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Linear superposition method for (2+1)-dimensional nonlinear wave equations*

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New forms of different-periodic travelling wave solutions for the (2+1)-dimensional Zakharov–Kuznetsov (ZK) equation and the Davey–Stewartson (DS) equation are obtained by the linear superposition approach of Jacobi elliptic function. A sequence of cyclic identities plays an important role in these procedures.

Keywords: linear superposition, nonlinear equation, travelling wave solution
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1. Introduction

Modern soliton theory has been widely applied to almost all natural sciences, especially to all physical branches such as fluid physics, condensed matter, biophysics, quantum field theory, etc., and there has been a diversity of excellent books to review the topic. Since the nonlinear physical systems are normally characterized by nonlinear partial differential equations (PDEs), the construction of the exact solutions of nonlinear physical equations is one of the most important and essential tasks in the soliton theory. During the past decades, many significant methods have been established by many famous physicists and mathematicians, such as the inverse scattering transformation (IST),[1,2] Bäcklund and Darboux transformations,[2−4] the bilinear method,[5] the Painlevé analysis method,[6−8] symmetry reduction method,[9,10] separated variables method,[11] etc.

It is well known that the Jacobi elliptic functions are closely related to nonlinear differential equations.[12] Moreover many nonlinear evolution equations have been shown to possess the elliptic function solutions.[12−15] Recently, Liu *et al* developed a Jacobi elliptic function expansion method to express exact solutions of some nonlinear evolution equations by using three Jacobi elliptic functions. Motivated by Liu's idea and the properties of Jacobi elliptic functions, Yan developed an extended expansion method by using ten Jacobi elliptic functions.[15] However the solutions given by these two methods obviously were merely some combinations of Jacobi elliptic functions with the same period for a PDE. In order to find new forms of different-periodic travelling wave solutions, Khare and Sukhatme[16] further provided an extended Jacobi elliptic function method via proper linear superpositions of the original known Jacobi elliptic functions. Based on this method, some nonlinear PDEs[17,18] have been shown to possess new forms of exact periodic travelling wave solutions.

The rest of the paper is organized as follows. In Section 2, the (2+1)-dimensional Zakharov–Kuznetsov (ZK) equation is studied by the above Jacobi elliptic function linear superposition approach and new forms of different-periodic travelling wave solutions are obtained. Following the same calculation method, we discuss the (2+1)-dimensional Davey–Stewartson (DS) system in Section 3. The last section contains a short summary and discussion.

2. Zakharov-Kuznetsov (ZK) equation

In the magnetized plasma case, Zakharov and

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Kuznetsov[19] first studied the ZK equation by using the fluid description of plasma. Recently, there has been increasing interest in the ZK equation because of its form similar to the well-known KdV equation and the physical applications in plasma physics and compressible fluid.

It is easy to find that the (2+1)-dimensional ZK equation

$$u_t + uu_x + u_{xxx} + u_{xyy} = 0 \tag{1}$$

has a simplest periodic travelling wave solution, which can be expressed by a Jacobi elliptic function

$$u_1(x,\ y,\ t) = a_0\alpha^2 + 12\alpha^2(1+\gamma^2)\mathrm{dn}^2(\xi_1,\ m),$$
$$\xi_1 \equiv \alpha(x + \gamma y - b_1\alpha^2 t), \tag{2}$$

where a_0, α and γ are arbitrary constants and the velocity b_1 is given by

$$b_1 = a_0 + 4(1+\gamma^2)(2-m^2).$$

In this paper, the Jacobi elliptic functions are denoted by the standard notations $\mathrm{dn}(\xi,\ m), \mathrm{cn}(\xi,\ m)$, and $\mathrm{sn}(\xi,\ m)$, which are doubly periodic functions with periods $2K(m)$, $4K(m)$, and $4K(m)$, respectively. Here, $K(m)$ denotes the complete elliptic integral of the first kind, and m represents the elliptic modulus parameter ($0 \le\ m\ \le 1$). The $m = 0$ limit gives $K(0) = \pi/2$ and trigonometric functions: $\mathrm{sn}(\xi,\ 0)$=$\sin\xi$, $\mathrm{cn}(\xi,\ 0)$=$\cos\xi$, and $\mathrm{dn}(\xi,\ 0)$=1. While the $m = 1$ limit gives $K(1) \longrightarrow \infty$ and hyperbolic functions: $\mathrm{sn}(\xi,\ 1) \longrightarrow \tanh\xi$, $\mathrm{cn}(\xi,\ 1) \longrightarrow \mathrm{sech}\ \xi$, and $\mathrm{dn}(\xi,\ 1) \longrightarrow \mathrm{sech}\xi$.

