Vol 15 No 3, March 2006
1009-1963/2006/15(03)/0507-06

Chinese Physics

© 2006 Chin. Phys. Soc.
and IOP Publishing Ltd

Robust synchronization of uncertain chaotic systems*

Li Fang(李 芳)[a)†], Hu Ai-Hua(胡爱花)[a)], and Xu Zhen-Yuan(徐振源)[b)]

[a)] *School of Communication and Control Engineering, Southern Yangtze University, Wuxi 214122, China*

[b)] *School of Science, Southern Yangtze University, Wuxi 214122, China*

(Received 30 June 2005; revised manuscript received 16 December 2005)

This paper investigates robust unified (lag, anticipated, and complete) synchronization of two coupled chaotic systems. By introducing the concepts of positive definite symmetrical matrix and Riccati inequality and the theory of robust stability, several criteria on robust synchronization are established. Extensive numerical simulations are also used to confirm the results.

Keywords: robust unified synchronization, complete synchronization, lag synchronization, anticipated synchronization

PACC: 0545

1. Introduction

Over the last decade, chaos synchronization has attracted much attention[1−10] because of its role in understanding the basic features of manmade and natural systems and its potential application in secure communication, telecommunications and others. For example, experiments on the electric activity of information transmission in brain[11,12] have shown a pattern which is a special kind of synchronization. Many experiments on mammals[13] have indicated complex synchronism between two firing neurons. Various kinds of generalized synchronization have also been observed[14−16] in many other numerical and laboratory experiments. Both the evidences of chaotic motion in the brain and the importance of synchronization in perceptive process of mammals indicate a possible role of chaos in neural ensembles and its potential application in communication. Many different synchronization states have been studied in the past ten years, namely, complete or identical synchronization (CS or IS), phase synchronization (PS), lag synchronization (LS), anticipated synchronization (AS), generalized synchronization (GS), etc. Correspondingly, a great number of methods have also been studied for all kinds of synchronization such as feedback control, adaptive control, impulsive control, etc. Meanwhile, a unified definition of synchronization, including IS, AS, LS and GS, has been presented,[17] which develops a framework to investigate different kinds of synchronization.

On the other hand, uncertainties happen frequently in various engineering, biological, and economical systems, owing to model errors, measurement inaccuracy, linear approximation, external noise, and so on. It is well known that uncertainties often result in instability. Therefore, robustness analysis of uncertain systems is very important. Several interesting results have been established in Refs.[18–20] for continuous dynamical systems. But so far there are few results for synchronization of coupled uncertain chaotic systems. Generally speaking, systems are always hoped to be asymptotically stable under the action of a proper controller. However, it needs expenditure of great control energy which even may be of no use sometimes. In addition, it is neither possible nor necessary to make the system's actual state exactly equal to the perfect state. In practical problems, errors between reality and ideal are permitted if they are within given limitations. Owing to these reasons, we bring forward the concept of robust synchronization.

In this paper, based on the robust control theory, we investigate the unified synchronization (including

*Project supported by the National Natural Science Foundation of China (Grant No 10372054).

†Corresponding author. E-mail: yianlee2005@yahoo.com.cn

lag, anticipated, and identical synchronization) and the robust unified synchronization. We discuss different cases of uncertainty and present a systematic and powerful control scheme. The organization of this paper is as follows. In Section 2, we introduce the concepts of unified synchronization and robust unified synchronization. For simplicity and clarity, in Section 3 we state and prove our main results in the cases with linear uncertainty and nonlinear uncertainty, respectively. Finally, we work through some examples to illustrate the effectiveness and applicability of our results.

2. Definition of unified synchronization and robust unified synchronization

Consider the following two chaotic systems:

$$\begin{cases} \dot{x} = Ax + f(x(t)), \\ \dot{y} = Ay + f(y(t)) + U(x, y, t), \end{cases} \tag{1}$$

where $x, y \in R^n$ are the state vectors of the drive system and response system, respectively, $A \in R^{n\times n}$ is a constant matrix, $f : R^n \to R^n$ is a known nonlinear continuous vector function satisfying

$$f(x(t_1)) - f(y(t_2)) = M_{x(t_1),y(t_2)}(x(t_1) - y(t_2)), \tag{2}$$

and $U(x, y, t)$ is the controller to be designed later.

