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Second-order random wave solutions for interfacial internal waves in N-layer density-stratified fluid*

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This paper studies the random internal wave equations describing the density interface displacements and the velocity potentials of N-layer stratified fluid contained between two rigid walls at the top and bottom. The density interface displacements and the velocity potentials were solved to the second-order by an expansion approach used by Longuet-Higgins (1963) and Dean (1979) in the study of random surface waves and by Song (2004) in the study of second-order random wave solutions for internal waves in a two-layer fluid. The obtained results indicate that the first-order solutions are a linear superposition of many wave components with different amplitudes, wave numbers and frequencies, and that the amplitudes of first-order wave components with the same wave numbers and frequencies between the adjacent density interfaces are modulated by each other. They also show that the second-order solutions consist of two parts: the first one is the first-order solutions, and the second one is the solutions of the second-order asymptotic equations, which describe the second-order nonlinear modification and the second-order wave–wave interactions not only among the wave components on same density interfaces but also among the wave components between the adjacent density interfaces. Both the first-order and second-order solutions depend on the density and depth of each layer. It is also deduced that the results of the present work include those derived by Song (2004) for second-order random wave solutions for internal waves in a two-layer fluid as a particular case.

Keywords: N-layer density-stratified fluid, interfacial internal waves, second-order random wave solutions

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1. Introduction

Stably stratified fluids give rise to many fascinating phenomena, which owe their existence to the interaction between density variations and gravity. One of the best known of these is the internal wave, presumably so named because, in contrast to more commonly encountered waves, it is formed deep within stratified fluids rather than on their surface.[1] Over the past decades a large amount of work has been done on free surface waves (see Refs.[2–4]). However, waves which occur in nature are never, in fact, free surface waves, since they are always beneath a fluid of some finite density, if only air.[5] Keulegan[6] and Long[7] did the initial work on internal waves. After that, numerous authors have pursued studies of internal waves from various points of view (see Refs.[8–10]).

Interfacial internal waves in a density-stratified fluid are important phenomena in the ocean, and many authors have investigated them. For example, using Padé approximations, Holyer[5] considered large amplitude interfacial waves moving between two infinite fluid of different densities, which were at rest far from the interface. Breyiannis *et al*[10] studied analytically and numerically the properties of two-dimensional steady periodic interfacial gravity waves between two fluids in relative motion and of constant vorticities and finite depths. In addition, the random behaviours of the free surface waves and internal waves have been well recognized in many field measurements in deep to shallow water (see Refs.[11–15]). Longuet-Higgins[13] established the second-order random wave theory of surface waves for deep water by using a per-

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turbation method. Later, Sharma and Dean[14] extended the analysis of Longuet-Higgins[13] for deep water to water of finite depth. However, the researches on random interfacial and internal waves are very few. By using an expansion method, analogous to that of Longuet-Higgins[13] and Sharma and Dean[14] in the study of random surface waves, Song[15] derived the second-order solutions for random internal waves in a two-layer fluid contained between two rigid walls at the top and bottom.

The wave equations describing interfacial internal waves in N- layer stratified fluid are nonlinear wave equations and finding their solutions are important for nonlinear problems. At present, there are many methods for solving nonlinear wave equations: for example, the perturbation method,[15−18] the homogeneous balance principle,[19−21] expansion method about the Jacobi elliptic function,[22] various tanh methods,[23] the general projective Riccati method[24] and so on.

It is also noticed that the phenomena of density stratification are very complicated in the ocean, and sometimes the density stratification is obviously in state of multi-layers. In present study, second-order random solutions for internal waves in N-layer stratified fluid contained between two rigid walls at the top and bottom are derived by an expansion approach, which is used by Longuet-Higgins[13] and Sharma and Dean[14] in the study of random surface waves and by Song[15] in the study of second-order random wave solutions for internal waves in a two-layer fluid. The solutions obtained here would have potentially important applications in the study of internal waves.

2. Basic equations and boundary conditions

We consider N-layer stratified fluids of different densities contained between two rigid walls at the top and bottom as shown in Fig. 1, which are homogeneous, inviscid, incompressible and irrotational. The second-order random solutions of internal wave propagating in the interfaces are derived.

The Cartesian coordinates $(\boldsymbol{x}, z) = (x, y, z)$ are introduced with origin at the upper boundary, where x and y are horizontal coordinates, and $z_i\,(i = 1, 2, \cdots, N)$ is the coordinate of depth of the undisturbed ith layer fluid, in which $z_0 = 0$ is the upper boundary and z_N is the bottom. $\rho^{(i)}$ is the density of the ith layer fluid, and $h_i = z_{i-1} - z_i\,(i = 1, 2, \cdots, N)$ is the thickness of the undisturbed ith layer fluid, and $\eta^{(i)}(x, y, t)(i = 1, 2, \cdots, N-1)$ is the vertical displacement of the ith density interface.

The velocity potential $\Phi^{(i)}$ satisfies the Laplace equations

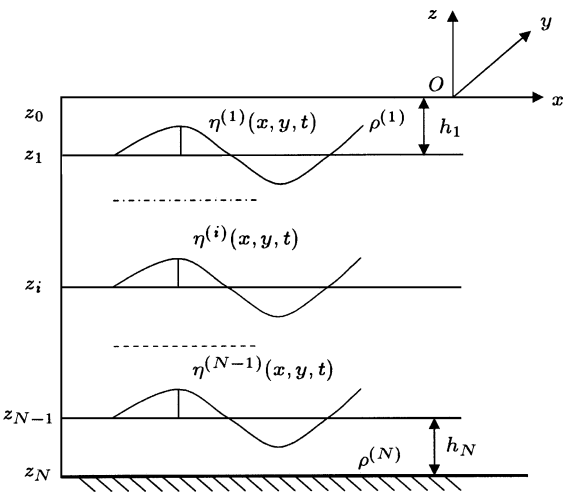


Fig.1. Definition sketch.

$$\nabla^2\Phi^{(i)} + \frac{\partial^2\Phi^{(i)}}{\partial z^2} = 0,$$
$$z_i + \eta^{(i)}(x, y, t) \le z \le z_i + \eta^{(i-1)}(x, y, t),$$
$$i = 1, 2, 3, \cdots, N, \tag{1}$$

where $\nabla = \left(\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right)$ is horizontal gradient operator, $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$, $\eta^{(0)} = 0$, and $\eta^{(N)} = 0$.

