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Direct method of finding first integral of two-dimensional autonomous systems in polar coordinates

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A direct method to find the first integral for two-dimensional autonomous system in polar coordinates is suggested. It is shown that if the equation of motion expressed by differential 1-forms for a given autonomous Hamiltonian system is multiplied by a set of multiplicative functions, then the general expression of the first integral can be obtained. An example is given to illustrate the application of the results.

Keywords: direct method, autonomous systems, first integral, polar coordinates
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1. Introduction

It is well known that research on the first integral of dynamical systems has already become an important and active field of modern mathematics, mechanics and physics, and some traditional methods, including the Lie symmetry, the Noether symmetry, the form invariance,[1−7] the direct method[8−12] and the Ermakov method,[13,14] have been used. It should be noticed that the direct method to obtain the first integral of autonomous system in Cartesian coordinates has been given. However, many dynamical systems, such as Kepler problem etc., cannot be discussed easily in Cartesian coordinates but in polar coordinates. Thus it is very important to obtain the first integral of autonomous system in polar coordinates. In this paper, we present a direct method to obtain the first integral for two-dimensional autonomous systems in polar coordinates. It is shown that if the equation of motion expressed by differential 1-forms for a given autonomous Hamiltonian system is multiplied by a set of multiplicative functions, then the general expression of the first integral can be derived. An example is given to illustrate the application of the results.

2. Theory and method of finding first integrals in polar coordinates

The Hamiltonian for two-dimensional autonomous systems in polar coordinates is

$$H=\frac{1}{2}\left(p_r^2+\frac{p_\theta^2}{r^2}\right)+V(r,\theta), \tag{1}$$

where $V(r,\theta)$ is the given potential of the system. The Hamiltonian generates the following equations of motion:

$$\frac{\mathrm{d}r}{\mathrm{d}t}=\frac{\partial H}{\partial p_r}=p_r, \tag{2a}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{\partial H}{\partial p_\theta}=\frac{p_\theta}{r^2}, \tag{2b}$$

$$\frac{\mathrm{d}p_r}{\mathrm{d}t}=-\frac{\partial H}{\partial r}=\frac{p_\theta^2}{r^3}-V_r, \tag{2c}$$

$$\frac{\mathrm{d}p_\theta}{\mathrm{d}t}=-\frac{\partial H}{\partial \theta}=-V_\theta, \tag{2d}$$

where p_r, p_θ are generalized momenta, subscripts r and θ of V denote the partial derivatives. Equations (2a)–(2d) can be equivalently written in term of the following six differential 1-forms

$$p_\theta \mathrm{d}r-r^2p_r\mathrm{d}\theta=0, \tag{3a}$$

$$(p_\theta^2-r^3V_r)\mathrm{d}r-r^3p_r\mathrm{d}p_r=0, \tag{3b}$$

$$V_\theta \mathrm{d}r+p_r\mathrm{d}p_\theta=0, \tag{3c}$$

$$(p_\theta^2-r^3V_r)\mathrm{d}\theta-rp_\theta\mathrm{d}p_r=0, \tag{3d}$$

$$r^2V_\theta \mathrm{d}\theta+p_\theta\mathrm{d}p_\theta=0, \tag{3e}$$

$$r^3V_\theta \mathrm{d}p_r+(p_\theta^2-r^3V_r)\mathrm{d}p_\theta=0. \tag{3f}$$

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Multiplying Eqs.(3a)–(3f) by various unknowns W, Q, R, S, T and U which are functions of r, θ, p_r, p_θ, and then summing up the resulting equations, we obtain

$$\begin{aligned}\mathrm{d}I =&W(p_\theta \mathrm{d}r - r^2 p_r \mathrm{d}\theta)\\ &+ Q[(p_\theta^2 - r^3 V_r)\mathrm{d}r - r^3 p_r \mathrm{d}p_r]\\ &+ R(V_\theta \mathrm{d}r + p_r \mathrm{d}p_\theta)\\ &+ S[(p_\theta^2 - r^3 V_r)\mathrm{d}\theta - r p_\theta \mathrm{d}p_r]\\ &+ T(r^2 V_\theta \mathrm{d}\theta + p_\theta \mathrm{d}p_\theta)\\ &+ U[r^3 V_\theta \mathrm{d}p_r + (p_\theta^2 - r^3 V_r)\mathrm{d}p_\theta] = 0, \end{aligned} \tag{4}$$

where $I = I(r, \theta, p_r, p_\theta)$ is the first integral.

