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Variational iteration method for solving the mechanism of the Equatorial Eastern Pacific El Niño–Southern Oscillation*

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A class of coupled system for the El Niño–Southern Oscillation (ENSO) mechanism is studied. Using the method of variational iteration for perturbation theory, the asymptotic expansions of the solution for ENSO model are obtained and the asymptotic behaviour of solution for corresponding problem is considered.

Keywords: nonlinear, method of variational iteration, perturbation theory, El Niño–Southern Oscillation model
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1. Introduction

El Niño and Southern Oscillation (ENSO) are the abnormal events happening in the atmosphere and the ocean of tropical Pacific. The phenomenon of the ENSO is a very attractive object in the international academic circles.[1−6] Lin and Mo *et al* have studied a class of problems in atmospheric physics, ocean weather, fluid dynamic systems and so on.[7−24] The oscillatory nature of ENSO requires both positive and negative ocean–atmosphere feedbacks. And a positive sea surface temperature (SST) anomaly in the equatorial eastern Pacific region is considered. This anomaly reduces the zonal SST gradient and the strength of the southern oscillation circulation, resulting in weaker trade winds around the equator. The weaker trade winds in turn drive the ocean circulation changes and then reinforce the SST anomaly.

Recently, many scientists have investigated the nonlinear singular perturbation problem. Approximation methods have been developed and refined, including the method of averaging, boundary layer method, matched asymptotic expansion method, and multiple scales method.[25−27] Using the perturbation method, Mo *et al* considered a class of singularly perturbation problems.[28−30] In this paper we study an ENSO model by using a simple and special method for variational iteration solution.[31]

The ENSO oscillator model should take into consideration the variations of both the eastern and western Pacific anomaly patterns. Because of the complexity of ENSO, we only study the oscillator model that overlooks the ENSO western Pacific anomaly patterns and does not consider the effect of the western Pacific on ENSO. As will be shown below, the oscillator model can be reduced to a simple oscillator model by further simplification with certain assumptions.

2. Model of Equatorial Pacific SST

It is assumed that winds in the western Pacific do not affect the SST anomalies in the eastern Pacific for

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regions 14°W–85°W and 10°S–10°N. Thus the coupled system requires equations as follows:[4]

$$\frac{\mathrm{d}T_\mathrm{a}}{\mathrm{d}t} = a_{11}T_\mathrm{a} + a_{12}T_\mathrm{s} + b_{11}T_\mathrm{a}T_\mathrm{s} + b_{12}T_\mathrm{s}^2, \quad (1)$$

$$\frac{\mathrm{d}T_\mathrm{s}}{\mathrm{d}t} = a_{21}T_\mathrm{a} + a_{22}T_\mathrm{s} + b_{21}T_\mathrm{a}T_\mathrm{s} + b_{22}T_\mathrm{s}^2, \quad (2)$$

where T_a is the locally averaged temperature anomaly of the air and T_s is the locally averaged temperature anomaly of the sea surface in the equatorial eastern Pacific region; the coefficients a_{ij} and b_{ij} are real nonnegative (for the expansions and their physical senses see Ref.[4]).

3. Variational iteration

Introduce functionals F_1 and F_2:

$$F_1 = T_\mathrm{a} + \int_0^t \lambda_1 \left[\frac{\mathrm{d}T_\mathrm{a}}{\mathrm{d}\tau} + a_{11}T_\mathrm{a} + a_{12}\bar{T}_\mathrm{s} + b_{11}\bar{T}_\mathrm{a}\bar{T}_\mathrm{s} + b_{12}\bar{T}_\mathrm{s}^2\right]\mathrm{d}\tau, \quad (3)$$

$$F_2 = T_\mathrm{s} + \int_0^t \lambda_2 \left[\frac{\mathrm{d}T_\mathrm{s}}{\mathrm{d}\tau} + a_{21}\bar{T}_\mathrm{a} + a_{22}T_\mathrm{s} + b_{21}\bar{T}_\mathrm{a}\bar{T}_\mathrm{s} + b_{22}\bar{T}_\mathrm{s}^2\right]\mathrm{d}\tau, \quad (4)$$

where $\bar{T}_\mathrm{a}$, $\bar{T}_\mathrm{s}$ are T_a, T_s restricted variables respectively, and λ_i $(i = 1, 2)$ are Lagrange multiplicators.