Remark 1 Many known chaotic or hyperchaotic systems satisfy (1) and (2). For example, Chua's circuit, the Lorenz system and the Rossler system all belong to the class defined by (1) and (2).

Definition 1 For the two systems described by (1), we say they have unified synchronization if $\lim\limits_{t+\infty} ||y(t) - x(t - \tau)|| = 0$, where τ is a given time lag. Specially, when $\tau = 0$, the two systems achieve CS. The case of $\tau > 0$ corresponds to LS, and $\tau < 0$ corresponds to AS.

Consider systems (1) with uncertainty, described as follows:

$$\begin{cases} \dot{x} = Ax + f(x(t)) + F(x(t), t), \\ \dot{y} = Ay + f(y(t)) + G(y(t), t) + U(x, y, t), \end{cases} \tag{3}$$

where $F(x(t), t)$ and $G(y(t), t)$ represent structural uncertainty or uncertain perturbation.

Definition 2 For the two systems described by (3), we say they possess the property of robust unified synchronization with tolerance σ (a given small positive constant) if there exists a control function $U(x, y, t)$ such that $\lim\limits_{t\to+\infty} \|y(t) - x(t - \tau)\| \le \sigma$.

3. Robust synchronization

The present section consists of two parts. At first, we investigate the robust synchronization in the case of linear uncertain dynamical systems. The corresponding results for nonlinear uncertain dynamical systems are given in the second part. All criteria in these two parts are suitable for $\tau \in R$, no matter whether $\tau > 0$,$\tau < 0$ or $\tau = 0$.

3.1. Robust synchronization with linear uncertain terms

Study robust synchronization of the following two coupled chaotic systems satisfying (2) with linear uncertain terms

$$\begin{cases} \dot{x} = Ax + f(x(t)) + A_1(t)x, \\ \dot{y} = Ay + f(y(t)) + A_2(t)y + U(x, y, t), \end{cases} \tag{4}$$

where both $A_1(t) = (a_1(t)_{ij})_{n\times n}$ and $A_2(t) = (a_2(t)_{ij})_{n\times n}$ are uncertain but bounded matrices satisfying $A_1(t), A_2(t) \in N[P, Q]$, i.e., $p_{ij} \le a_1(t)_{ij}, a_2(t)_{ij} \le q_{ij}$, where $P = (p_{ij})_{n\times n}$, $Q = (q_{ij})_{n\times n}$ are known constant matrices.

In order to obtain the robust synchronization controller of systems (4), we first introduce some lemmas.

Lemma 1 Let $A_1(t), A_2(t) \in N[P, Q]$, where $P = (p_{ij})_{n\times n}$, $Q = (q_{ij})_{n\times n}$ are known constant matrices. Then $A_1(t)$, $A_2(t)$ can be written as

$$A_1(t) = A_0 + R\Sigma_1(t)S, \tag{5}$$

$$A_2(t) = A_0 + R\Sigma_2(t)S, \tag{6}$$

where $A_0 = \dfrac{1}{2}(P + Q)$, $L = (l_{ij})_{n\times n} = \dfrac{1}{2}(Q - P)$,

$$\Sigma_1(t) = \mathrm{diag}(\delta_1(t)_{11}, \cdots, \delta_1(t)_{1n}, \cdots, \delta_1(t)_{n1}, \cdots, \delta_1(t)_{nn}) \in R^{n^2\times n^2}, |\delta_1(t)_{ij}| < 1, i, j = 1, 2, \cdots, n,$$
$$\Sigma_2(t) = \mathrm{diag}(\delta_2(t)_{11}, \cdots, \delta_2(t)_{1n}, \cdots, \delta_2(t)_{n1}, \cdots, \delta_2(t)_{nn}) \in R^{n^2\times n^2}, |\delta_2(t)_{ij}| < 1, i, j = 1, 2, \cdots, n,$$
$$R = (\sqrt{l_{11}}\varepsilon_1, \cdots, \sqrt{l_{1n}}\varepsilon_1, \cdots, \sqrt{l_{n1}}\varepsilon_n, \cdots, \sqrt{l_{nn}}\varepsilon_n) \in R^{n\times n^2},$$
$$S = (\sqrt{l_{11}}\varepsilon_1, \cdots, \sqrt{l_{1n}}\varepsilon_n, \cdots, \sqrt{l_{n1}}\varepsilon_1, \cdots, \sqrt{l_{nn}}\varepsilon_n)^{\mathrm{T}} \in R^{n^2\times n},$$
$$\varepsilon_i = (0, \cdots, 0, 1, 0, \cdots, 0)^{\mathrm{T}} \in R^n, \quad i = 1, \cdots, n.$$