In addition, we see that when $m \longrightarrow 1$ ($a_0 = 0$), the periodic travelling wave solution (2) naturally degenerates to the familiar one soliton solution

$$u_1(x,\ y,\ t) = 12\alpha^2(1+\gamma^2)\mathrm{sech}^2[\alpha(x + \gamma y - q_1\alpha^2 t)]. \tag{3}$$

According to Khare and Sukhatme's method,[9] we assume that the solution (2) has the linear superposition form

$$u_p(x,\ y,\ t) = a_0\alpha^2 + 12\alpha^2(1+\gamma^2)\sum_{i=1}^{p} d_i^2,$$
$$d_i \equiv \mathrm{dn}\left[\xi_p + \frac{2(i\ -\ 1)K(m)}{p}, m\right],$$
$$\xi_p \equiv \alpha(x + \gamma y - b_p\alpha^2 t), \tag{4}$$

where p is an arbitrary integer. For convenience, the relevant quantities s_i and c_i are defined in the same form as that of the quantity d_i,

$$s_i \equiv \mathrm{sn}\left[\xi_p + \frac{2(i\ -\ 1)K(m)}{p}, m\right],$$
$$c_i \equiv \mathrm{cn}\left[\xi_p + \frac{2(i\ -\ 1)K(m)}{p}, m\right]. \tag{5}$$

When $p = 1$, the solution (4) obviously is the original standard periodic solution (2). While for any other given choice of p, we can obtain the linear superposition solution of period $2K(m)/p$ for the ZK equation (1).

Substitution of Eq.(4) into the ZK equation (1) gives a simplification

$$24\alpha^5 m^2(1+\gamma^2)\Big\{[\ b_p - a_0 - 4(1+\gamma^2)(2-m^2)]$$
$$\times \sum_{i=1}^{p} d_i c_i s_i - 12(1+\gamma^2)\sum_{i=1}^{p} d_i c_i s_i \sum_{j\neq i}^{p} d_j^2\Big\} = 0. \tag{6}$$

The relevant identity[20] for an arbitrary p is

$$\sum_{i=1}^{p} d_i c_i s_i \sum_{j\neq i}^{p} d_j^2 = A(p,\ m)\sum_{k=1}^{p} d_k c_k s_k. \tag{7}$$

Some explicitly computed values of the constant $A(p,\ m)$ are

$$A(2,\ m) = 0,\ A(3,\ m) = \left(2 - \frac{2}{Q}\right),$$

and

$$A(4, m) = -2\sqrt{1-m^2}, \tag{8}$$

where

$$Q \equiv \mathrm{sn}^2[2K(m)/3,\ m]. \tag{9}$$

Meanwhile, the values at $m = 0$, and 1 are particularly simple:

$$A(p,\ 0) = -\frac{1}{3}(p-1)(p-2),\ \text{and}\ A(p,\ 1) = 0. \tag{10}$$

Therefore, inserting the identity (7) into the expression (6), it is obvious that Eq.(4) is a solution of the ZK equation with the general wave velocity

$$b_p = a_0 + 4(1+\gamma^2)[2 - m^2 + 3A(p,\ m)]. \tag{11}$$

Other than the solution (2), the additional solution of period $4K(m)$ of the ZK equation (1) is

$$u_1'(x,\ y,\ t) = a_0\alpha^2 - 12\alpha^2 m^2(1+\gamma^2)\mathrm{sn}^2(\eta_1,\ m), \tag{12}$$

