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Design of quantum VQ iteration and quantum VQ encoding algorithm taking $O(\sqrt{N})$ steps for data compression*

Pang Chao-Yang(庞朝阳)[a)b)†], Zhou Zheng-Wei(周正威)[a)‡],
Chen Ping-Xing(陈平形)[a)], and Guo Guang-Can(郭光灿)[a)§]

[a)] *Key Laboratory of Quantum Information, University of Science and Technology of China, Hefei 230026, China*

[b)] *College of Mathematics and Software Science, Sichuan Normal University, Chengdu 610066, China*

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Vector quantization (VQ) is an important data compression method. The key of the encoding of VQ is to find the closest vector among N vectors for a feature vector. Many classical linear search algorithms take $O(N)$ steps of distance computing between two vectors. The quantum VQ iteration and corresponding quantum VQ encoding algorithm that takes $O(\sqrt{N})$ steps are presented in this paper. The unitary operation of distance computing can be performed on a number of vectors simultaneously because the quantum state exists in a superposition of states. The quantum VQ iteration comprises three oracles, by contrast many quantum algorithms have only one oracle, such as Shor's factorization algorithm and Grover's algorithm. Entanglement state is generated and used, by contrast the state in Grover's algorithm is not an entanglement state. The quantum VQ iteration is a rotation over subspace, by contrast the Grover iteration is a rotation over global space. The quantum VQ iteration extends the Grover iteration to the more complex search that requires more oracles. The method of the quantum VQ iteration is universal.

Keywords: data compression, vector quantization, Grover's algorithm, quantum VQ iteration
PACC: 7110, 0270, 2980C, 4250

1. Introduction

1.1. The introduction of vector quantization (VQ)[1−5]

Image compression technologies become more and more important in consequence of spreading digital still camera and digital documentation system applications. In this area, vector quantization (VQ) arouses considerable interest because it is an effective technique to compress images. Figure 1 illustrates the concept of VQ image encoding and decoding. In image encoding using VQ, distances between an input vector, which is typically a block of 4×4 pixels taken from the input image, and template vectors in the codebook are calculated first, and then the most similar template vector having the shortest distance is determined. Next, only the index of that template vector is output and transmitted. To decompress the image, the decoder pastes the corresponding template pattern addressed by the received index at each location.[1]

The two main research topics of VQ are to design the codebook generation algorithm and to design the real-time encoding algorithm,[2,3] The encoding is described as follows.[2−5]

Encoding Suppose the codebook $C = \{c[i]|0 \leq i < N, c[i] \in R^k, k \geq 1\}$ be a table of $N = 2^n$ distinct codevectors. For an arbitrary input vector $x \in R^k$, we have to find the index i_0 such that $d(x, c[i_0])$ is minimum, where R^k represents the k dimensional Euclidean space. Then the index i_0 is transmitted.

To meet the requirements of real-time encoding, the feature vector must be encoded as fast as possible after arrival. Because sorting high-dimensional vectors is very difficult, the full-search algorithm taking

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†E-mail: cyp_900@hotmail.com or cypang@ustc.edu.cn
‡E-mail: zwzhou@ustc.edu.cn
§E-mail: gcguo@ustc.edu.cn

$O(N)$ steps of distance computation between two vectors, i.e. the linear search method, is adopted widely although there are some structured algorithms taking $O(\log_2 N)$ steps such as tree-structure VQ.[2−4]

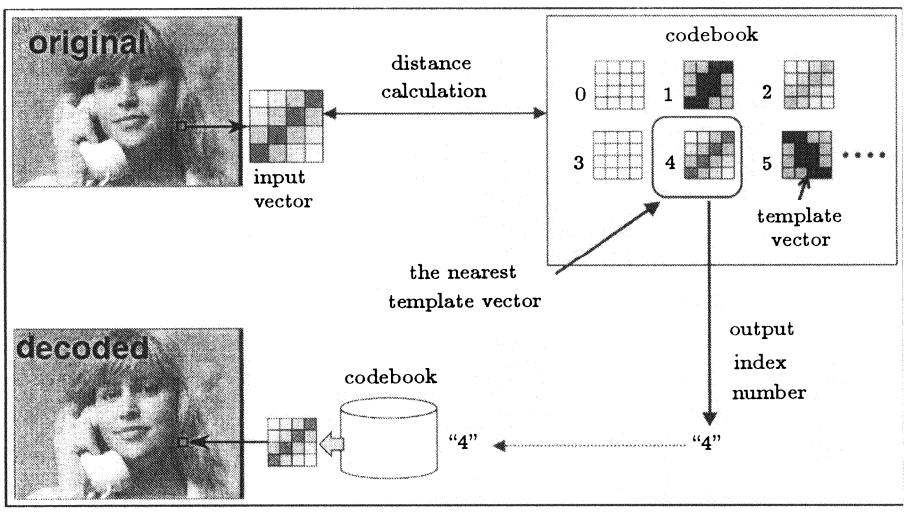


Fig.1. Concept of VQ encoding and decoding.[1]