At the density interfaces, the kinematic boundary conditions are

$$\frac{\partial\Phi^{(i)}}{\partial z} = \frac{\partial\eta^{(i)}}{\partial t} + \nabla\eta^{(i)}\cdot\nabla\Phi^{(i)},$$
$$z = z_i + \eta^{(i)}(x, y, t), \quad i = 1, 2, \cdots, N-1, \tag{2}$$
$$\frac{\partial\Phi^{(i)}}{\partial z} = \frac{\partial\eta^{(i-1)}}{\partial t} + \nabla\eta^{(i-1)}\cdot\nabla\Phi^{(i)},$$
$$z = z_{i-1} + \eta^{(i-1)}(x, y, t), \quad i = 2, 3, \cdots, N, \tag{3}$$

and the dynamic boundary conditions are

$$\rho^{(i)}\left\{g\eta^{(i)} + \frac{\partial\Phi^{(i)}}{\partial t} + \frac{1}{2}\left[\left|\nabla\Phi^{(i)}\right|^2 + \left(\frac{\partial\Phi^{(i)}}{\partial z}\right)^2\right]\right\} = \rho^{(i+1)}\left\{g\eta^{(i)} + \frac{\partial\Phi^{(i+1)}}{\partial t} + \frac{1}{2}\left[\left|\nabla\Phi^{(i+1)}\right|^2 + \left(\frac{\partial\Phi^{(i+1)}}{\partial z}\right)^2\right]\right\},$$
$$z = z_i + \eta^{(i)}(x, y, t), \quad i = 1, 2, \cdots, N-1. \tag{4}$$

The top and bottom boundary conditions are

$$\frac{\partial \Phi^{(1)}}{\partial z} = 0, \quad z = z_0 = 0, \tag{5}$$

$$\frac{\partial \Phi^{(N)}}{\partial z} = 0, \quad z = z_N. \tag{6}$$

3. First-order random internal wave solutions

As Longuet-Higgins (1963) and Song (2004) in the studies of random surface waves and random internal waves, the perturbation method is used to solve the above basic equations and boundary conditions. The velocity potentials and vertical displacements of the density interfaces can be expressed as

$$\Phi^{(i)} = \varepsilon \Phi_{\rm I}^{(i)} + \varepsilon^2 \Phi_{\rm II}^{(i)} + O(\varepsilon^3), \quad i = 1, 2, \cdots, N, \tag{7}$$

$$\eta^{(i)} = \varepsilon \eta_{\rm I}^{(i)} + \varepsilon^2 \eta_{\rm II}^{(i)} + O(\varepsilon^3), \quad i = 1, 2, \cdots, N-1, \tag{8}$$

where ε is the perturbation parameter, O is the order symbol, and the subscripts I and II denote quantities corresponding to the first-order and second-order perturbation solutions respectively. Also the velocity potential $\Phi^{(i)} (i = 1, 2, \cdots, N)$ at the density interfaces $z = z_{i-1} + \eta^{(i-1)}$ and $z = z_i + \eta^{(i)} (i = 1, 2, \cdots, N-1)$ can be expanded in Taylor series about $z = z_{i-1}$ and $z = z_i$ as

$$\begin{aligned} \Phi^{(i)} \big|_{z=z_{i-1}+\eta^{(i-1)}} =& \Phi^{(i)} \big|_{z=z_{i-1}} + \eta^{(i-1)} \frac{\partial \Phi^{(i)}}{\partial z} \big|_{z=z_{i-1}} \\ &+ O((\eta^{(i-1)})^2), \\ &(i = 2, 3, \cdots, N), \end{aligned} \tag{9}$$

$$\begin{aligned} \Phi^{(i)} \big|_{z=z_i+\eta^{(i)}} =& \Phi^{(i)} \big|_{z=z_i} + \eta^{(i)} \frac{\partial \Phi^{(i)}}{\partial z} \big|_{z=z_{i-1}} \\ &+ O((\eta^{(i)})^2), \\ &(i = 1, 2, \cdots, N-1). \end{aligned} \tag{10}$$

So we can replace the density interface boundary conditions in Eqs.(2)–(4) by the conditions to be satisfied for $z = z_i (i = 1, 2, \cdots, N-1)$ instead of $z = z_i + \eta^{(i)}$. Substituting (7)–(8) into (1)–(6) and (9)–(10) into (2)–(6), we derive the governing equations for the first and second powers as follows.

The first-order equations and the boundary conditions are

$$\begin{aligned} &\nabla^2 \Phi_{\rm I}^{(i)} + \frac{\partial^2 \Phi_{\rm I}^{(i)}}{\partial z^2} = 0, \\ &z_i + \eta^{(i)}(x, y, t) \le z \le z_{i-1} + \eta^{(i-1)}(x, y, t), \\ &i = 1, 2, \cdots, N, \end{aligned} \tag{11}$$

$$\frac{\partial \Phi_{\rm I}^{(1)}}{\partial z} = 0, \quad z = z_0 = 0, \tag{12}$$

$$\frac{\partial \Phi_{\rm I}^{(i)}}{\partial z} = \frac{\partial \eta_{\rm I}^{(i)}}{\partial t}, \quad z = z_i, \quad i = 1, 2, \cdots, N-1, \tag{13}$$

$$\frac{\partial \Phi_{\rm I}^{(i)}}{\partial z} = \frac{\partial \eta_{\rm I}^{(i-1)}}{\partial t}, \quad z = z_{i-1}, \quad i = 2, 3, \cdots, N, \tag{14}$$

$$\frac{\partial \Phi_{\rm I}^{(N)}}{\partial z} = 0, \quad z = z_N, \tag{15}$$

$$\begin{aligned} &\rho^{(i)} \left\{ g\eta_{\rm I}^{(i)} + \frac{\partial \Phi_{\rm I}^{(i)}}{\partial {\rm t}} \right\} = \rho^{(i+1)} \left\{ g\eta_{\rm I}^{(i)} + \frac{\partial \Phi_{\rm I}^{(i+1)}}{\partial {\rm t}} \right\}, \\ &z = z_i, \quad i = 1, 2, \cdots, N-1. \end{aligned} \tag{16}$$

The second-order equations and the boundary conditions are

$$\begin{aligned} &\nabla^2 \Phi_{\rm II}^{(i)} + \frac{\partial^2 \Phi_{\rm II}^{(i)}}{\partial z^2} = 0, \\ &z_i + \eta^{(i)}(x, y, t) \le z \le z_{i-1} + \eta^{(i-1)}(x, y, t), \\ &i = 1, 2, \cdots, N, \end{aligned} \tag{17}$$

$$\frac{\partial \Phi_{\rm II}^{(1)}}{\partial z} = 0, \quad z = z_0 = 0, \tag{18}$$

$$\begin{aligned} &\frac{\partial \Phi_{\rm II}^{(i)}}{\partial z} + \eta_{\rm I}^{(i)} \frac{\partial^2 \Phi_{\rm I}^{(i)}}{\partial z^2} = \frac{\partial \eta_{\rm II}^{(i)}}{\partial t} + \nabla \eta_{\rm I}^{(i)} \cdot \nabla \Phi_{\rm I}^{(i)}, \\ &z = z_i, \quad i = 1, 2, \cdots, N-1, \end{aligned} \tag{19}$$

$$\begin{aligned} &\frac{\partial \Phi_{\rm II}^{(i)}}{\partial z} + \eta_{\rm I}^{(i-1)} \frac{\partial^2 \Phi_{\rm I}^{(i)}}{\partial z^2} = \frac{\partial \eta_{\rm II}^{(i-1)}}{\partial t} + \nabla \eta_{\rm I}^{(i-1)} \cdot \nabla \Phi_{\rm I}^{(i)}, \\ &z = z_{i-1}, \quad i = 2, 3, \cdots, N, \end{aligned} \tag{20}$$

$$\frac{\partial \Phi_{\rm II}^{(N)}}{\partial z} = 0, \quad z = z_N, \tag{21}$$

$$\begin{aligned} &\rho^{(i)} \left\{ g\eta_{\rm II}^{(i)} + \frac{\partial \Phi_{\rm II}^{(i)}}{\partial {\rm t}} + \eta_{\rm I}^{(i)} \cdot \frac{\partial^2 \Phi_{\rm I}^{(i)}}{\partial {\rm t} \partial {\rm z}} + \frac{1}{2} \left[\left| \nabla \Phi_{\rm I}^{(i)} \right|^2 + \left(\frac{\partial \Phi_{\rm I}^{(i)}}{\partial z} \right)^2 \right] \right\} \\ =& \rho^{(i+1)} \left\{ g\eta_{\rm II}^{(i)} + \frac{\partial \Phi_{\rm II}^{(i+1)}}{\partial {\rm t}} + \eta_{\rm I}^{(i)} \cdot \frac{\partial^2 \Phi_{\rm I}^{(i+1)}}{\partial {\rm t} \partial {\rm z}} + \frac{1}{2} \left[\left| \nabla \Phi_{\rm I}^{(i+1)} \right|^2 + \left(\frac{\partial \Phi_{\rm I}^{(i+1)}}{\partial z} \right)^2 \right] \right\}, \\ &z = z_i, \quad i = 1, 2, \cdots, N-1. \end{aligned} \tag{22}$$