Lemma 2 Suppose $\Sigma(t) = \mathrm{diag}(\delta(t)_{11}, \cdots, \delta(t)_{1n}, \cdots, \delta(t)_{n1}, \cdots, \delta(t)_{nn}) \in R^{n^2 \times n^2}$, $|\delta(t)_{ij}| < 1, i, j = 1, \cdots, n$, thus for any positive number $\alpha > 0$ and for any vectors ξ, η with proper dimension, the following inequality holds:

$$\xi^{\mathrm{T}} \Sigma(t) \eta + \eta^{\mathrm{T}} \Sigma^{\mathrm{T}}(t) \xi \leq \alpha \xi^{\mathrm{T}} \xi + \frac{1}{\alpha} \eta^{\mathrm{T}} \eta. \tag{7}$$

By Lemma 1, (4) can be rewritten as

$$\begin{cases} \dot{x} = \bar{A}x + f(x(t)) + R\Sigma_1(t)Sx, \\ \dot{y} = \bar{A}y + f(y(t)) + R\Sigma_2(t)Sy + U(x, y, t), \end{cases} \tag{8}$$

where $\bar{A} \triangleq A + A_0$, and $R, S, \Sigma_1(t), \Sigma_2(t)$ are defined as in Lemma 1.

Let $e(t) = y(t) - x(t - \tau)$, then the error dynamical system is

$$\begin{aligned} \dot{e}(t) =& \bar{A}e(t) + f(y(t)) - f(x(t-\tau)) \\ & + U(x(t-\tau), y(t), t) + R\Sigma_2(t)Sy(t) \\ & - R\Sigma_1(t-\tau)Sx(t-\tau) \\ =& (\bar{A} + M_{y(t),x(t-\tau)} + R\Sigma_2(t)S)e(t) \\ & + U(x(t-\tau), y(t), t) + R(\Sigma_2(t) \\ & - \Sigma_1(t-\tau))Sx(t-\tau). \end{aligned} \tag{9}$$

Theorem 1 Assume there exist two positive numbers ρ and α, a positive definite symmetrical matrix $P \in R^{n \times n}$ and $\Sigma_2(t) = \Sigma_1(t - \tau)$. Thus $e = 0$ is a uniformly asymptotically stable equilibrium point of the error dynamical system (9) if the control function is chosen as $U(x(t-\tau), y(t), t) = Ke(t)$ such that for all possible $y(t)$ and $x(t-\tau)$ the following inequality always holds:

$$\begin{aligned} & P\bar{A} + \bar{A}^{\mathrm{T}}P + PK + K^{\mathrm{T}}P + PM_{y(t),x(t-\tau)} \\ & + M^{\mathrm{T}}_{y(t),x(t-\tau)}P + \alpha PRR^{\mathrm{T}}P + \frac{1}{\alpha}S^{\mathrm{T}}S \leq -\rho P. \end{aligned} \tag{10}$$

Proof Construct a Lyapunov function as $V = e(t)^{\mathrm{T}} P e(t)$.

According to assumption $\Sigma_2(t) = \Sigma_1(t - \tau)$, Eq.(9) can be rewritten as

$$\dot{e}(t) = (\bar{A} + M_{y(t),x(t-\tau)} + R\Sigma_2(t)S + K)e(t).$$

Based on lemma 2, the derivative of V with respect to time is

$$\begin{aligned} \dot{V} =& \dot{e}(t)^{\mathrm{T}}Pe(t) + e(t)^{\mathrm{T}}P\dot{e}(t) \\ =& e(t)^{\mathrm{T}}(\bar{A}^{\mathrm{T}}P + K^{\mathrm{T}}P + M^{\mathrm{T}}_{y(t),x(t-\tau)}P \\ & + S^{\mathrm{T}}\Sigma_2^{\mathrm{T}}(t)R^{\mathrm{T}}P + P\bar{A} + PK + PM_{y(t),x(t-\tau)} \\ & + PR\Sigma_2(t)S)e(t) \\ \leq& e(t)^{\mathrm{T}}(P\bar{A} + \bar{A}^{\mathrm{T}}P + PK + K^{\mathrm{T}}P \\ & + M^{\mathrm{T}}_{y(t),x(t-\tau)}P + PM_{y(t),x(t-\tau)} \\ & + \alpha PRR^{\mathrm{T}}P + \frac{1}{\alpha}S^{\mathrm{T}}S)e(t). \end{aligned}$$