1.2. The introduction of Grover's algorithm[6−18]

The power of quantum computation is due to the fact that the state of a quantum computer can be a superposition of basis states and we can perform an operation on multiple quantum states simultaneously.[6−8] The Grover iteration is defined as a rotation $G = (2|\Psi\rangle\langle\Psi| - I)O$ over the global N-dimensional Hilbert space, where $|\Psi\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle$ and O is the computing oracle.[6,7] And the kernel of Grover's algorithm is repeating the iteration until $|\Psi\rangle$ is rotated near the solution $|\beta\rangle$. Grover's algorithm takes $O(\sqrt{N})$ steps with successful probability $1 - O\left(\frac{1}{N}\right)$,[9] where N is the number of elements in the database. Refs.[10,11] show that Grover's algorithm is suitable for the case of an arbitrary initial amplitude distribution.[10,11] Some special cases of Grover's algorithm are discussed.[10−14]

Boyer, Dürr and Ahuja *et al* apply the Grover iteration to find a minimum or maximum value from a list of numbers with a certain probability $1 - p$ and the expected time is $O(\sqrt{N})$.[8,14,15] The probability of success can be improved to $1 - p^c \to 1$ by running the algorithms c times.[8,14,15]

2. The quantum VQ iteration G_{VQ}

2.1. The two oracles O_{d} and O_{f}

Being similar to other quantum algorithms,[6−9,12,16−18] the addressing unitary operations and the ancilla registers saving the fetched data $c[i]$ are omitted in the following discussion. We often use expression $d(x, c[i]) = \sum_{j=1}^{k} |x_j - c[i]_j|^2$ as a substitute for distance formula in classical algorithms,[2] where $x \in R^k$,$x = (x_1, x_2, ..., x_k)$ and $y \in R^k$, $y = (y_1, y_2, ..., y_k)$.

The quantum VQ encoding algorithm below calls two oracles. One of the oracles denoted by O_{d} is used to compute the distance $d(x, c[i])$, $0 \le i < N$, i.e.

$$O_{\mathrm{d}} : |x_1\rangle ... |x_k\rangle |i\rangle |p\rangle |q\rangle \\ \to |x_1\rangle ... |x_k\rangle |i\rangle |p \oplus d(x, c[i])\rangle |q\rangle.$$

In vector notation,

$$O_{\mathrm{d}} : |x\rangle |i\rangle |p\rangle |q\rangle \to |x|i\rangle |p \oplus d(x, c[i])\rangle |q\rangle,$$

where each ket represents a register rather than a single qubit. The length of each component x_j, $|i\rangle$, $|p\rangle$ and $|q\rangle$ is m, n, n_0 and n_0, respectively. The register $|p\rangle$ is used to save the distance value $d(x, c[i])$, $|i\rangle$ corresponds to the index of the codevector. The register $|q\rangle$ is used to save the smaller distance value for comparison of distance values. For convenience, the register $|p\rangle$ and $|q\rangle$ should be initialized to zero and a very large number respectively. Then the oracle O_{d} can be simplified as

$$O_{\mathrm{d}} : \ |x\rangle |i\rangle |0\rangle |q\rangle \to |x|i\rangle |d(x, c[i])\rangle |q\rangle.$$

The other oracle denoted by $O_{\rm f}$ is used to compare two distance values:

$$f(i)=\begin{cases}1 & \text{if } d(x,c[i])<q\\ 0 & \text{otherwise.}\end{cases}$$

By Deutsch's algorithm,[6,7] we have

$$O_{\rm f}:|x\rangle|i\rangle|d(x,c[i])\rangle|q\rangle \rightarrow(-1)^{f(i)}|x\rangle|i\rangle|d(x,c[i])\rangle|q\rangle,$$

where the ancilla qubits are omitted.

