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The Lagrangian and the Lie symmetries of charged particle motion in homogeneous electromagnetic field

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In this paper, a constant of motion of charged particle motion in homogeneous electromagnetic field is derived from Newton's equations and the characteristics of partial differential equation, the related Lagrangian is also given by means of the obtained constant of motion. By discussing the Lie symmetry for this classical system, this paper obtains the general expression of the conserved quantity. It is shown that the conserved quantity is the same as the constant of motion in essence.

Keywords: constant of motion, Lagrangian, Lie symmetry, conserved quantity
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1. Introduction

It is well-known that if the classical Hamiltonian usually expressed by the Lagrangian is obtained, then this mechanical system can be quantized. Therefore, it is important to extract the correct Lagrangian for a given mechanical system. On the other hand, the corresponding inverse problem that the Lagrangian is derived from the differential equations of motion has already become an important and active field of the modern mathematical, mechanical and physical science, some important works have been done.[1−7] It can be seen that the symmetry and conserved quantity of a system can be discussed from the Lagrangian with the help of the Noether and Lie methods and the form invariance, which have resulted in a series of important results in past twenty years.[8−17]

The motion of non-relativistic charged particle in electromagnetic field is a very basic dynamical problem and the related symmetry structure has attracted attention in recently years.[18−20] In Refs.[18–20], the Noether, Lie and Mei symmetries for this classical system have been studied under the condition when the Lagrangian is given and the partial differential equations for the electromagnetic field have been also obtained. However, in this paper we will obtain a constant of motion for the charged particle in a given homogeneous electromagnetic field from the Newton's equations and the characteristics of partial differential equation and therefore derive the Lagrangian by means of the obtained constant of motion. On the other hand, by discussing Lie symmetries the general expression of conserved quantity is also obtained. The result shows that the conserved quantity is the same as the constant of motion in essence.

2. The constant of motion and Lagrangian

The Newton's equations for the non-relativistic charged particle in homogeneous electromagnetic field are

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = qBv_y, \tag{1a}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = v_x, \tag{1b}$$

$$m\frac{\mathrm{d}v_y}{\mathrm{d}t} = q(E - Bv_x), \tag{1c}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = v_y, \tag{1d}$$

where m is the mass and q is the charge of the charged particle, $\boldsymbol{E} = (0, E, 0)$ is the electric field, $\boldsymbol{B} = (0, 0, B)$ is the magnetic field. Let $a = qB/m$

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and $b = qe/m$, the Eqs.(1a) and (1c) can be rewritten as

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = av_y = g_1, \tag{1e}$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = b - av_x = g_2. \tag{1f}$$

From the first integral of Eq.(1) the constant of motion for this classical system can be obtained as follows:

$$v_x\frac{\partial K}{\partial x} + v_y\frac{\partial K}{\partial y} + g_1\frac{\partial K}{\partial v_x} + g_2\frac{\partial K}{\partial v_y} = 0, \tag{2}$$

where K is the constant of motion. Solution of Eq.(2) can be derived from its characteristic equation as follows:

$$\frac{\mathrm{d}x}{v_x} = \frac{\mathrm{d}y}{v_y} = \frac{\mathrm{d}v_x}{g_1} = \frac{\mathrm{d}v_y}{g_2} = \frac{\mathrm{d}K}{0}. \tag{3}$$

According to Eqs.(1e), (1f) and (3), we can obtain

$$\mathrm{d}\left[\frac{1}{2}(v_x^2 + v_y^2) - by\right] = 0. \tag{4}$$

So, the constant of motion can be obtained by

$$K = \frac{1}{2}(v_x^2 + v_y^2) - by. \tag{5}$$

From the first two-terms of Eq.(3) we can deduce $C_2^1 = v_x/v_y$ and $C_1^2 = v_y/v_x$. Hence, Eq.(5) can be rewritten as

$$K_1 = \frac{1}{2}v_x^2[1 + (C_1^2)^2] - by, \tag{6a}$$

$$K_2 = \frac{1}{2}v_y^2[1 + (C_2^1)^2] - by, \tag{6b}$$

where K_1 and K_2 are the constants of motion which are expressed in terms of the components v_x and v_y respectively.