Take variationals δF_i, $i = 1, 2$, of the functionals (3) and (4):

$$\delta F_1 = \delta T_\mathrm{a} + (\lambda_1 \delta T_\mathrm{a})\,|_{\tau=t} - \int_0^t (\lambda_1' - a_{11}\lambda_1)\delta T_\mathrm{a}\mathrm{d}\tau,$$

$$\delta F_2 = \delta T_\mathrm{s} + (\lambda_2 \delta h)\,|_{\tau=t} - \int_0^t (\lambda_2' - a\lambda_{22})\delta T_\mathrm{s}\mathrm{d}\tau.$$

Let $\delta_i F = 0$. Then we have

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}\tau} - a_{11}\lambda_1 = 0, \quad \tau > t, \quad (5)$$

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}\tau} - a_{22}\lambda_2 = 0, \quad \tau > t, \quad (6)$$

$$\lambda_i(\tau)\,|_{\tau=t} = -1, \quad i = 1, 2. \quad (7)$$

From Eqs.(5) and (7), we obtain

$$\lambda_1 = -\exp(a_{11}(\tau - t)), \quad (8)$$

$$\lambda_2 = -\exp(a_{22}(\tau - t)). \quad (9)$$

From Eqs.(3), (4) and (8), (9), we can construct the following variational iteration:

$$(T_\mathrm{a})_{n+1} = (T_\mathrm{a})_n - \int_0^t \exp(a_{11}(\tau - t)) \times \left[\frac{\mathrm{d}(T_\mathrm{a})_n}{\mathrm{d}\tau} + a_{11}(T_\mathrm{a})_n + a_{12}(T_\mathrm{s})_n + b_{12}(T_\mathrm{a})_n(T_\mathrm{s})_n + b_{12}(T_\mathrm{s}^2)_n\right]\mathrm{d}\tau, \quad (10)$$

$$(T_\mathrm{s})_{n+1} = (T_\mathrm{s})_n - \int_0^t \exp(a_{22}(\tau - t)) \times \left[\frac{\mathrm{d}(T_\mathrm{s})_n}{\mathrm{d}\tau} + a_{21}(T_\mathrm{a})_n + a_{22}(T_\mathrm{s})_n + b_{21}(T_\mathrm{a})_n(T_\mathrm{s})_n + b_{22}(T_\mathrm{s}^2)_n\right]\mathrm{d}\tau. \quad (11)$$

Obviously, if the series of $\{(T_\mathrm{a}(t))_n\}$ and $\{(T_\mathrm{s}(t))_n\}$ derived from Eqs.(10) and (11) are convergent, then $T_\mathrm{a}(t) = \lim\limits_{n\to\infty}(T_\mathrm{a}(t))_n$, and $T_\mathrm{s}(t) = \lim\limits_{n\to\infty}(h_\mathrm{s}(t))_n$ will be the series of solution to the system (1) and (2).

4. Approximate solution for the model

Now we first select the initial approximation $(T_0(t),\ h_0(t))$ such that it satisfies the system

$$\frac{\mathrm{d}T_\mathrm{a}}{\mathrm{d}t} = a_{11}T_\mathrm{a} + a_{12}T_\mathrm{s}, \quad (12)$$

$$\frac{\mathrm{d}T_\mathrm{s}}{\mathrm{d}t} = a_{21}T_\mathrm{a} + a_{22}T_\mathrm{s}. \quad (13)$$

From Eqs.(12) and (13), we have

$$(T_\mathrm{a}(t))_0 = D_{10}\exp(\lambda_1 t) + D_{20}\exp(\lambda_2 t), \quad (14)$$

$$(T_\mathrm{s}(t))_0 = C_{10}\exp(\lambda_1 t) + C_{20}\exp(\lambda_2 t), \quad (15)$$

where

$$\lambda_i = \frac{1}{2}[(a_{1q} + a_{22}) \pm \sqrt{(a_{11} - a_{22})^2 + 4(a_{12}a_{21})}, \quad i = 1, 2, \quad (16)$$

$$D_{10} = \frac{a_{22}C_{10}}{\lambda_1 - a_{11}}, \quad D_{20} = \frac{a_{22}C_{20}}{\lambda_2 - a_{11}}, \quad (17)$$

and C_{10} and C_{20} are arbitrary constants.