2.2. The concept of quantum VQ iteration $G_{\rm VQ}$

Definition The quantum VQ iteration $G_{\rm VQ}$ is defined as $G_{\rm VQ}=(2|\xi\rangle\langle\xi|-I)(O_{\rm d})^{-1}O_{\rm f}O_{\rm d}$, where $|\xi\rangle=\frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|i\rangle$, $(O_{\rm d})^{-1}$ is the inverse transform of $O_{\rm d}$. For any input state $|{\rm input}\rangle=\sum_{i=0}^{N-1}A_i|x\rangle|i\rangle|0\rangle|y\rangle$, we have

$$\begin{aligned}&(O_{\rm d})^{-1}O_{\rm f}O_{\rm d}|x\rangle|i\rangle|0\rangle|y\rangle\\ =&(-1)^{f(i)}|x\rangle|i\rangle|0\rangle|y\rangle,\end{aligned}\tag{1}$$

i.e., $|x\rangle|i\rangle|0\rangle|y\rangle$ is the eigenstate of $(O_{\rm d})^{-1}O_{\rm f}O_{\rm d}$,

$$G_{\rm VQ}|{\rm input}\rangle=|x\rangle\Big\{(2|\xi\rangle\langle\xi|-I)\times\Big[\sum_{i=0}^{N-1}(-1)^{f(i)}A_i|i\rangle\Big]\Big\}|0\rangle|y\rangle,\tag{2}$$

where A_i is the probability amplitude of the state $|i\rangle$ and $\sum_{i=0}^{N-1}|A_i|^2=1$.

Figures 2 and 3 illustrate the concept of VQ quantum iteration $G_{\rm VQ}$.

By Eqs.(1) and (2), being similar to Grover iteration,[6,7] the quantum VQ iteration $G_{\rm VQ}$ is a rotation over N-dimensional subspace decided by state $|\xi\rangle$ because the contents of the first, third and fourth registers are constant and only the second register is variable ($0\le i<N$) (Fig.3). The dimension of the global space is $2^{n_0}N$ which is decided by the lengths in the second and the third register. Comparing the Grover iteration, the quantum VQ iteration is not a rotation over global space.

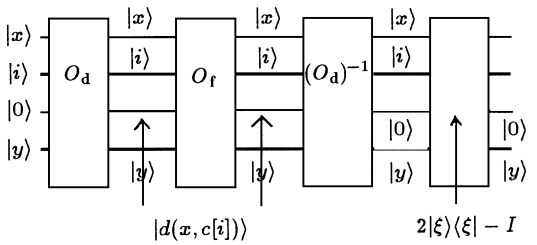


Fig.2. Concept of VQ quantum iteration $G_{\rm VQ}$.

Theorem Suppose t is the number of elements in the set $\{j|d(x,c[j])<y, j=0,1,2,...N-1\}$ for given value y.

Let $|\Psi_0\rangle=\frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|x\rangle|i\rangle|0\rangle|y\rangle$ be the initial state. Running the quantum VQ iteration $G_{\rm VQ}$ $O\left(\sqrt{\frac{N}{t}}\right)$ times, we can find a special i_0 such that $d(x,c[i_0])<y$.

Figure 3 shows the action of a single quantum VQ iteration $G_{\rm VQ}$.

Proof Let

$$|\alpha\rangle=\frac{1}{\sqrt{N-t}}|x\rangle\Big(\sum_u|u\rangle\Big)|0\rangle|y\rangle$$

and

$$|\beta\rangle=\frac{1}{\sqrt{t}}|x\rangle\Big(\sum_v|v\rangle\Big)|0\rangle|y\rangle,$$

where $d(x,c[u])\ge y$ and $d(x,c[v])<y$.

Hence,

$$|\Psi_0\rangle=\sqrt{\frac{N-t}{N}}|\alpha\rangle+\sqrt{\frac{t}{N}}|\beta\rangle.$$

The quantum VQ iteration $G_{\rm VQ}$ is a rotation over subspace span$\{|\alpha\rangle,|\beta\rangle\}$ by Eqs.(1) and (2). Let $\cos\frac{\theta}{2}=\sqrt{\frac{N-t}{N}}$. The rotation angle is θ. Therefore, being similar to Grover iteration,[6,7] only applying $G_{\rm VQ}$ $O\left(\sqrt{\frac{N}{t}}\right)$ times, $|\Psi_0\rangle$ is rotated to $|\beta\rangle$.