Now, we assume that the first-order random internal wave solutions of equations (11)–(16) are

$$\eta_{\mathrm{I}}^{(i)} = \sum_{j=1}^{\infty} b_j^{(i)} \cos\psi_j, \quad i = 1, 2, \cdots, N-1, \tag{23}$$

$$\Phi_{\mathrm{I}}^{(i)} = \sum_{j=1}^{\infty} A_j^{(i)}(z) \cos\psi_j, \quad i = 1, 2, \cdots, N, \tag{24}$$

where $b_j^{(i)}$ denotes the amplitude of the jth wave component of the ith layer density interface, $\psi_j = \boldsymbol{k}_j \cdot \boldsymbol{x} - \omega_j t + \varepsilon_j$, $\boldsymbol{x} = (x, y)$, $\omega_j$is the angular frequency, $\boldsymbol{k}_j = (k_{j1}, k_{j2})$ is the horizontal wave number vector with the moduli k_j, ε_j denotes the random phase. If we assume that ε_j is the independent random variable and consider that it is uniformly distributed over the interval $(0,\ 2\pi)$, then we can obtain that $\eta_j^{(i)}$ obeys the Gaussian distribution. Substituting (23), (24) into (11)–(16), we have

$$A_j^{(1)} = -\frac{b_j^{(1)}\omega_j}{k_j \sinh(k_j h_1)} \cosh k_j z, \quad j = 1, 2, \cdots, \tag{25}$$

$$A_j^{(i)} = \frac{\omega_j}{k_j} \cdot \frac{b_j^{(i-1)} \cosh k_j(z - z_i) - b_j^{(i)} \cosh k_j(z - z_{i-1})}{\sinh k_j h_i}, \quad j = 1, 2, \cdots, \quad i = 2, 3, \cdots, N-1, \tag{26}$$

$$A_j^{(N)} = \frac{b_j^{(N-1)}\omega_j}{k_j \sinh(k_j h_N)} \cosh k_j(z - z_N), \quad j = 1, 2, \cdots. \tag{27}$$

Therefore, we obtain $\Phi_{\mathrm{I}}^{(i)} (i = 1, 2, \cdots, N)$ as follows:

$$\Phi_{\mathrm{I}}^{(1)} = -\sum_{j=1}^{\infty} \frac{b_j^{(1)}\omega_j}{k_j \sinh(k_j h_1)} \cosh k_j z \sin\psi_j, \tag{28}$$

$$\Phi_{\mathrm{I}}^{(i)} = \sum_{j=1}^{\infty} \frac{\omega_j}{k_j} \cdot \frac{b_j^{(i-1)} \cosh k_j(z - z_i) - b_j^{(i)} \cosh k_j(z - z_{i-1})}{\sinh k_j h_i} \sin\psi_j, \quad i = 2, 3, \cdots, N-1, \tag{29}$$

$$\Phi_{\mathrm{I}}^{(N)} = \sum_{j=1}^{\infty} \frac{b_j^{(N-1)}\omega_j}{k_j \sinh(k_j h_N)} \cosh k_j(z - z_N) \sin\psi_j. \tag{30}$$

Substituting (23) and (28)–(30) into (16), and manipulating these equations we get

$$a_j^{(1)} b_j^{(1)} + c_j^{(1)} b_j^{(2)} = 0, \quad j = 1, 2, \cdots, \tag{31}$$

$$c_j^{(i-1)} b_j^{(i-1)} + a_j^{(i)} b_j^{(i)} + c_j^{(i)} b_j^{(i+1)} = 0, \quad i = 2, 3, \cdots, N-2; \quad j = 1, 2, \cdots, \tag{32}$$

$$c_j^{(N-2)} b_j^{(N-2)} + a_j^{(N-1)} b_j^{(N-1)} = 0, \quad j = 1, 2, \cdots, \tag{33}$$

where

$$a_j^{(i)} = g\left(\rho^{(i)} - \rho^{(i+1)}\right) + \frac{\omega_j^2}{k_j}\left(\rho^{(i)} \coth k_j h_i + \rho^{(i+1)} \coth k_j h_{i+1}\right), \quad i = 1, 2, \cdots, N-1, \quad j = 1, 2, \cdots, \tag{34}$$

$$c_j^{(i)} = -\frac{\rho^{(i+1)}\omega_j^2}{k_j \sinh k_j h_{i+1}}, \quad i = 1, 2, \cdots, N-2, \quad j = 1, 2, \cdots. \tag{35}$$

Thus, we can derive

$$\begin{vmatrix} a_j^{(1)} & c_j^{(1)} & 0 & 0 & \cdots & 0 & 0 & 0 \\ c_j^{(1)} & a_j^{(2)} & c_j^{(2)} & 0 & \cdots & 0 & 0 & 0 \\ 0 & c_j^{(2)} & a_j^{(3)} & c_j^{(3)} & \cdots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & c_j^{(N-3)} & a_j^{(N-2)} & c_j^{(N-2)} \\ 0 & 0 & 0 & 0 & \cdots & 0 & c_j^{(N-2)} & a_j^{(N-1)} \end{vmatrix} = 0, \quad j = 1, 2, \cdots. \tag{36}$$

Equations (31)–(33) represent linear modulations among the amplitudes of the first-order wave components with same wave numbers and frequencies between the adjacent density interfaces.

4. Second-order random internal wave solutions

In order to find the second-order random velocity potentials and the displacements at the fluid interfaces, we use the second-order equations and boundary conditions (17)–(22). Since boundary conditions (19), (20) and (22) have nonhomogeneous terms depending on the first-order solutions, these terms must be derived from the first-order solutions (23) and (28)–(30), so the boundary conditions (19), (20) and (22) have the following forms:

$$\frac{\partial \Phi_{\mathrm{II}}^{(1)}}{\partial z} = \frac{\partial \eta_{\mathrm{II}}^{(1)}}{\partial t} + \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{2}b_j^{(1)}b_l^{(1)}\frac{\omega_j}{k_j\tanh k_j h_1}\left[\left(k_j^2+\boldsymbol{k}_j\cdot\boldsymbol{k}_l\right)\sin(\psi_j+\psi_l)\right.$$
$$\left.+\left(k_j^2-\boldsymbol{k}_j\cdot\boldsymbol{k}_l\right)\sin(\psi_j-\psi_l)\right], \quad z=z_1, \tag{37}$$