Substituting the initial approximate solution (14) and (15) into Eqs.(10) and (11), respectively, we obtain the first-order approximate solution to the system (1) and (2):

$$
\begin{aligned}
(T_{\mathrm{a}})_1 =&(T_{\mathrm{a}})_0 - \int_0^t \exp(a_{11}(\tau - t))\left[\frac{\mathrm{d}(T_{\mathrm{a}})_0}{\mathrm{d}\tau} + a_{11}(T_{\mathrm{a}})_0 + a_{12}(T_{\mathrm{s}})_0 + b_{11}(T_{\mathrm{a}})_0(T_{\mathrm{s}})_0 + b_{12}(T_{\mathrm{s}}^2)_0\right]\mathrm{d}\tau \\
=&D_{10}\exp(\lambda_1 t) + D_{20}\exp(\lambda_2 t) \\
&- \int_0^t \exp(a_{11}(\tau - t))[(\lambda_1 + a_{11})D_{10}\exp(\lambda_1\tau) + (\lambda_2 + a_{11})D_{20}\exp(\lambda_2\tau)) \\
&+ a_{12}(C_{10}\exp(\lambda_1\tau) + C_{20}\exp(\lambda_2\tau)) \\
&+ b_{11}(D_{10}\exp(\lambda_1\tau) + D_{20}\exp(\lambda_2\tau))(C_{10}\exp(\lambda_1\tau) + C_{20}\exp(\lambda_2\tau)) \\
&+ b_{12}(C_{10}\exp(\lambda_1\tau) + C_{20}\exp(\lambda_2\tau))^2]\mathrm{d}\tau; \\
(T_{\mathrm{s}})_1 =&(T_{\mathrm{s}})_0 - \int_0^t \exp(a_{22}(\tau - t))\left[\frac{\mathrm{d}(T_{\mathrm{s}})_0}{\mathrm{d}\tau} + a_{21}(T_{\mathrm{a}})_0 + a_{22}(T_{\mathrm{s}})_0 + b_{21}(T_{\mathrm{a}})_0(T_{\mathrm{s}})_0 + b_{22}(T_{\mathrm{s}}^2)_0\right]\mathrm{d}\tau \\
=&C_{10}\exp(\lambda_1 t) + C_{20}\exp(\lambda_2 t) \\
&- \int_0^t \exp(a_{22}(\tau - t))[a_{21}(D_{10}\exp(\lambda_1\tau) + D_{20}\exp(\lambda_2\tau)) \\
&+ (\lambda_1 + a_{22})C_{10}\exp(\lambda_1\tau) + (\lambda_2 + a_{22})C_{20}(\exp(\lambda_2\tau)) \\
&+ b_{21}(D_{10}\exp(\lambda_1\tau) + D_{20}\exp(\lambda_2\tau))(C_{10}\exp(\lambda_1\tau) + C_{20}\exp(\lambda_2\tau)) \\
&+ b_{22}(C_{10}\exp(\lambda_1\tau) + C_{20}\exp(\lambda_2\tau))^2]\mathrm{d}\tau.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
(T_{\mathrm{a}}(t))_1 =&D_{10}\exp(\lambda_1 t) + D_{20}\exp(\lambda_2 t) + \left(D_{10} + \frac{a_{12}C_{10}}{\lambda_1 - a_{11}}\right)(\exp(-a_{11}t) - \exp(\lambda_1 t)) \\
&+ \left(D_{20} + \frac{a_{12}C_{20}}{\lambda_2 - a_{11}}\right)(\exp(-a_{11}t) - \exp(\lambda_2\tau)) \\
&+ \frac{(b_{11}D_{10} + b_{12}C_{10})C_{10}}{2\lambda_1 + a_{11}}(\exp(-a_{11}t) - \exp(2\lambda_1 t)) \\
&+ \frac{b_{11}(C_{10}D_{20} + C_{20}D_{10}) + 2b_{12}C_{10}C_{20}}{\lambda_1 + \lambda_2 + a_{11}}(\exp(-a_{11}t) - \exp((\lambda_1 + \lambda_2)t)) \\
&+ \frac{(b_{11}D_{20} + b_{12}C_{20})C_{20}}{2\lambda_2 + a_{11}}(\exp(-a_{11}t) - \exp(2\lambda_2 t)), \qquad (18) \\
(T_{\mathrm{s}}(t))_1 =&C_{10}\exp(\lambda_1 t) + C_{20}\exp(\lambda_2 t) + \left(C_{10} + \frac{a_{22}D_{10}}{\lambda_1 + a_{22}}\right)(\exp(-a_{22}t) - \exp(\lambda_1 t)) \\
&+ \left(C_{20} + \frac{a_{22}D_{20}}{\lambda_2 + a_{22}}\right)(\exp(-a_{22}t) - \exp(\lambda_2\tau)) \\
&+ \frac{(b_{21}D_{10} + b_{22}C_{10})C_{10}}{2\lambda_1 + a_{22}}(\exp(-a_{22}t) - \exp(2\lambda_1 t)) \\
&+ \frac{b_{21}(C_{10}D_{20} + C_{20}D_{10}) + 2b_{22}C_{10}C_{20}}{\lambda_1 + \lambda_2 + a_{22}}(\exp(-a_{22}t) - \exp((\lambda_1 + \lambda_2)t)) \\
&+ \frac{(b_{21}D_{20} + b_{22}C_{20})C_{20}}{2\lambda_2 + a_{22}}(\exp(-a_{22}t) - \exp(2\lambda_2 t)), \qquad (19)
\end{aligned}
$$