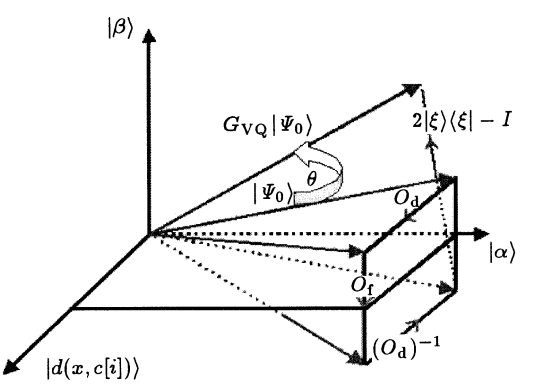


Fig.3. The action of a single quantum VQ iteration $G_{\rm VQ}$.

2.3. The significance of using two or more oracles

The quantum VQ iteration $G_{\rm VQ}$ comprises three oracles $O_{\rm d}$, $O_{\rm f}$ and $(O_{\rm d})^{-1}$; by contrast many algorithms have only one oracle such as the famous Shor's factorization algorithm, Grover's algorithm, Deutsch's algorithm, the Deutsch–Jozsa algorithm,[6,7,9,17,18] the algorithm of Bernstein–Vazirani's problem,[6,7,17] the algorithm of Simon's problem,[17] the minimum search algorithm,[14,15] the maximum search algorithm,[8] and so on.

It will bring convenience by designing a quantum algorithm that uses of two or more oracles. It is the basic requirement of classical database search to find a recorder satisfying some given conditions.[19] The requirement should be a basic requirement for the future quantum searching also, and each complex condition corresponds to one oracle of quantum computation. Grover's algorithm cannot perform more complex searching because it has only one oracle; by contrast $G_{\rm VQ}$ has three oracles and can perform more complex searching. On the other hand, measuring is strictly prohibited during the process of quantum computation in most cases and the application of mores oracles evades the measurement problem.

2.4. Comparison between the quantum VQ iteration and the Grover iteration

The quantum VQ iteration $G_{\rm VQ} = (2|\xi\rangle\langle\xi| - I)(O_{\rm d})^{-1}O_{\rm f}O_{\rm d}$ is derived from the Grover iteration $G = (2|\Psi\rangle\langle\Psi| - I)O$; they have the same idea that the input state is rotated to the solutions (Fig.3). The VQ iteration $G_{\rm VQ}$ extends the Grover iteration to the more complex search requiring more oracles. The entanglement state $O_{\rm d}|\Psi_0\rangle = \frac{1}{\sqrt{N}}|x\rangle\left(\sum_{i=0}^{N-1}|i\rangle|d(x,c[i])\right)|y\rangle$ is generated and used; by contrast the state in Grover's algorithm is not an entanglement state. The quantum VQ iteration $G_{\rm VQ}$ is a rotation in subspace; by contrast the Grover iteration is a rotation over global space. Rotating over subspace reduces the search range and speeds up the computing. The method of the quantum VQ iteration is universal because the oracle $O_{\rm d}$ can be substituted by any other function. The feasibility of realizing the quantum VQ iteration depends on realization of the Grover iteration and the oracle $O_{\rm d}$. Table 1 shows the difference between the quantum VQ iteration and the Grover iteration.

Table 1. The difference between the quantum VQ iteration and the Grover iteration.

The quantum VQ iteration (Fig.3.) $G_{\rm VQ} = (2\|\xi\rangle\langle\xi\| - I)(O_{\rm d})^{-1}O_{\rm f}O_{\rm d}$	The Grover iteration[6,7,16] $G = (2\|\Psi\rangle\langle\Psi\| - I)O$.
Has three oracles. Can perform more complex search.	Has only one oracle. Perform simple search.
The entanglement states is generated and used. Can perform complex computation.	There are no entanglement states. Cannot perform complex computation.
The rotation is over subspace. So the time complexity is $O(\sqrt{N})$.	The rotation is over global space. The time complexity is greater than $O(\sqrt{N})$.

3. The quantum encoding algorithm

3.1. Sub-procedure

The function of the sub-procedure is to find a smaller distance value $y_0 = d(x, c[i_0])$ such that $y_0 < y$ for the given y. The Grover iteration in Ref.[14] is substituted by $G_{\rm VQ}$.

The steps of the sub-procedure are:

1) Initialize $\Gamma = 1$ and $\lambda = \frac{6}{5}$.

2) Choose j uniformly at random among the non-negative integers not greater than Γ.