where $\eta_1 \equiv \alpha(x + \gamma y - c_1\alpha^2 t)$ with the velocity $c_1 = a_0 - 4(1+\gamma^2)(1+m^2)$. In the case that p is

odd, the linear superposition of the previous solution (12) is

$$u'_p(x,\ y,\ t) = a_0\alpha^2 - 12\alpha^2 m^2(1+\gamma^2)\sum_{i=1}^{p}\tilde{s}_i^2,$$

$$\tilde{s}_i \equiv \mathrm{sn}\left[\eta_p + \frac{4(i-1)K(m)}{p},\ m\right],$$

$$\eta_p \equiv \alpha(x+\gamma y - c_p\alpha^2 t). \tag{13}$$

Substituting the solution (13) into the left-hand side of Eq.(1), we obtain

$$24\alpha^5 m^2(1+\gamma^2)\Bigg\{[c_p - a_0 + 4(1+\gamma^2)(1+m^2)] \times \sum_{i=1}^{p}\tilde{s}_i\tilde{d}_i\tilde{c}_i + 12m^2(1+\gamma^2)\sum_{i=1}^{p}\tilde{s}_i^2\sum_{j\neq i}^{p}\tilde{s}_j\tilde{d}_j\tilde{c}_j\Bigg\}. \tag{14}$$

Similarly, the identity analogous to the identity (7) for an odd p is

$$\sum_{i=1}^{p}\tilde{s}_i^2\sum_{j\neq i}^{p}\tilde{s}_j\tilde{c}_j\tilde{d}_j = B(p,\ m)\sum_{k=1}^{p}\tilde{s}_k\tilde{c}_k\tilde{d}_k. \tag{15}$$

Thus one easily acquires that Eq.(13) is an exact solution of the ZK equation with the velocity

$$c_p = a_0 - 4(1+\gamma^2)(1+m^2) - 12m^2(1+\gamma^2)B(p,\ m).$$

Note that for an even integer p, we do not obtain any linear superposition solution of the basic solution (12). As an illustration, for $p=3$, it is shown that[18]

$$B(3,\ m) = \frac{2}{1-q^2}, \tag{16}$$

and the relevant velocity of the ZK solution is

$$c_3 = a_0 - 4(1+\gamma^2)(1+m^2) - \frac{24m^2(1+\gamma^2)}{1-q^2}, \tag{17}$$

where q is defined by

$$q \equiv \mathrm{dn}[2K(m)/3,\ m]. \tag{18}$$

3. Davey–Stewartson (DS) equation

The (2+1)-dimensional Davey–Stewartson (DS) equation is given by

$$\mathrm{i}Q_t + \frac{1}{2}(Q_{xx}+Q_{yy}) = -\sigma|Q|^2Q + \psi Q,$$

$$\psi_{xx} - \psi_{yy} = 2\sigma(|Q|^2)_{xx},$$

$$\sigma = \pm 1, \tag{19}$$

where $Q(x,\ y,\ t)$ is complex while $\psi(x,\ y,\ t)$ is real. Equation (19) is of physical interest, since it describes the propagation of two-dimensional water waves with a finite depth. The DS equation is studied in different theoretical frameworks. For example, the initial value problem is linearized by Fokas and Ablowitz.[21]

To seek the doubly periodic solutions of Eq.(19), we introduce the following transformations:

$$Q(x,\ y,\ t) = u(\xi)\mathrm{e}^{\mathrm{i}[r(\xi)+nt]},$$

$$\psi(x,\ y,\ t) = \psi(\xi),$$

$$\xi \equiv x + ky - vt, \tag{20}$$

with $u(\xi),\ r(\xi),\ \psi(\xi)$ all being real and $|\ k\ | > 1$. Substitution of the expressions (20) in to Eq.(19) yields

$$\psi = \frac{2\sigma}{1-k^2}|Q|^2 = \frac{2\sigma}{1-k^2}u^2, \tag{21}$$

$$r'(\xi) = \frac{1}{1+k^2}\left(v + \frac{A}{u^2}\right), \tag{22}$$

$$u'^2(\xi) = -\frac{\sigma u^4}{k^2-1} + \frac{2}{1+k^2}\left(n - \frac{v^2}{2(1+k^2)}\right)u^2 + \frac{2B}{1+k^2} - \frac{A^2}{(1+k^2)^2u^2}, \tag{23}$$

where the symbol "$'$" denotes a derivative with respect to the argument ξ, and A and B are constants of integration. Therefore the whole problem reduces to finding the solution of Eq.(23).