$$\frac{\partial \Phi_{\mathrm{II}}^{(i)}}{\partial z} = \frac{\partial \eta_{\mathrm{II}}^{(i-1)}}{\partial t} - \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{2}\frac{\omega_j}{k_j\sin k_j h_i}b_l^{(i-1)}\left(b_j^{(i-1)}\cosh k_j h_i - b_j^{(i)}\right)\left[\left(k_j^2+\boldsymbol{k}_j\cdot\boldsymbol{k}_l\right)\sin(\psi_j+\psi_l)\right.$$
$$\left.+\left(k_j^2-\boldsymbol{k}_j\cdot\boldsymbol{k}_l\right)\sin(\psi_j-\psi_l)\right], \quad z=z_{i-1}, \quad i=2,3,\cdots,N-1, \tag{38}$$

$$\frac{\partial \Phi_{\mathrm{II}}^{(i)}}{\partial z} = \frac{\partial \eta_{\mathrm{II}}^{(i)}}{\partial t} - \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{2}\frac{\omega_j}{k_j\sin k_j h_i}b_l^{(i)}\left(b_j^{(i-1)}-b_j^{(i)}\cosh k_j h_i\right)\left[\left(k_j^2+\boldsymbol{k}_j\cdot\boldsymbol{k}_l\right)\sin(\psi_j+\psi_l)\right.$$
$$\left.+\left(k_j^2-\boldsymbol{k}_j\cdot\boldsymbol{k}_l\right)\sin(\psi_j-\psi_l)\right], \quad z=z_i, \quad i=2,3,\cdots,N-1, \tag{39}$$

$$\frac{\partial \Phi_{\mathrm{II}}^{(N)}}{\partial z} = \frac{\partial \eta_{\mathrm{II}}^{(N-1)}}{\partial t} - \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{2}b_j^{(N-1)}b_l^{(N-1)}\frac{\omega_j}{k_j\tanh k_j h_N}\left[\left(k_j^2+\boldsymbol{k}_j\cdot\boldsymbol{k}_l\right)\sin(\psi_j+\psi_l)\right.$$
$$\left.+\left(k_j^2-\boldsymbol{k}_j\cdot\boldsymbol{k}_l\right)\sin(\psi_j-\psi_l)\right], \quad z=z_{N-1}, \tag{40}$$

$$\rho^{(1)}\left\{g\eta_{\mathrm{II}}^{(1)}+\frac{\partial \Phi_{\mathrm{II}}^{(1)}}{\partial t}+\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{4}b_j^{(1)}b_l^{(1)}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{k_jk_l\tanh k_jh_1\tanh k_lh_1}-\omega_j\omega_l-2\omega_l^2\right]\right.$$
$$\left.\times\cos(\psi_j+\psi_l)+\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{4}b_j^{(1)}b_l^{(1)}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{k_jk_l\tanh k_jh_1\tanh k_lh_1}+\omega_j\omega_l-2\omega_l^2\right]\cos(\psi_j-\psi_l)\right\}$$
$$=\rho^{(2)}\left\{g\eta_{\mathrm{II}}^{(1)}+\frac{\partial \Phi_{\mathrm{II}}^{(2)}}{\partial t}+\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{4}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{k_jk_l}\cdot\frac{b_j^{(1)}\cosh k_jh_2-b_j^{(2)}}{\sinh k_jh_2}\right.\right.$$
$$\times\left.\frac{b_l^{(1)}\cosh k_lh_2-b_l^{(2)}}{\sinh k_lh_2}-b_j^{(1)}b_l^{(1)}\omega_j\omega_l-2b_j^{(1)}b_l^{(1)}\omega_l^2\right]\cos(\psi_j+\psi_l)+\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{4}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{k_jk_l}\right.$$
$$\left.\left.\times\frac{b_j^{(1)}\cosh k_jh_2-b_j^{(2)}}{\sinh k_jh_2}\cdot\frac{b_l^{(1)}\cosh k_lh_2-b_l^{(2)}}{\sinh k_lh_2}+b_j^{(1)}b_l^{(1)}\omega_j\omega_l-2b_j^{(1)}b_l^{(1)}\omega_l^2\right]\cos(\psi_j-\psi_l)\right\},$$
$$z=z_1, \tag{41}$$

$$\rho^{(i)}\left\{g\eta_{\mathrm{II}}^{(i)}+\frac{\partial \Phi_{\mathrm{II}}^{(i)}}{\partial t}+\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{4}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{k_jk_l}\cdot\frac{b_j^{(i-1)}-b_j^{(i)}\cosh k_jh_i}{\sinh k_jh_i}\cdot\frac{b_l^{(i-1)}-b_l^{(i)}\cosh k_lh_i}{\sinh k_lh_i}\right.\right.$$
$$\left.-b_j^{(i)}b_l^{(i)}\omega_j\omega_l-2b_j^{(i)}b_l^{(i)}\omega_l^2\right]\cos(\psi_j+\psi_l)+\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{4}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{k_jk_l}\cdot\frac{b_j^{(i-1)}-b_j^{(i)}\cosh k_jh_i}{\sinh k_jh_i}\right.$$
$$\left.\left.\times\frac{b_l^{(i-1)}-b_l^{(i)}\cosh k_lh_i}{\sinh k_lh_i}+b_j^{(i)}b_l^{(i)}\omega_j\omega_l-2b_j^{(i)}b_l^{(i)}\omega_l^2\right]\cos(\psi_j-\psi_l)\right\}$$

$$
\begin{aligned}
=&\rho^{(i+1)}\left\{g\eta_{\mathrm{II}}^{(i)}+\frac{\partial\Phi_{\mathrm{II}}^{(i+1)}}{\partial t}+\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{4}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{k_jk_l}\cdot\frac{b_j^{(i)}\cosh k_jh_{i+1}-b_j^{(i+1)}}{\sinh k_jh_{i+1}}\cdot\frac{b_l^{(i)}\cosh k_lh_{i+1}-b_l^{(i+1)}}{\sinh k_lh_{i+1}}\right.\right.\\
&\left.-b_j^{(i)}b_l^{(i)}\omega_j\omega_l-2b_j^{(i)}b_l^{(i)}\omega_l^2\right]\cos(\psi_j+\psi_l)+\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{4}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{k_jk_l}\cdot\frac{b_j^{(i)}\cosh k_jh_{i+1}-b_j^{(i+1)}}{\sinh k_jh_{i+1}}\right.\\
&\left.\left.\times\frac{b_l^{(i)}\cosh k_lh_{i+1}-b_l^{(i+1)}}{\sin k_lh_{i+1}}+b_j^{(i)}b_l^{(i)}\omega_j\omega_l-2b_j^{(i)}b_l^{(i)}\omega_l^2\right]\cos(\psi_j-\psi_l)\right\},\\
&z=z_i,\quad i=2,3,\cdots,N-2,
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
&\rho^{(N-1)}\left\{g\eta_{\mathrm{II}}^{(N-1)}+\frac{\partial\Phi_{\mathrm{II}}^{(N-1)}}{\partial t}+\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{4}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{k_jk_l}\cdot\frac{b_j^{(N-2)}-b_j^{(N-1)}\cosh k_jh_{N-1}}{\sinh k_jh_{N-1}}\right.\right.\\
&\left.\times\frac{b_l^{(N-2)}-b_l^{(N-1)}\cosh k_lh_{N-1}}{\sinh k_lh_{N-1}}-b_j^{(N-1)}b_l^{(N-1)}\omega_j\omega_l-2b_j^{(N-1)}b_l^{(N-1)}\omega_l^2\right]\cos(\psi_j+\psi_l)\\
&+\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{4}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{k_jk_l}\cdot\frac{b_j^{(N-2)}-b_j^{(N-1)}\cosh k_jh_{N-1}}{\sinh k_jh_{N-1}}\cdot\frac{b_l^{(N-2)}-b_l^{(N-1)}\cosh k_lh_{N-1}}{\sinh k_lh_{N-1}}\right.\\
&\left.\left.+b_j^{(N-1)}b_l^{(N-1)}\omega_j\omega_l-2b_j^{(N-1)}b_l^{(N-1)}\omega_l^2\right]\cos(\psi_j-\psi_l)\right\}\\
=&\rho^{(N)}\left\{g\eta_{\mathrm{II}}^{(N-1)}+\frac{\partial\Phi_{\mathrm{II}}^{(N)}}{\partial t}+\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{4}b_j^{(N-1)}b_l^{(N-1)}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{k_jk_l\tanh k_jh_N\tanh k_lh_N}\right.\right.\\
&\left.-\omega_j\omega_l-2\omega_l^2\right]\cos(\psi_j+\psi_l)+\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{4}b_j^{(N-1)}b_l^{(N-1)}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{k_jk_l\tanh k_jh_N\tanh k_lh_N}\right.\\
&\left.\left.+\omega_j\omega_l-2\omega_l^2\right]\cos(\psi_j-\psi_l)\right\},\quad z=z_{N-1}.
\end{aligned}
\tag{43}
$$

