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Potential method of integration for solving the equations of mechanical systems*

Wu Hui-Bin(吴惠彬)†

Faculty of Science, Beijing Institute of Technology, Beijing 100081, *China*

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This paper is intended to apply a potential method of integration to solving the equations of holonomic and nonholonomic systems. For a holonomic system, the differential equations of motion can be written as a system of differential equations of first order and its fundamental partial differential equation is solved by using the potential method of integration. For a nonholonomic system, the equations of the corresponding holonomic system are solved by using the method and then the restriction of the nonholonomic constraints on the initial conditions of motion is added.

Keywords: potential method of integration, holonomic system, nonholonomic system, equation of motion

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1. Introduction

The method of integration for solving the equations of mechanical systems is a very important and difficult. Most of the classical methods of integration in analytical mechanics are devoted to holonomic and conservative systems, say the canonical transformation, the invariant of integration, the Hamilton–Jacobi method, the Poisson method, etc.[1−3] In general, these traditional methods are unsuitable for nonconservative systems and nonholonomic systems. But the potential method of integration,[4] the field method of integration[5−8] posed after 1960s, the Noether method[9−18] developed lately, and so on provide powerful tools for integrating the differential equations of motion of nonconservative systems and nonholonomic systems. This paper is to apply the potential method of integration[4] raised by Arzhanyh in 1965 for solving the differential equations of motion of holonomic and nonholonomic systems.

2. Potential method of integration

Study the solution of differential equations

$$\dot{x}_i = F_i(t, x_j), \quad (i, j = 1, \ldots, n). \tag{1}$$

The potential method of integration[4] shows that the solution of Eq.(1) can be expressed as

$$x_i = \frac{\partial \psi}{\partial p_i}, \tag{2}$$

where $\psi(t, p_i, a_i)$ is a complete integral of the partial differential equation of first order

$$\frac{\partial \psi}{\partial t} = p_j F_j \left(t, \frac{\partial \psi}{\partial p_i} \right). \tag{3}$$

When expressing the solution (2) as an explicit function of time t, p_i must be replaced by its value that can be found from the following expression:

$$\frac{\partial \psi}{\partial a_i} = b_i = \text{const.} \tag{4}$$

The result is

$$x_i = f_i(t, c_j), \tag{5}$$

where

$$c_i = \omega_i(a_j, b_j), \tag{6}$$

in which a_i, b_i, c_i are all constants.

The main difficulty in applying the above potential method of integration is to find the complete integral of the partial differential equation (3) that is

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†E-mail: huibinwu@bit.edu.cn

called the fundamental partial differential equation. In Ref.[4], the above result was obtained, but not applied to any specific mechanical systems.

3. Application of the potential method of integration to holonomic systems

The differential equations of motion of a general holonomic mechanical system have the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s,\quad (s=1,\ldots,n), \tag{7}$$

where $L=L(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ is the Lagrangian of the system, $Q_s=Q_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})$ are the generalized forces and q_s the generalized coordinates. When

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_k}\right)\neq 0, \tag{8}$$

all the generalized accelerations can be solved from Eq.(7), and written as

$$\ddot{q}_s=g_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\quad (s=1,\ldots,n), \tag{9}$$

which can be transformed into equations of first order, and then solved by using the above potential method of integration.

4. Application of the potential method of integration to nonholonomic systems

The differential equations of motion and the constraint equations of a general nonholonomic mechanical system are respectively given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s+\lambda_\beta\frac{\partial f_\beta}{\partial \dot{q}_s}\quad (s=1,\ldots,n), \tag{10}$$

$$f_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0\quad (\beta=1,\ldots,g). \tag{11}$$

Eliminating multipliers λ_β by using Eqs.(10) and (11),[19] one can obtain the equations of the corresponding holonomic system

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s+\Lambda_s, \tag{12}$$

where

$$\Lambda_s=\Lambda_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})=\lambda_\beta(t,\boldsymbol{q},\dot{\boldsymbol{q}})\frac{\partial f_\beta}{\partial \dot{q}_s} \tag{13}$$

are explicit functions of $t,\boldsymbol{q},\dot{\boldsymbol{q}}$, and called generalized nonholonomic constraint reaction forces. In order to find the solution of Eqs.(10) and (11) for the nonholonomic system, firstly, we transform Eqs.(12) into a system of differential equations of first order, and find its solution by using the potential method of integration; secondly we add the restriction of nonholonomic constraints (11) on the initial conditions of motion

$$f_\beta(t_0,\boldsymbol{q}_0,\dot{\boldsymbol{q}}_0)=0 \tag{14}$$

to the solution. Thus, there are $(2n-g)$ arbitrary constants in the solution of Eqs.(10) and (11) for the nonholonomic system.