In the case $\sigma = 1$, the two periodic travelling wave solutions of Eq.(23) are ($\xi_1 = x + ky - v_1 t$),

$$u_1(\xi_1) = \sqrt{k^2-1}\,\mathrm{dn}\xi_1,\ r_1 = \frac{v_1\xi_1}{k^2+1},\ \psi_1 = -2\mathrm{dn}^2\xi_1,$$

$$A\ =\ 0,\ B\ =\ \frac{1}{2}(m^2-1)(k^4-1),$$

$$v_1^2 = 2n(1+k^2) + (m^2-2)(1+k^2)^2, \tag{24}$$

and

$$u_2(\xi_1) = \sqrt{k^2-1}\ m\ \mathrm{cn}\xi_1,$$

$$r_2 = \frac{v_1\xi_1}{k^2+1},\ \psi_2 = -2m^2\mathrm{cn}^2\xi_1,$$

$$A\ =\ 0,\ B\ =\ \frac{1}{2}m^2(1-m^2)(k^4-1),$$

$$v_2^2 = 2n(1+k^2) + (1-2m^2)(1+k^2)^2. \tag{25}$$

While for $\sigma = -1$, the travelling wave solution of Eq.(23) is

$$u_3(\xi_1) = \sqrt{k^2-1}\ m\ \mathrm{sn}\xi_1,$$

$$r_3 = \frac{v_1\xi_1}{k^2+1},\ \psi_3 = 2m^2\mathrm{sn}^2\xi_1,$$

$$A = 0,\ B = \frac{1}{2}m^2(k^4-1),$$
$$v_3^2 = 2n(1+k^2)+(1+m^2)(1+k^2)^2. \quad (26)$$

In the limiting case $m=1$, Eqs.(24)–(26) degenerate to the familiar soliton solutions respectively

$$u_1(\xi_1) = \sqrt{k^2-1}\,\mathrm{sech}\xi_1,$$
$$r_1 = \frac{v_1\xi_1}{k^2+1},\ \psi_1 = -2\mathrm{sech}^2\xi_1,$$
$$A = B = 0,$$
$$v_1^2 = 2n(1+k^2)-(1+k^2)^2,\quad \left(n \geq \frac{(k^2+1)}{2}\right), \quad (27)$$

$$u_2(\xi_1) = \sqrt{k^2-1}\tanh\xi_1,$$
$$r_2 = \frac{v_1\xi_1}{k^2+1},\ \psi_1 = 2\tanh^2\xi_1,$$
$$A = 0,\ B = \frac{1}{2}(k^4-1),$$
$$v_2^2 = 2n(1+k^2)+2(1+k^2)^2. \quad (28)$$

Based on the above-mentioned method, we assume the linear superposition solution along the period $2K(m)/p$ of Eq.(23) to be

$$u_p(x,\ y,\ t) = \sqrt{k^2-1}\sum_{i=1}^{p} d_i,$$
$$d_i = \mathrm{dn}\left[\xi_p + \frac{2(i-1)K(m)}{p},\ m\right],$$
$$\xi_p \equiv x + ky - v_p t. \quad (29)$$

Obviously, $p=1$ is the original solution (24), while for any other choice of p, we obtain solutions of period $2K(m)/p$.