As the procedure used in the first-order approximation, the solutions of the second-order equations can be assumed as

$$
\Phi_{\mathrm{II}}^{(i)}=\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left[B_{ijl}^{(-)}(z)\sin(\psi_j-\psi_l)+B_{ijl}^{(+)}(z)\sin(\psi_j+\psi_l)\right],\quad i=1,2,\cdots,N,
\tag{44}
$$

$$
\eta_{\mathrm{II}}^{(i)}=\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left[d_{ijl}^{-}\cos(\psi_j-\psi_l)+d_{ijl}^{+}\cos(\psi_j+\psi_l)\right],\quad i=1,2,\cdots,N-1.
\tag{45}
$$

Substituting Eqs.(44), (45) into Eqs.(17), (18), (37)–(40) and (21), we can determine $B_{ilj}^{(-)}$ and $B_{ilj}^{(+)}$ as follows:

$$
B_{1jl}^{(\pm)}(z)=A_{1jl}^{\pm}\cosh\left(|\boldsymbol{k}_j\pm\boldsymbol{k}_l|\,z\right),\quad j,l=1,2,\cdots,
\tag{46}
$$

$$
\begin{aligned}
B_{ijl}^{(\pm)}(z)=&A_{1ijl}^{\pm}\cosh\left(|\boldsymbol{k}_j\pm\boldsymbol{k}_l|\,(z-z_i)\right)-A_{2ijl}^{\pm}\cosh\left(|\boldsymbol{k}_j\pm\boldsymbol{k}_l|\,(z-z_{i-1})\right),\\
&i=2,3,\cdots,N-1,\quad j,l=1,2,\cdots,
\end{aligned}
\tag{47}
$$

$$
B_{Njl}^{(\pm)}(z)=A_{N\,jl}^{\pm}\cosh\left(|\boldsymbol{k}_j\pm\boldsymbol{k}_l|\,(z-z_N)\right),\quad j,l=1,2,\cdots,
\tag{48}
$$

where

$$
A_{1jl}^{-}=\begin{cases}-\dfrac{C_{1jl}^{-}}{k_{jl}^{-}\sinh(k_{jl}^{-}h_1)}, & \text{for}\quad j\neq l,\\ 0, & \text{for}\quad j=l,\end{cases}
\tag{49}
$$

$$A^{+}_{1jl} = -\frac{C^{+}_{1jl}}{k^{+}_{jl}\sinh(k^{+}_{jl}h_1)}, \tag{50}$$

$$A^{-}_{1ijl} = \begin{cases} \dfrac{C^{-}_{1ijl}}{k^{-}_{jl}\sinh(k^{-}_{jl}h_i)}, & \text{for} \quad j \neq l, \\ 0, & \text{for} \quad j = l, \end{cases} \qquad i = 2, 3, \cdots, N-1, \tag{51}$$

$$A^{-}_{2ijl} = \begin{cases} \dfrac{C^{-}_{2ijl}}{k^{-}_{jl}\sinh(k^{-}_{jl}h_i)}, & \text{for} \quad j \neq l, \\ 0, & \text{for} \quad j = l, \end{cases} \qquad i = 2, 3, \cdots, N-1, \tag{52}$$

$$A^{+}_{1ijl} = \frac{C^{+}_{1ijl}}{k^{+}_{jl}\sinh(k^{+}_{jl}h_i)}, \quad i = 2, 3, \cdots, N-1, \tag{53}$$

$$A^{+}_{2ijl} = \frac{C^{+}_{2ijl}}{k^{+}_{jl}\sinh(k^{+}_{jl}h_i)}, \quad i = 2, 3, \cdots, N-1, \tag{54}$$

$$A^{-}_{Njl} = \begin{cases} \dfrac{C^{-}_{Njl}}{k^{-}_{jl}\sinh(k^{-}_{jl}h_N)}, & \text{for} \quad j \neq l, \\ 0, & \text{for} \quad j = l, \end{cases} \tag{55}$$

$$A^{+}_{Njl} = \frac{C^{+}_{Njl}}{k^{+}_{jl}\sinh(k^{+}_{jl}h_N)}, \tag{56}$$

$$C^{\pm}_{1jl} = (\omega_j \pm \omega_l)d^{\pm}_{1jl} + \frac{1}{4}b^{(1)}_j b^{(1)}_l \left[\omega_j\left(k_j^2 \pm \boldsymbol{k}_j\cdot\boldsymbol{k}_l\right)/S_{1j} + \omega_l\left(\boldsymbol{k}_j\cdot\boldsymbol{k}_l \pm k_l^2\right)/S_{1l}\right], \tag{57}$$

$$\begin{aligned} C^{\pm}_{1ijl} =& (\omega_j \pm \omega_l)d^{\pm}_{(i-1)jl} - \frac{1}{4}b^{(i-1)}_j b^{(i-1)}_l \left[\omega_j\left(k_j^2 \pm \boldsymbol{k}_j\cdot\boldsymbol{k}_l\right)/S_{ij} + \omega_l\left(\boldsymbol{k}_j\cdot\boldsymbol{k}_l \pm k_l^2\right)/S_{il}\right] \\ &+ \frac{1}{2}b^{(i-1)}_j b^{(i)}_l \omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l \pm k_l^2)/R_{il}, \end{aligned} \tag{58}$$

$$\begin{aligned} C^{\pm}_{2ijl} =& (\omega_j \pm \omega_l)d^{\pm}_{ijl} + \frac{1}{4}b^{(i)}_j b^{(i)}_l \left[\omega_j\left(k_j^2 \pm \boldsymbol{k}_j\cdot\boldsymbol{k}_l\right)/S_{ij} + \omega_l\left(\boldsymbol{k}_j\cdot\boldsymbol{k}_l \pm k_l^2\right)/S_{il}\right] \\ &- \frac{1}{2}b^{(i-1)}_j b^{(i)}_l \omega_j(k_j^2 \pm \boldsymbol{k}_j\cdot\boldsymbol{k}_l)/R_{ij}, \end{aligned} \tag{59}$$

$$C^{\pm}_{Njl} = (\omega_j \pm \omega_l)d^{\pm}_{(N-1)jl} - \frac{1}{4}b^{(N-1)}_j b^{(N-1)}_l \left[\omega_j\left(k_j^2 \pm \boldsymbol{k}_j\cdot\boldsymbol{k}_l\right)/S_{Nj} + \omega_l\left(\boldsymbol{k}_j\cdot\boldsymbol{k}_l \pm k_l^2\right)/S_{Nl}\right], \tag{60}$$

$$k^{\pm}_{jl} = |\boldsymbol{k}_j \pm \boldsymbol{k}_l|, k_j = |\boldsymbol{k}_j|, \quad S_{ij} = k_j\tanh k_j h_i, \quad R_{ij} = k_j\sinh k_j h_i. \tag{61}$$