Expanding the left hand side of Eq.(4) and equating the coefficients of $\mathrm{d}r, \mathrm{d}\theta, \mathrm{d}p_r, \mathrm{d}p_\theta$ in Eq.(4) on both sides, we have

$$I_r = p_\theta W + (p_\theta^2 - r^3 V_r)Q + V_\theta R, \tag{5a}$$

$$I_\theta = -r^2 p_r W + (p_\theta^2 - r^3 V_r)S + r^2 V_\theta T, \tag{5b}$$

$$I_{p_r} = -r^3 p_r Q - r p_\theta S + r^3 V_\theta U, \tag{5c}$$

$$I_{p_\theta} = p_r R + p_\theta T + (p_\theta^2 - r^3 V_r)U. \tag{5d}$$

Since

$$I_{r\theta} = I_{\theta r}, I_{rp_r} = I_{p_r r}, I_{rp_\theta} = I_{p_\theta r},$$

$$I_{\theta p_r} = I_{p_r \theta}, I_{\theta p_\theta} = I_{p_\theta \theta}, I_{p_r p_\theta} = I_{p_\theta p_r}, \tag{6}$$

and

$$V_{rp_r} = V_{rp_\theta} = V_{\theta p_r} = V_{\theta p_\theta} = V_{p_r} = V_{p_\theta} = 0, \tag{7}$$

we have the following six coupled partial differential equations

$$p_\theta \frac{\partial W}{\partial \theta} + r^2 p_r \frac{\partial W}{\partial r} + (p_\theta^2 - r^3 V_r)\frac{\partial Q}{\partial \theta} + V_\theta \frac{\partial R}{\partial \theta} - (p_\theta^2 - r^3 V_r)\frac{\partial S}{\partial r} - r^2 V_\theta \frac{\partial T}{\partial r}$$

$$= -2r p_r W + r^3 V_{r\theta} Q - V_{\theta\theta} R - (3r^2 V_r + r^3 V_{rr})S + (2r V_\theta + r^2 V_{r\theta})T, \tag{8a}$$

$$p_\theta \frac{\partial W}{\partial p_r} + (p_\theta^2 - r^3 V_r)\frac{\partial Q}{\partial p_r} + r^3 p_r \frac{\partial Q}{\partial r} + V_\theta \frac{\partial R}{\partial p_r} + r p_\theta \frac{\partial S}{\partial r} - r^3 V_\theta \frac{\partial U}{\partial r} = -3r^2 p_r Q - p_\theta S + (3r^2 V_\theta + r^3 V_{r\theta})U, \tag{8b}$$

$$p_\theta \frac{\partial W}{\partial p_\theta} + (p_\theta^2 - r^3 V_r)\frac{\partial Q}{\partial p_\theta} - p_r \frac{\partial R}{\partial r} + V_\theta \frac{\partial R}{\partial p_\theta} - p_\theta \frac{\partial T}{\partial r} - (p_\theta^2 - r^3 V_r)\frac{\partial U}{\partial r} = -W - 2p_\theta Q - (3r^2 V_r + r^3 V_{rr})U, \tag{8c}$$

$$-r^2 p_r \frac{\partial W}{\partial p_r} + r^3 p_r \frac{\partial Q}{\partial \theta} + r p_\theta \frac{\partial S}{\partial \theta} + (p_\theta^2 - r^3 V_r)\frac{\partial S}{\partial p_r} + r^2 V_\theta \frac{\partial T}{\partial p_r} - r^3 V_\theta \frac{\partial U}{\partial \theta} = r^2 W + r^3 V_{\theta\theta} U, \tag{8d}$$

$$-r^2 p_r \frac{\partial W}{\partial p_\theta} - p_r \frac{\partial R}{\partial \theta} + (p_\theta^2 - r^3 V_r)\frac{\partial S}{\partial p_\theta} - p_\theta \frac{\partial T}{\partial \theta} + r^2 V_\theta \frac{\partial T}{\partial p_\theta} - (p_\theta^2 - r^3 V_r)\frac{\partial U}{\partial \theta} = -2p_\theta S - r^3 V_{r\theta} U, \tag{8e}$$

$$-r^3 p_r \frac{\partial Q}{\partial p_\theta} - p_r \frac{\partial R}{\partial p_r} - r p_\theta \frac{\partial S}{\partial p_\theta} - p_\theta \frac{\partial T}{\partial p_r} - (p_\theta^2 - r^3 V_r)\frac{\partial U}{\partial p_r} + r^3 V_\theta \frac{\partial U}{\partial p_\theta} = R + rS. \tag{8f}$$