According to the properties of the Jacobi elliptic functions, it is easy to derive identities[20]

$$\left(\sum_{i=1}^{p} d_i\right)^2 = \sum_{i=1}^{p} d_i^2 + C(p,\ m),$$
$$\left(\sum_{i=1}^{p} d_i\right)^4 = \sum_{i=1}^{p} d_i^4 + D(p,\ m)\sum_{i=1}^{p} d_i^2 + E(p,\ m),$$
$$m^4\sum_{i<j}^{p} s_i c_i s_j c_j = F(p,\ m)\sum_{i=1}^{p} d_i^2 + G(p,\ m). \quad (30)$$

Some exactly computed values of the constants $C(p,\ m)$, $D(p,\ m)$, $E(p,\ m)$, $F(p,\ m)$, and $G(p,\ m)$ are

$$C(3,\ m) = 2q(q+2),\ C(4,\ m) = 4\tilde{t}\ (1+\tilde{t}^{\ 2}\ ) + 4\tilde{t}^{\ 2},$$
$$D(2,\ m) = 2C(2,\ m) = 4F(2,\ m) = 4\sqrt{1-m^2},$$
$$D(3,\ m) = 4F(3,\ m) = \frac{8m^2q}{1-q^2},$$
$$D(4,\ m) = 4F(4,\ m) = 4[\ 2\tilde{t}\ (1+\tilde{t}^{\ 2}\ ) + \tilde{t}^{\ 2}\ ],$$
$$E(2,\ m) = 6(1-m^2),$$
$$E(3,\ m) = 6q^2(q+2)^2 - 8(1-m^2),$$
$$E(4,\ m) = 16[\ \tilde{t}\ (1+\tilde{t}^{\ 2}\ ) + \tilde{t}^{\ 2}\ ]^2 + 8\tilde{t}^{\ 2}\ (1+\tilde{t}^{\ 2}\ )^2 - 4\tilde{t}^{\ 4},$$
$$G(2,\ m) = \sqrt{1-m^2}(\ m^2-2),$$
$$G(3,\ m) = m^2 - (2-m^2)(1+q)^2,$$
$$G(4,\ m) = -4\tilde{t}^{\ 3}\ (1+\tilde{t}^{\ 2}\ ) - 2(2-m^2)[\ \tilde{t}^{\ 2} + \tilde{t}\ (1+\tilde{t}^{\ 2}\ )\ ], \quad (31)$$

where $\tilde{t}$ is given by

$$\tilde{t} \equiv (1-m^2)^{1/4}. \quad (32)$$

Substituting Eq.(29) together with identities (30) into Eq.(23), we have

$$\begin{aligned}&\Big[\ (2-m^2) + 2F(p,\ m) + D(p,\ m)\\ &+ \frac{2}{1+k^2}\left(\frac{v_p^2}{2(1+k^2)} - n\right)\ \Big]\ \sum_{i=1}^{p} d_i^2 + p(m^2-1)\\ &+ 2G(p,\ m) + E(p,\ m)\\ &+ \frac{2}{1+k^2}\left(\frac{v_p^2}{2(1+k^2)} - n\right)C(p,\ m)\\ &- \frac{2B}{k^4-1} = 0, \end{aligned} \quad (33)$$

where $p = 2,\ 3,\ \ldots$. Thus the general expression for the velocity v_p is

$$\begin{aligned}v_p^2 =& 2n(1+k^2) + (1+k^2)^2[\ m^2 - 2 - 2F(p,\ m)\\ &- D(p,\ m)\ ], \end{aligned} \quad (34)$$

and the constant B should satisfy

$$\begin{aligned}B =& \frac{k^4-1}{2}\Big[\ p\ (m^2-1) + 2G(p,\ m) + E(p,\ m)\\ &+ \frac{2}{1+k^2}\left(\frac{v_p^2}{2(1+k^2)} - n\right)C(p,\ m)\ \Big]. \end{aligned}$$

Moreover, for an arbitrary p at $m=0$,

$$D(p,\ 0) = 4F(p,\ 0) = \frac{4(p^2-1)}{3},$$

and at $m=1$,

$$D(p,\ 1) = F(p,\ 1) = 0.$$

Thus from Eq.(34), it is obvious that the solution u_p is valid only if $n \geq p^2(1+k^2)$, and in this case v_p^2 varies from $2(1+k^2)[n-p^2(1+k^2)]$ to $(1+k^2)[2n-(1+k^2)]$

as the elliptic modulus parameter m changes from 0 to 1.