Inserting (46)–(48) into (44), we obtain $\Phi^{(i)}_{\rm II}(i = 1, 2, \cdots, N)$ as follows:

$$\Phi^{(1)}_{\rm II} = \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left[A^{-}_{1jl}\cosh\left(|\boldsymbol{k}_j - \boldsymbol{k}_l|\,z\right)\sin(\psi_j - \psi_l) + A^{+}_{1jl}\cosh\left(|\boldsymbol{k}_j + \boldsymbol{k}_l|\,z\right)\sin(\psi_j + \psi_l)\right], \tag{62}$$

$$\begin{aligned} \Phi^{(i)}_{\rm II} =& \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left\{\left[A^{-}_{1ijl}\cosh\left(|\boldsymbol{k}_j - \boldsymbol{k}_l|\,(z - z_i)\right) - A^{-}_{2ijl}\cosh\left(|\boldsymbol{k}_j - \boldsymbol{k}_l|\,(z - z_{i-1})\right)\right]\sin(\psi_j - \psi_l)\right. \\ &\left. + \left[A^{+}_{1ijl}\cosh\left(|\boldsymbol{k}_j + \boldsymbol{k}_l|\,(z - z_i)\right) - A^{+}_{2ijl}\cosh\left(|\boldsymbol{k}_j + \boldsymbol{k}_l|\,(z - z_{i-1})\right)\right]\sin(\psi_j + \psi_l)\right\}, \\ & i = 2, 3, \cdots, N-1, \end{aligned} \tag{63}$$

$$\begin{aligned} \Phi^{(N)}_{\rm II} =& \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left[A^{-}_{Njl}\cosh\left(|\boldsymbol{k}_j - \boldsymbol{k}_l|\,(z - z_N)\right)\sin(\psi_j - \psi_l)\right. \\ &\left. + A^{+}_{Njl}\cosh\left(|\boldsymbol{k}_j + \boldsymbol{k}_l|\,(z - z_N)\right)\sin(\psi_j + \psi_l)\right]. \end{aligned} \tag{64}$$

Substituting (45) and (62)–(64) into (41) –(43), we have

$$d^{-}_{ijl} = 0, \quad j, l = 1, 2, \cdots \text{ and } \ j = l, \tag{65}$$

and

$$\begin{cases} m_{1jl}^{-}d_{1jl}^{-}+n_{1jl}^{-}d_{2jl}^{-}=p_{1jl}^{-}, \\ n_{(i-1)jl}^{-}d_{(i-1)jl}^{-}+m_{ijl}^{-}d_{ijl}^{-}+n_{ijl}^{-}d_{(i+1)jl}^{-}=p_{ijl}^{-}, \quad i=2,3,\cdots,N-2, \\ n_{(N-2)jl}^{-}d_{(N-2)jl}^{-}+m_{(N-1)jl}^{-}d_{(N-1)jl}^{-}=p_{(N-1)jl}^{-}, \\ \qquad j,l=1,2,\cdots \text{and} j\neq l, \end{cases} \tag{66}$$

$$\begin{cases} m_{1jl}^{+}d_{1jl}^{+}+n_{1jl}^{+}d_{2jl}^{+}=p_{1jl}^{+}, \\ n_{(i-1)jl}^{+}d_{(i-1)jl}^{+}+m_{ijl}^{+}d_{ijl}^{+}+n_{ijl}^{+}d_{(i+1)jl}^{+}=p_{ijl}^{+}, \quad i=2,3,\cdots,N-2, \\ n_{(N-2)jl}^{+}d_{(N-2)jl}^{+}+m_{(N-1)jl}^{+}d_{(N-1)jl}^{+}=p_{(N-1)jl}^{+}, \\ \qquad j,l=1,2,\cdots \end{cases} \tag{67}$$

in which

$$m_{ijl}^{\pm}=g(\rho^{(i)}-\rho^{(i+1)})+\left[\frac{\rho^{(i)}}{\tanh(k_{jl}^{\pm}h_i)}+\frac{\rho^{(i+1)}}{\tanh(k_{jl}^{\pm}h_{i+1})}\right]\frac{(\omega_j\pm\omega_l)^2}{k_{jl}^{\pm}}, i=1,2,\cdots,N-1, \tag{68}$$

$$n_{ijl}^{\pm}=-\frac{\rho^{(i+1)}(\omega_j\pm\omega_l)^2}{k_{jl}^{\pm}\sinh(k_{jl}^{\pm}h_{i+1})}, \quad i=1,2,\cdots,N-2, \tag{69}$$

$$\begin{aligned} p_{1jl}^{\pm}=&-\frac{1}{4}\rho^{(1)}b_j^{(1)}b_l^{(1)}\left\{\frac{(\omega_j\pm\omega_l)\left[\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/S_{1j}+\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/S_{1l}\right]}{k_{jl}^{\pm}\tanh(k_{jl}^{\pm}h_1)}+\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{S_{1j}S_{1l}}\right. \\ &\left.\mp\omega_j\omega_l-2\omega_l^2\right\}+\rho^{(2)}\left\{\frac{1}{4}b_j^{(1)}b_l^{(1)}(\omega_j\pm\omega_l)\frac{\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/S_{2j}+\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/S_{2l}}{k_{jl}^{\pm}\tanh(k_{jl}^{\pm}h_2)}\right. \\ &+\frac{1}{4}b_j^{(2)}b_l^{(2)}(\omega_j\pm\omega_l)\frac{\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/S_{2j}+\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/S_{2l}}{k_{jl}^{\pm}\sinh(k_{jl}^{\pm}h_2)}-\frac{1}{2}b_j^{(1)}b_l^{(2)}(\omega_j\pm\omega_l) \\ &\times\left[\frac{\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/R_{2l}}{k_{jl}^{\pm}\tanh(k_{jl}^{\pm}h_2)}+\frac{\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/R_{2j}}{k_{jl}^{\pm}\sinh(k_{jl}^{\pm}h_2)}\right]+\frac{1}{4}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\omega_j\omega_l\left(b_j^{(1)}\Big/S_{2j}-b_j^{(2)}\Big/R_{2j}\right)\right. \\ &\left.\left.\times\left(b_l^{(1)}\Big/S_{2l}-b_l^{(2)}\Big/R_{2l}\right)\mp b_j^{(1)}b_l^{(1)}\omega_j\omega_l-2b_j^{(1)}b_l^{(1)}\omega_l^2\right]\right\}, \end{aligned} \tag{70}$$