In order to solve the above equations, we only consider some special cases, for example, we take

$$W = W(r, \theta, p_r, p_\theta), Q = 0, T = T(r, \theta, p_r, p_\theta),$$

$$R = R(r, \theta, p_r, p_\theta), S = 0, U = 0. \tag{9}$$

Substituting Eqs.(19) into Eqs.(5) and (8), we obtain

$$I_r = p_\theta W + V_\theta R, \tag{10a}$$

$$I_\theta = r^2(V_\theta T - p_r W), \tag{10b}$$

$$I_{p_r} = 0, \tag{10c}$$

$$I_{p_\theta} = p_r R + p_\theta T, \tag{10d}$$

and

$$p_\theta \frac{\partial W}{\partial \theta} + r^2 p_r \frac{\partial W}{\partial r} + V_\theta \frac{\partial R}{\partial \theta} - r^2 V_\theta \frac{\partial T}{\partial r}$$

$$= -2r p_r W - V_{\theta\theta} R + (2r V_\theta + r^2 V_{r\theta})T, \tag{11a}$$

$$p_\theta \frac{\partial W}{\partial p_r} + V_\theta \frac{\partial R}{\partial p_r} = 0, \tag{11b}$$

$$p_\theta \frac{\partial W}{\partial p_\theta} - p_r \frac{\partial R}{\partial r} + V_\theta \frac{\partial R}{\partial p_\theta} - p_\theta \frac{\partial T}{\partial r} = -W, \tag{11c}$$

$$-p_r \frac{\partial W}{\partial p_r} + V_\theta \frac{\partial T}{\partial p_r} = W, \tag{11d}$$

$$-r^2 p_r \frac{\partial W}{\partial p_\theta} - p_r \frac{\partial R}{\partial \theta} - p_\theta \frac{\partial T}{\partial \theta} + r^2 V_\theta \frac{\partial T}{\partial p_\theta} = 0, \tag{11e}$$

$$-p_r \frac{\partial R}{\partial p_r} - p_\theta \frac{\partial T}{\partial p_r} = R. \tag{11f}$$

According to the theorem of partial differential equations, using Eq.(11b), we have

$$p_\theta W + V_\theta R = c_1(r, \theta, p_\theta). \tag{12}$$

From Eq.(11d), we have

$$r^2(V_\theta T - p_r W) = c_2(r, \theta, p_\theta). \tag{13}$$

From Eq.(11f), we obtain

$$p_r R + p_\theta T = c_3(r, \theta, p_\theta). \tag{14}$$

In term of Eqs.(12), (13) and (14),we obtain

$$p_r \frac{c_1 - p_\theta W}{V_\theta} + p_\theta \frac{r^2 p_r W + c_2}{r^2 V_\theta} = c_3, \tag{15}$$

that is

$$r^2 p_r c_1 + p_\theta c_2 - r^2 V_\theta c_3 = 0, \tag{16}$$

and Eqs.(10a), (10b) and (10d) can be turned into the following forms

$$I_r = p_\theta W + V_\theta R = c_1(r, \theta, p_\theta), \tag{17a}$$

$$I_\theta = r^2(V_\theta T - p_r W) = c_2(r, \theta, p_\theta), \tag{17b}$$

$$I_{p_\theta} = p_r R + p_\theta T = c_3(r, \theta, p_\theta). \tag{17c}$$

According to the above three equations, we obtain a partial differential equation,

$$r^2 p_r \frac{\partial I}{\partial r} + p_\theta \frac{\partial I}{\partial \theta} - r^2 V_\theta \frac{\partial I}{\partial p_\theta} = 0. \tag{18}$$