3) Apply j iterations of the quantum VQ iteration $G_{\rm VQ} = (2|\xi\rangle\langle\xi| - I)O_{\rm d}^{-1}O_{\rm f}O_{\rm d}$ starting from the state $|\Psi_0\rangle = \frac{1}{\sqrt{N}}|x\rangle\left(\sum_{i=0}^{N-1}|i\rangle\right)|0\rangle|y\rangle$.

4) Observe the second register: let i_0 be the outcome.

5) Perform classical computing $y_0 = d(x, c[i_0])$. If $y_0 < y$, the problem is solved. Save the old distance value and the corresponding codevector $c[l]$ and exit.

6) Otherwise, set Γ to $\min\{\lambda\Gamma, \sqrt{N}\}$ and go to step 3).

The time complexity of the sub-procedure is $O\left(\sqrt{\frac{N}{t}}\right)$ by Ref.[14], where t is the number of elements of set $\{i|d(x, c[i]) < y\}$.

3.2. The quantum VQ encoding algorithm

The form of the encoding algorithm is similar to the minimum search algorithm[15] and the maximum search algorithm.[8] The Grover iteration in Ref.[15] is substituted by the quantum VQ iteration $G_{\rm VQ}$ and the sub-procedure is called repeatedly until obtaining the solution:

1) Obtain a random distance value $y \in \{d(x, c[i])|1 \leq i \leq N\}$ and the corresponding index l of the codevector. This can be realized by classical computing.

2) Repeat the following steps and interrupt it when the total running time is longer than $22.5\sqrt{N} + 1.4(\log_2 N)^2$. Then go to step 2)b.

a) Apply the sub-procedure above.

b) Observe the second register and the third register: let i_0, y_0 be the outcomes respectively. That is, $y_0 = d(x, c[i_0])$. If $y_0 < y$, then set y_0 to y and i_0 to l.

3) Return the corresponding classical index $l_{\rm classical}$ of the codevector $c[l]$ as the encoding result for x and send the $l_{\rm classical}$ to the receiver for decoding.

Clearly the total qubits $\tau = km + 2n_0 + n$ if ancilla qubits are omitted. The time complexity of the algorithm is $O(\sqrt{N})$ steps of distance computing by Ref.[15].

4. Conclusion

Vector quantization (VQ, Fig.1) is regarded as an important data compression method, especially for image compression and speech compression.[1−3] The encoding of VQ problem is defined as finding the particular codevector index $i_0 = \arg \min_{0 \leq i < N} d(x, c[i])$ from the table of the codebook $C = \{c[i]|0 \leq i < N, c[i] \in R^k, k \geq 1\}$ and send the index i_0 to the receiver, where $d(x, c[i]) = \sum_{j=1}^{k} |x_j - c[i]_j|^2$ generally.[2,3] $O(N)$ steps of distance computing are required for obtaining the *exact* index i_0 generally in classical case.[2,3] This paper presents the quantum VQ iteration $G_{\rm VQ} = (2|\xi\rangle\langle\xi| - I)O_{\rm d}^{-1}O_{\rm f}O_{\rm d}$ (Fig.2 and Fig.3) and applies it to realize quantum VQ encoding algorithm that takes $O(\sqrt{N})$ steps. The quantum VQ iteration is derived from the Grover iteration, they have the same idea that input state is the superposition of states and it is rotated to the solutions (Fig.3). The VQ quantum iteration $G_{\rm VQ}$ comprises three oracles $O_{\rm d}$, $O_{\rm f}$ and $(O_{\rm d})^{-1}$; by contrast many algorithms have only one oracle such as the well known Shor factorization algorithm, Grover's algorithm, Deutsch's algorithm, Deutsch–Jozsa algorithm, the algorithm of Bernstein–Vazirani's problem, the algorithm of Simon's problem, the minimum search algorithm, the maximum search algorithm, and so on.[6−8,14−18] The application of two or more oracles is important for future quantum computation because complex computation is required to decompose into many simple oracles and measuring is strictly prohibited during the process of quantum computation in most cases. The quantum VQ iteration extends the Grover iteration to the more complex search that requires more oracles. Entanglement state is generated and used; by contrast the state in Grover's algorithm is not an entanglement state. The quantum VQ iteration is a rotation over subspace; by contrast the Grover iteration is a rotation over global space. Rotating over subspace reduces the search range and speeds up the process of algorithm. The method of $G_{\rm VQ}$ is universal. The feasibility of realizing of the quantum VQ iteration depends on the realization of the Grover iteration and the oracle $O_{\rm d}$.

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