For an odd integer p, we obtain an additional solution of Eq.(23) by using the linear superposition of the solution (25)

$$u_p(x,\ y,\ t) = \sqrt{k^2-1}\ m\sum_{i=1}^{p}\tilde{c}_i,$$

$$\tilde{c}_i = \mathrm{cn}\left[\zeta_p + \frac{4(i-1)K(m)}{p},\ m\right],$$

$$\zeta_p \equiv x + ky - v_p t. \tag{35}$$

Similarly, one derives the identities[20]

$$\left(\sum_{i=1}^{p}\tilde{c}_i\right)^2 = \sum_{i=1}^{p}\tilde{c}_i^2 + H(p,\ m),$$

$$\left(\sum_{i=1}^{p}\tilde{c}_i\right)^4 = \sum_{i=1}^{p}\tilde{c}_i^4 + I(p,\ m)\sum_{i=1}^{p}\tilde{c}_i^2 + J(p,\ m),$$

$$\sum_{i<j}^{p}\tilde{s}_i\tilde{d}_i\tilde{s}_j\tilde{d}_j = L(p,\ m)\sum_{i=1}^{p}\tilde{c}_i^2 + M(p,\ m). \tag{36}$$

Evidently, substitution of Eq.(35) with identities (36) into Eq.(23) derives the general expression for the velocity v_p

$$\begin{aligned} v_p^2 =& 2n(1+k^2) + (1+k^2)^2\ [1 - 2m^2 - 2L(p,\ m) \\ & - m^2\ I(p,\ m)]. \end{aligned} \tag{37}$$

And the constant B is given by

$$\begin{aligned} B =& \frac{m^2\ (k^4-1)}{2B}\Bigg[\ p\ (1-m^2) \\ & + 2M(p,\ m) + m^2 J(p,\ m) \\ & + \frac{2}{1+k^2}\left(\frac{v_p^2}{2(1+k^2)} - n\right)H(p,\ m)\Bigg]. \end{aligned}$$

Some explicitly calculated values of the constants $H(p,\ m)$, $I(p,\ m)$, $J(p,\ m)$, $L(p,\ m)$, and $M(p,\ m)$ are

$$H(3,\ m) = -\frac{2q(q+2)}{(1+q)^2},$$

$$\begin{aligned} m^2 I(3,\ m) =& 4L(3,\ m) \\ =& -4m^2 q\left[\frac{q+2}{(1+q)^2} + \frac{q}{1-q^2}\right], \end{aligned}$$

$$J(3,\ m) = \frac{8q^3(q+2)}{(1+q)^2(1-q^2)} + \frac{6q^2(q+2)^2}{(1+q^4)},$$

$$M(3,m) = 2m^2\frac{q^3(q+2)}{(1+q)^2(1-q^2)} - (1-2m^2)\frac{q(q+2)}{(1+q)^2}. \tag{38}$$

Therefore v_p^2 changes from $(1+k^2)(2n+1+k^2)$ to $(1+k^2)(2n-1-k^2)$ as m goes from 0 to 1.

On the other hand, for an odd p, using the solution (26), we obtain the connected solution of Eq.(23) via linear superposition

$$u_p(x,\ y,\ t) = \sqrt{k^2-1}\ m\sum_{i=1}^{p}\tilde{s}_i,$$

$$\tilde{s}_i = \mathrm{sn}\left[\eta_p + \frac{4(i-1)K(m)}{p},\ m\right],$$

$$\eta_p \equiv x + ky - v_p t. \tag{39}$$

In this case, the identities analogous to Eq.(36) for an odd p are

$$\left(\sum_{i=1}^{p}\tilde{s}_i\right)^2 = \sum_{i=1}^{p}\tilde{s}_i^2 + N(p,\ m),$$

$$\left(\sum_{i=1}^{p}\tilde{s}_i\right)^4 = \sum_{i=1}^{p}\tilde{s}_i^4 + O(p,\ m)\sum_{i=1}^{p}\tilde{s}_i^2 + R(p,\ m),$$

$$\sum_{i<j}^{p}\tilde{c}_i\tilde{d}_i\tilde{c}_j\tilde{d}_j = S(p,\ m)\sum_{i=1}^{p}\tilde{s}_i^2 + T(p,\ m). \tag{40}$$