$$\begin{aligned} p_{ijl}^{\pm}=&-\rho^{(i)}\left\{\frac{1}{4}b_j^{(i-1)}b_l^{(i-1)}(\omega_j\pm\omega_l)\frac{\left[\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/S_{ij}+\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/S_{il}\right]}{k_{jl}^{\pm}\sinh(k_{jl}^{\pm}h_i)}+\frac{1}{4}b_j^{(i)}b_l^{(i)}\right. \\ &\times(\omega_j\pm\omega_l)\frac{\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/S_{ij}+\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/S_{il}}{k_{jl}^{\pm}\tanh(k_{jl}^{\pm}h_i)}-\frac{1}{2}b_j^{(i-1)}b_l^{(i)}(\omega_j\pm\omega_l) \\ &\times\left[\frac{\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/R_{il}}{k_{jl}^{\pm}\sinh(k_{jl}^{\pm}h_i)}+\frac{\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/R_{ij}}{k_{jl}^{\pm}\tanh(k_{jl}^{\pm}h_i)}\right]+\frac{1}{4}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\omega_j\omega_l\left(\frac{b_j^{(i-1)}}{R_{ij}}-\frac{b_j^{(i)}}{S_{ij}}\right)\right. \\ &\left.\left.\times\left(\frac{b_l^{(i-1)}}{R_{il}}-\frac{b_l^{(i)}}{S_{il}}\right)\mp b_j^{(i)}b_l^{(i)}\omega_j\omega_l-2b_j^{(i)}b_l^{(i)}\omega_l^2\right]\right\}+\rho^{(i+1)}\left\{\frac{1}{4}(\omega_j\pm\omega_l)b_j^{(i)}b_l^{(i)}\right. \\ &\times\frac{\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/S_{(i+1)j}+\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/S_{(i+1)l}}{k_{jl}^{\pm}\tanh(k_{jl}^{\pm}h_{i+1})}+\frac{1}{4}(\omega_j\pm\omega_l)b_j^{(i+1)}b_l^{(i+1)} \\ &\times\frac{\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/S_{(i+1)j}+\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/S_{(i+1)l}}{k_{jl}^{\pm}\sinh(k_{jl}^{\pm}h_{i+1})}-\frac{1}{2}b_j^{(i)}b_l^{(i+1)}(\omega_j\pm\omega_l) \\ &\times\left[\frac{\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/R_{(i+1)l}}{k_{jl}^{\pm}\tanh(k_{jl}^{\pm}h_{i+1})}+\frac{\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/R_{(i+1)j}}{k_{jl}^{\pm}\sinh(k_{jl}^{\pm}h_{i+1})}\right]+\frac{1}{4}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\omega_j\omega_l\left(\frac{b_j^{(i)}}{S_{(i+1)j}}-\frac{b_j^{(i+1)}}{R_{(i+1)j}}\right)\right. \\ &\left.\left.\times\left(\frac{b_l^{(i)}}{S_{(i+1)l}}-\frac{b_l^{(i+1)}}{R_{(i+1)l}}\right)\mp b_j^{(i)}b_l^{(i)}\omega_j\omega_l-2b_j^{(i)}b_l^{(i)}\omega_l^2\right]\right\}, \quad i=2,3,\cdots,N-2, \end{aligned} \tag{71}$$

$$
\begin{aligned}
p^{\pm}_{(N-1)jl} = &-\rho^{(N-1)}\left\{\frac{1}{4}b_j^{(N-2)}b_l^{(N-2)}(\omega_j\pm\omega_l)\frac{\left[\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/S_{(N-1)j}+\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/S_{(N-1)l}\right]}{k^{\pm}_{jl}\sinh(k^{\pm}_{jl}h_{N-1})}\right.\\
&+\frac{1}{4}b_j^{(N-1)}b_l^{(N-1)}(\omega_j\pm\omega_l)\frac{\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/S_{(N-1)j}+\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/S_{(N-1)l}}{k^{\pm}_{jl}\tanh(k^{\pm}_{jl}h_{N-1})}-\frac{1}{2}b_j^{(N-2)}b_l^{(N-1)}\\
&\times(\omega_j\pm\omega_l)\left[\frac{\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/R_{(N-1)l}}{k^{\pm}_{jl}\sinh(k^{\pm}_{jl}h_{N-1})}+\frac{\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/R_{(N-1)j}}{k^{\pm}_{jl}\tanh(k^{\pm}_{jl}h_{N-1})}\right]+\frac{1}{4}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\omega_j\omega_l\right.\\
&\times\left.\left(\frac{b_j^{(N-2)}}{R_{(N-1)j}}-\frac{b_j^{(N-1)}}{S_{(N-1)j}}\right)\left(\frac{b_l^{(N-2)}}{R_{(N-1)l}}-\frac{b_l^{(N-1)}}{S_{(N-1)l}}\right)\mp b_j^{(N-1)}b_l^{(N-1)}\omega_j\omega_l-2b_j^{(N-1)}b_l^{(N-1)}\omega_l^2\right]\Bigg\}\\
&+\frac{1}{4}\rho^{(N)}b_j^{(N-1)}b_l^{(N-1)}\left\{\frac{(\omega_j\pm\omega_l)\left[\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/S_{Nj}+\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/S_{Nl}\right]}{k^{\pm}_{jl}\tanh(k^{\pm}_{jl}h_N)}+\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{S_{Nj}S_{Nl}}\right.\\
&\left.\mp\omega_j\omega_l-2\omega_l^2\right\}. \qquad (72)
\end{aligned}
$$

So d^{-}_{ijl} and d^{+}_{ijl} can be determined by Eqs.(66) and (67).

5. Discussion

It is noticed that there exist some differences between second-order random wave solutions for internal waves in a two-fluid system[15] and in N-layer stratified fluid. The latter contains the nonlinear modifications and the nonlinear interactions among the wave components between the adjacent density interfaces, so in this case the latter is much more complicated than the former.

Now we consider a specific case. When assumption $N=2$ is satisfied, the first-order random internal waves solutions are

$$\Phi_{\rm I}^{(1)} = -\sum_{j=1}^{\infty}\frac{b_j^{(1)}\omega_j}{k_j\sinh(k_jh_1)}\cosh k_jz\sin\psi_j, \qquad (73)$$

$$\Phi_{\rm I}^{(2)} = \sum_{j=1}^{\infty}\frac{b_j^{(1)}\omega_j}{k_j\sinh(k_jh_2)}\cosh k_j(z-z_2)\sin\psi_j, \qquad (74)$$

$$\eta_{\rm I}^{(1)} = \sum_{j=1}^{\infty}b_j^{(1)}\cos\psi_j, \qquad (75)$$

$$g\left(\rho^{(1)}-\rho^{(2)}\right)+\frac{\omega_j^2}{k_j}\left(\rho^{(1)}\coth k_jh_1+\rho^{(2)}\coth k_jh_2\right)=0,\quad j=1,2,\cdots. \qquad (76)$$