The relationship between the Lagrangian and the constant of motion is carried out through the Legendre transformation[3] and can be written as

$$v_x\frac{\partial L}{\partial v_x} + v_y\frac{\partial L}{\partial v_y} - L = K, \tag{7}$$

whose characteristic equations is

$$\frac{\mathrm{d}v_x}{v_x} = \frac{\mathrm{d}v_y}{v_y} = \frac{\mathrm{d}L}{K + L} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}y}{0}. \tag{8}$$

From Eq.(8), we have

$$\frac{\mathrm{d}v_x}{v_x} = \frac{\mathrm{d}L_1}{K_1 + L_1}, \tag{9a}$$

$$\frac{\mathrm{d}v_y}{v_y} = \frac{\mathrm{d}L_2}{K_2 + L_2}. \tag{9b}$$

Substituting Eqs.(6a) and (6b) into Eqs.(9a) and (9b) respectively, we can obtain

$$L_1 = A_1v_x + v_x\int^{v_x}\frac{K_1(x, y, \xi)\mathrm{d}\xi}{\xi^2}, \tag{10a}$$

$$L_2 = A_2v_y + v_y\int^{v_y}\frac{K_2(x, y, \xi)\mathrm{d}\xi}{\xi^2}, \tag{10b}$$

where $A_1 = A_1(x, y, C_1^2)$ and $A_2 = A_2(x, y, C_2^1)$ are arbitrary functions, L_1 and L_2 are the Lagrangians which are expressed in terms of the components v_x and v_y respectively, and given by[3]

$$\begin{aligned} L =& A_1v_x + A_2v_y + \frac{1}{2}\left[v_x\int^{v_x}\frac{K_1(x, y, \xi)\mathrm{d}\xi}{\xi^2}\right. \\ &\left. + v_y\int^{v_y}\frac{K_2(x, y, \xi)d\xi}{\xi^2}\right] \\ =& \frac{1}{2}(v_x^2 + v_y^2) + A_1v_x + A_2v_y + by. \end{aligned} \tag{11}$$

In fact, $\boldsymbol{A} = (A_1, A_2, 0)$ is the vector potential and $V = -by$ is the scalar potential.[18] Substituting Eq.(11) into Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v_x}\right) - \frac{\partial L}{\partial x} = 0, \tag{12a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial v_y}\right) - \frac{\partial L}{\partial y} = 0, \tag{12b}$$

and comparing Eqs.(12a) and (12b) with Eqs.(1e) and (1f), we have

$$A_{2x} - A_{1y} = a, \tag{13}$$

where the subscripts x and y denote partial derivatives. According to $\boldsymbol{E} = -\nabla V - \dfrac{\partial \boldsymbol{A}}{\partial t}$, we obtain

$$A_{1t} = A_{2t} = 0. \tag{14}$$

From Eqs.(13) and (14), we can choose

$$A_1 = -ay, \quad A_2 = 0. \tag{15}$$

3. The Lie point symmetries and conserved quantities

Eqs.(1e) and (1f) can be rewritten as

$$\ddot{x} = a\dot{y} = g_1, \tag{16a}$$

$$\ddot{y} = b - a\dot{x} = g_2. \tag{16b}$$

Now, let us consider infinitesimal point transformations

$$\begin{aligned} t^* &= t + \varepsilon\tau(x, y, t), \\ x^* &= x + \varepsilon\eta_1(x, y, t), \\ y^* &= y + \varepsilon\eta_2(x, y, t), \end{aligned} \tag{17}$$

where ε is the infinitesimal parameter and τ, η_1 and η_2 are the generators of infinitesimal transformations. Under infinitesimal transformations (17), we introduce the generator vector for infinitesimal transformations

$$X^{(0)} = \tau\frac{\partial}{\partial t} + \eta_1\frac{\partial}{\partial x} + \eta_2\frac{\partial}{\partial y}. \tag{18}$$

The first extended infinitesimal generator is

$$X^{(1)} = X^{(0)} + (\dot{\eta}_1 - \dot{x}\dot{\tau})\frac{\partial}{\partial \dot{x}} + (\dot{\eta}_2 - \dot{y}\dot{\tau})\frac{\partial}{\partial \dot{y}}, \tag{19}$$

and the second is

$$\begin{aligned} X^{(2)} =& X^{(1)} + (\ddot{\eta}_1 - 2\ddot{x}\dot{\tau} - \dot{x}\ddot{\tau})\frac{\partial}{\partial \ddot{x}} \\ &+ (\ddot{\eta}_2 - 2\ddot{y}\dot{\tau} - \dot{y}\ddot{\tau})\frac{\partial}{\partial \ddot{y}}. \end{aligned} \tag{20}$$