Thus the general expression for the velocity of solution (39) is

$$\begin{aligned} v_p^2 =& 2n(1+k^2) + (1+k^2)^2\ [1 + m^2 - 2S(p,\ m) \\ & + m^2\ O(p,\ m)]. \end{aligned} \tag{41}$$

Particularly, some precisely calculated values of the constants $N(p,\ m), O(p,\ m)$, $R(p,\ m)$, $S(p,\ m)$, and $T(p,\ m)$ are

$$N(3,\ m) = \frac{2(q^2-1)}{m^2},$$

$$m^2 O(3,\ m) = -4S(3,\ m) = 4m^2\left[\frac{1}{1-q^2} + \frac{q^2-1}{m^2}\right],$$

$$R(3,\ m) = \frac{6(\ q^2-1)^2}{m^4} - \frac{8}{m^2},$$

$$T(3,\ m) = \frac{(1+m^2)(\ q^2-1)}{m^2} + 2. \tag{42}$$

Note that the quantity q in Eqs.(31), (38), and (42) possesses the same definition as Eq.(18). Naturally v_p^2 changes from $(1+k^2)(2n+1+k^2)$ to $2(1+k^2)(n+1+k^2)$ as m varies from 0 to 1.

Remarkably for an even integer p, there are no linear superposition forms of the original standard periodic solutions (25) and (26).

As an illustration, the solution u_p (29) is plotted in Fig.1. It is evidently shown in Fig.1 that u_p is a periodic solution and the solution obtained via the linear superposition method of Jacobi elliptic function possesses new properties.

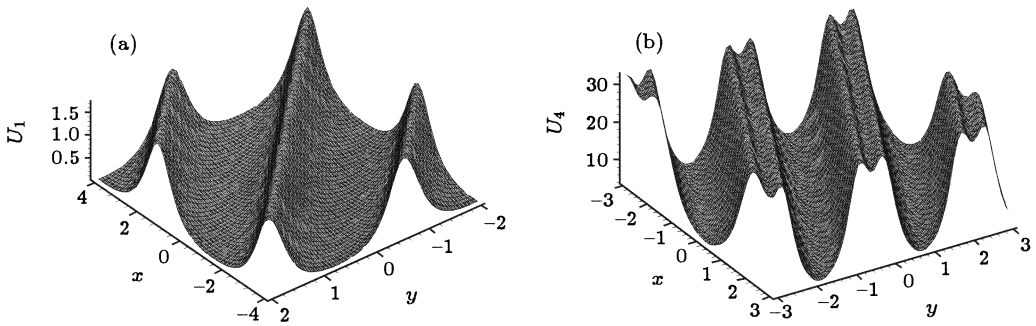


Fig.1. Solution (29) for DS equation with Eq.(23) and k=2, t=0, m=0.95: Fig.1(a) for u_1, Fig.1(b) for u_4.

4. Summary

In short, the linear superposition method based on different Jacobi elliptic functions is applied to two (2+1)-dimensional nonlinear wave solutions. And it is shown that new forms of different periodic travelling wave solutions of the ZK equation and the DS equation are obtained via the above-mentioned linear superposition approach. For these two considered (2+1)-dimensional equations, the obtained periodic travelling wave solutions are not only of period $2K(m)/p$ but also of period $4K(m)/p$ (in the case where p is odd). This linear superposition procedure is crucially dependent on a sequence of remarkable cyclic identities involving the Jacobi elliptic functions.

It is easy to see that the new forms of different-periodic travelling wave solutions, obtained via the above-mentioned method, degenerate to the one soliton solutions in the case where $m = 1$. In other words, the single soliton solutions are just the limiting case of the Jacobi elliptic function solutions. Besides the one soliton solution, it is well known that many nonlinear evolution equations possess two-soliton, three-soliton, or even multisoliton solutions. If we start from some proper forms of the known soliton solutions other than the one soliton solutions, can we derive some new linear superposition solutions expressed by Jacobi elliptic functions? This question is worth being studied further.

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