The solutions of the second-order equations are

$$\Phi_{\rm II}^{(1)} = \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left[A^{-}_{1jl}\cosh\left(|\boldsymbol{k}_j-\boldsymbol{k}_l|\,z\right)\sin(\psi_j-\psi_l)+A^{+}_{1jl}\cosh\left(|\boldsymbol{k}_j+\boldsymbol{k}_l|\,z\right)\sin(\psi_j+\psi_l)\right], \qquad (77)$$

$$\Phi_{\rm II}^{(2)} = \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left[A^{-}_{2jl}\cosh\left(|\boldsymbol{k}_j-\boldsymbol{k}_l|\,(z-z_2)\right)\sin(\psi_j-\psi_l)\ +A^{+}_{2jl}\cosh\left(|\boldsymbol{k}_j+\boldsymbol{k}_l|\,(z-z_2)\right)\sin(\psi_j+\psi_l)\right], \qquad (78)$$

$$\eta_{\rm II}^{(1)} = \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\left[d^{-}_{1jl}\cos(\psi_j-\psi_l)+d^{+}_{1jl}\cos(\psi_j+\psi_l)\right], \qquad (79)$$

$$d^{-}_{1jl} = \begin{cases} p^{-}_{1jl}/m^{-}_{1jl}, & j\neq l,\\ 0, & j=l,\end{cases} \qquad (80)$$

$$d^{+}_{1jl} = \frac{p^{+}_{1jl}}{m^{+}_{1jl}},\quad j,l=1,2,\cdots, \qquad (81)$$

in which

$$A_{1jl}^{-}=\begin{cases}-\dfrac{C_{1jl}^{-}}{k_{jl}^{-}\sinh(k_{jl}^{-}h_1)}, & \text{for} \quad j\neq l,\\ 0, & \text{for} \quad j=l,\end{cases} \tag{82}$$

$$A_{1jl}^{+}=-\frac{C_{1jl}^{+}}{k_{jl}^{+}\sinh(k_{jl}^{+}h_1)}, \tag{83}$$

$$A_{2jl}^{-}=\begin{cases}\dfrac{C_{2jl}^{-}}{k_{jl}^{-}\sinh(k_{jl}^{-}h_2)}, & \text{for} \quad j\neq l,\\ 0, & \text{for} \quad j=l,\end{cases} \tag{84}$$

$$A_{2jl}^{+}=\frac{C_{2jl}^{+}}{k_{jl}^{+}\sinh(k_{jl}^{+}h_2)}, \tag{85}$$

$$C_{1jl}^{\pm}=(\omega_j\pm\omega_l)d_{1jl}^{\pm}+\frac{1}{4}b_j^{(1)}b_l^{(1)}\left[\omega_j\left(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l\right)/S_{1j}+\omega_l\left(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2\right)/S_{1l}\right], \tag{86}$$

$$C_{2jl}^{\pm}=(\omega_j\pm\omega_l)d_{1jl}^{\pm}-\frac{1}{4}b_j^{(1)}b_l^{(1)}\left[\omega_j\left(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l\right)/S_{2j}+\omega_l\left(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2\right)/S_{2l}\right], \tag{87}$$

$$m_{1jl}^{\pm}=g(\rho^{(1)}-\rho^{(2)})+\frac{(\omega_j\pm\omega_l)^2}{k_{jl}^{\pm}}\left[\rho^{(1)}\coth(k_{jl}^{\pm}h_1)+\rho^{(2)}\coth(k_{jl}^{\pm}h_2)\right], \tag{88}$$

$$\begin{aligned}p_{1jl}^{\pm}=&-\frac{1}{4}\rho^{(1)}b_j^{(1)}b_l^{(1)}\left\{\frac{(\omega_j\pm\omega_l)\left[\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/S_{1j}+\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/S_{1l}\right]}{k_{jl}^{\pm}\tanh(k_{jl}^{\pm}h_1)}+\boldsymbol{k}_j\cdot\boldsymbol{k}_l\frac{\omega_j\omega_l}{S_{1j}S_{1l}}\right.\\&\left.\mp\omega_j\omega_l-2\omega_l^2\right\}+\rho^{(2)}\left\{\frac{1}{4}b_j^{(1)}b_l^{(1)}(\omega_j\pm\omega_l)\frac{\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)/S_{2j}+\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)/S_{2l}}{k_{jl}^{\pm}\tanh(k_{jl}^{\pm}h_2)}\right.\\&\left.+\frac{1}{4}\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\omega_j\omega_l\frac{b_j^{(1)}}{S_{2j}}\cdot\frac{b_l^{(1)}}{S_{2l}}\mp b_j^{(1)}b_l^{(1)}\omega_j\omega_l-2b_j^{(1)}b_l^{(1)}\omega_l^2\right]\right\}.\end{aligned} \tag{89}$$

By simple operations, we have

$$\begin{aligned}p_{1jl}^{\pm}=&\frac{1}{4}b_j^{(1)}b_l^{(1)}\left\{\frac{(\omega_j\pm\omega_l)}{k_{jl}^{\pm}}\left[\omega_j(k_j^2\pm\boldsymbol{k}_j\cdot\boldsymbol{k}_l)\left(\rho^{(2)}\coth(k_{jl}^{\pm}h_2)/S_{2j}-\rho^{(1)}\coth(k_{jl}^{\pm}h_1)/S_{1j}\right)\right.\right.\\&\left.+\omega_l(\boldsymbol{k}_j\cdot\boldsymbol{k}_l\pm k_l^2)\left(\rho^{(2)}\coth(k_{jl}^{\pm}h_2)/S_{2l}-\rho^{(1)}\coth(k_{jl}^{\pm}h_1)/S_{1l}\right)\right]-\left(\rho^{(2)}-\rho^{(1)}\right)\left(\omega_j^2+\omega_l^2\right)\\&\left.+\omega_j\omega_l\left[\boldsymbol{k}_j\cdot\boldsymbol{k}_l\left(\rho^{(2)}/(S_{2j}S_{2l})-\rho^{(1)}/(S_{1j}S_{1l})\right)\mp\rho^{(2)}\pm\rho^{(1)}\right]\right\}.\end{aligned} \tag{90}$$

It is noted that the first-order solutions (73)–(76) and the solutions (77)–(81) of the second-order equations are exactly the same as those derived by Song[15] for second-order random wave solutions for internal waves in a two-layer fluid, so the result of Song[15] is just the result for a particular case of the present results.

Furthermore, if the density of the upper layer is small enough and can be neglected (i. e.,$\rho^{(1)}=0$) and assumption $N=2$ is satisfied, the present results can reduce to those derived by Sharma and Dean[14] for surface waves in finite water depth, and they would further be simplified to those derived by Longuet-Higgins [13] for deep water if the depth of the lower layer is taken as infinite (for the details see Ref.[15]).

6. Conclusions

In this paper, second-order random solutions for interfacial internal waves in N-layer stratified fluid contained between two rigid walls at the top and the bottom are derived by an expansion approach, used

by Song[15] in the study of random internal waves in a two-layer fluid and by Longuet-Higgins[13] and Sharma and Dean[14] in the study of random surface waves.

From the obtained results, it is noted that the second-order random wave solutions consist of two parts which depend on the density and depth of each layer. The first one is the first-order random wave solutions which are a linear superposition of many waves with different amplitudes, wave numbers and frequencies. The second one is the solutions of the second-order asymptotic equations which represent the second-order nonlinear modification and the second-order wave-wave interactions among both the wave components on the same density interfaces and the wave components between the adjacent density interfaces. It is also noticed that the amplitudes of first-order wave components with same wave numbers and frequencies between the adjacent density interfaces are linearly modulated by each other. In addition, The results of the present work include those derived by Song[15] for second-order random wave solutions for internal waves in a two-layer fluid as a particular case.

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