5. Illustrative examples

Example 1 Study the integral of the following holonomic nonconservative system:

$$L=\frac{1}{2}\left(\dot{q}_1^2+\dot{q}_2^2\right)+\frac{1}{2}q_1^2,\quad Q_1=0,\quad Q_2=\dot{q}_1. \tag{15}$$

The differential equations of the system are

$$\ddot{q}_1=q_1,\quad \ddot{q}_2=\dot{q}_1. \tag{16}$$

Setting

$$x_1=q_1,\quad x_2=q_2,\quad x_3=\dot{q}_1,\quad x_4=\dot{q}_2,$$

Eq.(16) can be expressed as

$$\dot{x}_1=x_3,\quad \dot{x}_2=x_4,\quad \dot{x}_3=x_1,\ \text{and}\ \dot{x}_4=x_3. \tag{17}$$

Then the fundamental partial differential equation (3) gives

$$\frac{\partial \psi}{\partial t}=p_1\frac{\partial \psi}{\partial p_3}+p_2\frac{\partial \psi}{\partial p_4}+p_3\frac{\partial \psi}{\partial p_1}+p_4\frac{\partial \psi}{\partial p_3}. \tag{18}$$

The following complete integral of Eq.(18) can be found:

$$\begin{aligned}\psi =&a_1(-p_1+p_2+p_3-p_4)\exp(-t)\\&+a_2p_2+a_3(p_1+p_2+p_3+p_4)\exp(t)\\&+a_4(p_4+p_2t).\end{aligned}$$

Thus, the expression (2) gives

$$\begin{aligned}x_1&=\frac{\partial \psi}{\partial p_1}=-a_1\exp(-t)+a_3\exp(t),\\ x_2&=\frac{\partial \psi}{\partial p_2}=a_1\exp(-t)+a_3\exp(t)+a_2+a_4t,\\ x_3&=\frac{\partial \psi}{\partial p_3}=a_1\exp(-t)+a_3\exp(t),\\ x_4&=\frac{\partial \psi}{\partial p_4}=-a_1\exp(-t)+a_3\exp(t)+a_4.\end{aligned} \tag{19}$$

Example 2 A nonholonomic system is

$$L=\frac{1}{2}\left(\dot{q}_1^2+\dot{q}_2^2\right),\quad Q_1=Q_2=0, \tag{20}$$

$$f = \dot{q}_1 + bt\dot{q}_2 - bq_2 + t = 0. \tag{21}$$

Find its solution by using the Arzhanyh method(i.e. the potential method of integration).

The differential equations of the corresponding holonomic system of the above system are

$$\ddot{q}_1 = -\frac{1}{1+b^2t^2}, \quad \ddot{q}_2 = -\frac{bt}{1+b^2t^2}. \tag{22}$$

Setting

$$x_1 = q_1, \quad x_2 = q_2, \quad x_3 = \dot{q}_1, \quad x_4 = \dot{q}_2,$$

Eq.(22) can be expressed as

$$\dot{x}_1 = x_3, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = -\frac{1}{1+b^2t^2},$$

$$\dot{x}_4 = -\frac{bt}{1+b^2t^2}. \tag{23}$$

Then the fundamental partial differential equation (3) gives

$$\begin{aligned}\frac{\partial\psi}{\partial t} =& p_1\frac{\partial\psi}{\partial p_3} + p_2\frac{\partial\psi}{\partial p_4} + p_3\left(-\frac{1}{1+b^2t^2}\right) \\ &+ p_4\left(-\frac{bt}{1+b^2t^2}\right).\end{aligned} \tag{24}$$

The following complete integral of the above equation can be found:

$$\begin{aligned}\psi =& a_1p_1 + a_2p_2 + a_3(p_3 + p_1t) + a_4(p_4 + p_2t) \\ &+ p_1\frac{1}{2b^2}\ln(1+b^2t^2) - (p_3 + p_1t)\frac{1}{b}\arctan(bt) \\ &+ p_2\left(\frac{t}{b} - \frac{1}{b^2}\arctan(bt)\right) \\ &- (p_4 + p_2t)\frac{1}{2b}\ln(1+b^2t^2).\end{aligned}$$

Thus, the expression (2) shows

$$\begin{aligned}x_1 =& \frac{\partial\psi}{\partial p_1} = a_1 + a_3t + \frac{1}{2b^2}\ln(1+b^2t^2) \\ &- \frac{t}{b}\arctan(bt), \\ x_2 =& \frac{\partial\psi}{\partial p_2} = a_2 + a_4t + \frac{t}{b} - \frac{1}{b^2}\arctan(bt) \\ &- \frac{t}{2b}\ln(1+b^2t^2), \\ x_3 =& \frac{\partial\psi}{\partial p_3} = a_3 - \frac{1}{b}\arctan(bt), \\ x_4 =& \frac{\partial\psi}{\partial p_4} = a_4 - \frac{1}{2b}\ln(1+b^2t^2),\end{aligned} \tag{25}$$

which is a solution of the equations of the corresponding holonomic system. For finding the solution of the nonholonomic system, one must add the restriction of nonholonomic constraints (11) on the initial conditions of motion to the solution. Let the initial conditions be

$$t = 0, \quad x_s(0) = x_{s0}, \quad (s = 1, \cdots, 4),$$

then the restriction of nonholonomic constraint (21) will be

$$x_{30} - bx_{20} = 0, \tag{26}$$

namely

$$a_3 - ba_2 = 0. \tag{27}$$

Thus, the expressions (25) and (27) are the solution of the nonholonomic system, in which there are only 3 arbitrary constants.

6. Summary

The potential method of integration expounded in this paper was given by Arzhanyh in his book[4] in 1965. The purpose of this paper is to apply the existing method to integrating the differential equations of holonomic and nonholonomic mechanical systems. The two examples shown above are typical, the first one is the well-known Whittaker equation,[20] the second one was given by Novoselov.[21]

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