The solution of Eq.(18) can be deduced from the equation for the characteristic

$$\frac{\mathrm{d}r}{r^2 p_r} = \frac{\mathrm{d}\theta}{p_\theta}, \tag{19}$$

and we have

$$\frac{p_\theta}{r} + p_r \theta = a_1. \tag{20}$$

From the characteristic

$$\frac{\mathrm{d}\theta}{p_\theta} = \frac{\mathrm{d}p_\theta}{-r^2 V_\theta}, \tag{21}$$

we obtain

$$\frac{1}{2} p_\theta^2 + r^2 V + g(r) = a_2, \tag{22}$$

where $g(r)$ is an arbitrary function of r. From the characteristic

$$\frac{\mathrm{d}r}{r^2 p_r} = \frac{\mathrm{d}p_\theta}{-r^2 V_\theta}, \tag{23}$$

we obtain

$$p_r p_\theta + \int V_\theta \mathrm{d}r + h(\theta) = a_3, \tag{24}$$

where $h(\theta)$ is an arbitrary function of θ. So, the general expression of first integral is

$$I = \varphi\left(\frac{p_\theta}{r} + p_r\theta, \quad \frac{1}{2}p_\theta^2 + r^2 V + g(r), \right.$$
$$\left. p_r p_\theta + \int V_\theta \mathrm{d}r + h(\theta)\right), \tag{25}$$

where φ is an arbitrary differentiable function.

3. Example

The non-central dynamical system is one of the fundamental systems in mechanics and quantum mechanics: its Hamiltonian in polar coordinates is [14]

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\sigma}{r} + \frac{1}{r^2 \sin^2\theta}(g_1 + g_2 \cos\theta)$$
$$= \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r, \theta), \tag{26}$$

where σ, g_1 and g_2 are positive constants, and

$$V(r, \theta) = -\frac{\sigma}{r} + \frac{1}{r^2}\Phi(\theta), \tag{27}$$

$$\Phi(\theta) = \frac{1}{\sin^2\theta}(g_1 + g_2 \cos\theta). \tag{28}$$

Substituting Eq.(27) into Eqs.(22) and (24), separately, we can obtain

$$\frac{1}{2}p_\theta^2 + \Phi(\theta) - \sigma r + g(r) = a_2, \tag{29}$$

$$p_r p_\theta - \frac{1}{r}\frac{\partial \Phi(\theta)}{\partial \theta} + h(\theta) = a_3, \tag{30}$$

where $g(r)$, $h(\theta)$ are arbitrary functions of r or θ.

The general expression of first integral for non-central dynamical system is

$$I = \varphi\left(\frac{p_\theta}{r} + p_r\theta, \frac{1}{2}p_\theta^2 + \Phi(\theta) - \sigma r + g(r), \right.$$
$$\left. p_r p_\theta - \frac{1}{r}\frac{\partial \Phi(\theta)}{\partial \theta} + h(\theta)\right). \tag{31}$$

Specially, if we suppose that

$$I = \frac{1}{2}p_\theta^2 + \Phi(\theta) - \sigma r + g(r), \tag{32}$$

and

$$g(r) = \sigma r, \tag{33}$$

the first integral for non-central dynamical system is

$$I = \frac{1}{2}p_\theta^2 + \Phi(\theta). \tag{34}$$

It is shown that the first integral obtained from direct method is the same as the Ermakov invariant obtained from the Ermakov method in Ref.[14].

References

[1] Mei F X 1999 *Applications of Lie Groups and Lie Algebras to Constrained Mechanical Systems* (Beijing: Science Press) (in Chinese)

[2] Zhao Y Y and Mei F X 1999 *Symmetries and Invariants of Mechanical Systems* (Beijing: Science Press) (in Chinese)

[3] Mei F X 2004 *Symmetry and Conserved Quantity of Constrained Mechanical Systems* (Beijing: Beijing Institute of Technology Press) (in Chinese)

[4] Zhao Y Y 1994 *Acta Mech. Sin.* **26** 380 (in Chinese)

[5] Sarlet W 1981 *J. Phys.* A: *Math. Gen.* **14** 2227

[6] Lutzky M 1979 *J. Phys.* A: *Math. Gen.* **12** 973

[7] Zhang H B, Chen L Q and Liu R W 2005 *Chin. Phys.* **14** 238

[8] Annamalai A and Tamizhmani K M 1994 *Nonlinear Math. Phys.* **1** 309

[9] Gao P Y and Liu Z N 1998 *Phys. Lett.* A **244** 59

[10] Gao P Y 1999 *Phys. Lett.* A **255** 253

[11] Zhang Y 2001 *Acta Phys. Sin.* **50** 2059 (in Chinese)

[12] Shang M and Mei F X 2005 *Chin. Phys.* **14** 1707

[13] Haas F and Goedert J 1996 *J. Phys.* A: *Math. Gen.* **29** 4083

[14] Lou Z M, Chen Z D and Wang W L 2005 *Chin. Phys.* **14** 1483