The necessary and sufficient conditions that Eqs.(16a) and (16b) remain invariant are

$$\ddot{\eta}_1 - 2\ddot{x}\dot{\tau} - \dot{x}\ddot{\tau} = X^{(1)}(g_1), \tag{21a}$$

$$\ddot{\eta}_2 - 2\ddot{y}\dot{\tau} - \dot{y}\ddot{\tau} = X^{(1)}(g_2). \tag{21b}$$

Substituting Eqs.(16) and (19) into Eqs.(21) we can obtain

$$\ddot{\eta}_1 - a\dot{y}\dot{\tau} - \dot{x}\ddot{\tau} - a\dot{\eta}_2 = 0, \tag{22a}$$

$$\ddot{\eta}_2 - \dot{\tau}(2b - a\dot{x}) - \dot{y}\ddot{\tau} + a\dot{\eta}_1 = 0. \tag{22b}$$

By extending Eqs.(22) we can obtain the polynomial equations in velocity components and furthermore derive a series of partial differential equations about τ, η_1 and η_2, if the coefficients of all monomials of the form $\dot{x}^m\dot{y}^n$ are selected to be identically zero. Some detailed discussions are given as follows.

The coefficients of $\ddot{x}$ and $\ddot{y}$ in Eq.(22a) are easily given by

$$\eta_{1x} = 0, \tag{23a}$$

$$\eta_{1y} = 0. \tag{23b}$$

In terms of Eqs.(23a) and (23b) we have

$$\eta_1 = \rho_1(t), \tag{24}$$

where $\rho_1(t)$ is an arbitrary function of time.

The coefficients of $\dot{x}\ddot{x}$, $\dot{x}\ddot{y}$ and $\dot{y}$ in Eq.(22a) are also obtained by

$$\tau_x = 0, \tag{25a}$$

$$\tau_y = 0, \tag{25b}$$

$$\tau_t = 0. \tag{25c}$$

On account of Eqs.(25a), (25b) and (25c) we have

$$\tau = c_0, \tag{26}$$

where c_0 is an arbitrary constant.

The coefficients of $\ddot{x}$ and $\ddot{y}$ in Eq.(22b) are

$$\eta_{2x} = 0, \tag{27a}$$

$$\eta_{2y} = 0. \tag{27b}$$

According to Eqs.(27a) and (27b) we have

$$\eta_2 = \rho_2(t), \tag{28}$$

where $\rho_2(t)$ is an arbitrary function of time.

Substituting Eq.(26) into Eqs.(22a) and (22b) we obtain

$$\ddot{\eta}_1 - a\dot{\eta}_2 = 0, \tag{29a}$$

$$\ddot{\eta}_2 + a\dot{\eta}_1 = 0. \tag{29b}$$

Equations (29a) and (29b) lead to two equations

$$\eta_1 = D\sin(at + \phi) + c_1, \tag{30a}$$

$$\eta_2 = D\cos(at + \phi) + c_2, \tag{30b}$$

where D, c_1, c_2 are arbitrary constants

If there is a gauge function G satisfying the structure equations

$$L\dot{\tau} + X^{(1)}(L) + \dot{G} = 0, \tag{31}$$

then according to theorem of the Lie symmetry,[8] the system has the conserved quantity

$$I = L\tau + \frac{\partial L}{\partial \dot{x}}(\eta_1 - \dot{x}\tau) + \frac{\partial L}{\partial \dot{y}}(\eta_2 - \dot{y}\tau) + G. \tag{32}$$

Substituting Eqs.(11) and (26) into Eq.(31) and noticing Eq.(16), we obtain

$$G = -\eta_2\dot{y} - \eta_1(\dot{x} - ay). \tag{33}$$

Substituting Eqs.(11), (26) and (33) into Eq.(32), we obtain the general expression of conserved quantity of the system

$$I = -c_0\left[\frac{1}{2}(\dot{x}^2 + \dot{y}^2) - by\right] = -c_0K. \tag{34}$$

Obviously, the conserved quantity I is the same as the constant of motion K in essence.

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