where λ_i, D_i $(i = 1, 2)$ are determined by Eqs.(16) and (17), and C_{10} and C_{20} are arbitrary constants.

Using the same method, we can obtain the higher-order approximate solution.

It is not difficult to find that

(i) for $a_{11}a_{22} < a_{12}a_{22}$, we can decide that the two characteristic roots λ_i, $i = 1, 2$ from Eqs.(12) and (13) are real with different signs, thus from the theory of singular point, the zero point in the phase plane of the linear system (6) and (7) is an unstable saddle point;

(ii) for $a_{11}a_{22} > a_{12}a_{22}$, the two characteristic roots are different positive roots λ_i, $i = 1, 2$, thus there is an instable node or focus;

(iii) for $a_{11}a_{22} = a_{12}a_{22}$, there are a positive characteristic root and a zero root. Thus there exists a singular line in the phase plane, and this singular line is instable too.

And the trajectories in the phase plane for the system (12) and (13) are situated away from the zero point.

Because the nonlinear terms of the coupled system (1) and (2) are $R_1 = (b_{11}T_aT_s + b_{12}T_s^2)$ and $R_2 = (b_{21}T_aT_s + b_{22}T_s^2)$, we have

$$\lim_{\|y\|\to 0} \frac{\|R\|}{\|y\|} = 0,$$

where

$$\|R\| = (R_1^2 + R_2^2)^{1/2}, \quad \|y\| = (T_a^2 + T_s^2)^{1/2}.$$

Thus there appears the same stable behaviour for the solution of the nonlinear coupled system (1) and (2) and the linear system (5) and (6). Therefore, the solution of the system (1) and (2) is unstable.

From the above conclusion, we know that the sea-air oscillator of an ENSO model (1) and (2), the air temperature anomaly T_a and sea surface temperature anomaly T_s in the equatorial eastern Pacific region show unstable trend and an asymptotic unstable annual phenomenon.

5. Conclusion

A. From the constructions of the system (Eqs.(1) and (2)) and functional F_i $(i = 1, 2)$ (Eqs.(3) and (4)), we can prove that the sequences of functions $\{(T_1(t))_n\}$, and $\{(T_2(t))_n\}$ determined by Eqs.(10) and (11) are convergent sequences. Then $(T_1(t) = \lim_{n\to\infty}(T_1(t))_n$, $T_2(t) = \lim_{n\to\infty}(T_2(t))_n)$ is a series of solution to the system (1) and (2).

B. The convergence rate for the solution $(T_1(t), T_2(t))$ of the system (1) and (2) is determined by selection of the initial approximate solution. In this paper, we select the initial approximate solution (14) and (15), from which we can find a series of finer approximate solution.

C. Atmospheric physics is a study of very complicated natural phenomena. Hence we need to use some basic models for the sea-air oscillator. And then we solve them using the approximate method. The variational iteration method is a simple and valid method.

D. The variational iteration is an approximate analytic method which is different from the general numerical method. The expansions of the solution by using the variational iteration method can be analytic. Thus, from Eqs.(18) and (19), we can further study the qualitative and quantitative behaviours of the temperature anomaly in the equatorial eastern Pacific region, the thermocline anomaly and the zonal wind stress anomaly. We leave these problems in future studies.

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