_s)\right](\xi_s-\dot{q}_s\xi_0)\\&+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{F}}=0\end{aligned} \tag{19}$$

then the unified symmetry will lead to a new type of conserved quantity

$$I_{\mathrm{F}}=X^{(1)}(L)\xi_0+\frac{\partial X^{(1)}(L)}{\partial\dot{q}_s}(\xi_s-\dot{q}_s\xi_0)+G_{\mathrm{F}}=\text{const.} \tag{20}$$

Proof Because the unified symmetry is certainly a form invariance, so the criterion Eqs.(12) hold. Using relationships (12) and (19), we can prove that proposition 3 is true.[24]

4. Illustrative example

We consider a particle of variable mass, its mass is

$$m=m_0\exp(-t),(m_0=\text{const.}). \tag{21}$$

The Lagrangian is

$$L=\frac{1}{2}m(\dot{q}_1^2+\dot{q}_2^2+\dot{q}_3^2). \tag{22}$$

The nonpotential generalized forces are

$$Q_1=Q_2=Q_3=0. \tag{23}$$

The absolute velocity of separate corpuscle is zero, i.e.

$$\boldsymbol{\mu}=-\dot{\boldsymbol{r}}_i=-\dot{q}_1\boldsymbol{i}-\dot{q}_2\boldsymbol{j}-\dot{q}_3\boldsymbol{k}. \tag{24}$$

The motion of the particle is subjected to the following nonholonomic constraint:

$$\varphi=\dot{q}_1^2+\dot{q}_2^2-\dot{q}_3^2=0. \tag{25}$$

Let's study the unified symmetry of the system. According to Eqs.(21), (23) and (5), we have

$$P_1=P_2=P_2=0. \tag{26}$$

Choosing the infinitesimal generators as

$$\xi_0=1,\qquad\xi_1=\xi_2=\xi_3=0, \tag{27}$$

after calculation, we find that generators (27) satisfy Eqs.(2), Eqs.(9) and

$$\begin{cases}X^{(1)}(L)=\dot{m}\dot{q}_3^2,\quad X^{(1)}\left[X^{(1)}(L)\right]=0,\\Q_1+\Lambda_1+P_1=Q_2+\Lambda_2+P_2=0,\\X^{(1)}(Q_1+\Lambda_1)+X^{(2)}(P_1)=0,\\X^{(1)}(Q_2+\Lambda_2)+X^{(2)}(P_2)=0.\end{cases} \tag{28}$$

It is easy to verify that generators (27) are quantities with unified symmetry.

Substituting relationships (22), (23), (27) and (28) into the Noether identity (10), we obtain the gauge function

$$G_{\mathrm{N}} = -m\dot{q}_3^2, \tag{29}$$

and proposition 1 gives

$$I_{\mathrm{N}} = -\frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2 + 3\dot{q}_3^2) = \text{const.}, \tag{30}$$

which is the Noether conserved quantity deduced from the unified symmetry (27).

Formula (16) becomes

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = -3, \tag{31}$$

which has the solution

$$\mu = (\dot{q}_3 - q_3)\exp(-3t). \tag{32}$$

From Eqs.(27) and (32), using proposition 2, we can only obtain a trivial Hojman conserved quantity:

$$I_{\mathrm{H}} = -3. \tag{33}$$

Substituting Eqs.(27) and (28) into Eqs.(19), we obtain

$$G_{\mathrm{F}} = 0. \tag{34}$$

And proposition 3 gives

$$I_{\mathrm{F}} = -\dot{m}\dot{q}_3^2 = \text{const.}, \tag{35}$$

which is a new conserved quantity deduced from the unified symmetry (27).

It is obvious that both the Hojman conserved quantity (33) and the new conserved quantity (35) are non-Noether conserved quantities.

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