Therefore, by (10) we have $\dot{V} \leq -\rho e(t)^{\mathrm{T}} P e(t)$. Hence, the theorem is true, that is to say, given a τ, two systems of Eq.(4) achieve LS, AS or CS with $\tau > 0$,$\tau < 0$ or $\tau = 0$, respectively.

Remark 2 There are two cases satisfying assumption $\Sigma_2(t) = \Sigma_1(t - \tau)$. The first is that both Σ_1 and Σ_2 are identical constant matrices. In this case the parameter perturbation is time invariant and identical. We can regard them as two identical semi-determined systems while we only know part of the information of their parameters. Another case is that Σ_1 and Σ_2 are time-varying and different, but they have a certain relation with time lag τ. In this case, we may look upon system x and system y as systems with noise, which disturbs systems' parameters. However, if $\Sigma_2(t) = \Sigma_1(t - \tau)$ is satisfied and other conditions in theorem 1 hold, the two systems still can achieve synchronization.

Next we consider the case that $\Sigma_2(t) \neq \Sigma_1(t - \tau)$, and first introduce the following lemma.

Lemma 3[21] Consider the following uncertain system:

$$\begin{cases} \dot{x} = (A + \Delta A(t))x + Bw(t) + u(t), \\ z(t) = Cx(t), \end{cases} \tag{11}$$

where $x \in R^n$ is the state of the system, $z \in R^m$ is the output of the system, A, B and C are constant matrices of proper dimension, $\Delta A(t) \in R^{n \times n}$ is an uncertain perturbation matrix, $w(t) \in R^n$ is the external noise vector and $u(t)$ is the controller. Given a scalar quantity $\gamma > 0$, the system is said to be robust quadratic stable with degree of disturbed decay γ, i.e., $\|z\|_2 \leq \gamma \|w\|_2$, if there exists a positive definite symmetrical matrix P such that for all possible $\Delta A(t)$, the following inequality holds:

$$\begin{aligned} & (A + \Delta A(t) + K)^{\mathrm{T}}P + P(A + \Delta A(t) + K) \\ & + \gamma^{-2}PB^{\mathrm{T}}BP + C^{\mathrm{T}}C \leq 0, \end{aligned} \tag{12}$$

where the feedback controller is designed as $u = Kx$, which is named as H_∞ robust controller.

Theorem 2 Given a constant $\gamma > 0$, let $U(x(t-\tau), y(t), t) = Ke(t)$. Thus the two systems of (4) achieve robust unified synchronization with tolerance σ, if there exists a positive definite symmetrical matrix P such that for all possible $\Sigma_2(t)$, the following inequality is satisfied:

$$\begin{aligned}&(\bar{A} + M_{y(t),x(t-\tau)} + R\Sigma_2(t)S + K)^{\mathrm{T}}P\\&+ P(\bar{A} + M_{y(t),x(t-\tau)}\\&+ R\Sigma_2(t)S + K) + \gamma^{-2}P^2 \le -I. \end{aligned} \tag{13}$$

Proof Consider $R(\Sigma_2(t) - \Sigma_1(t-\tau))Sx(t-\tau)$ as the external noise $w(t)$ of system (9), choose $B = I$ and $C = I$, i.e., the error $e(t)$ is the output vector. Thus, according to lemma 3, we have $\lim\limits_{t\to+\infty}\|e(t)\| \le \gamma \lim\limits_{t\to+\infty}\|R(\Sigma_2(t) - \Sigma_1(t-\tau))Sx(t-\tau)\|$. For any $\Sigma_2(t) - \Sigma_1(t-\tau)$, there always exists a small positive number γ such that $\gamma \lim\limits_{t\to+\infty}||R(\Sigma_2(t) - \Sigma_1(t-\tau))SX(t-\tau)|| \le \sigma$. So $\lim\limits_{t\to+\infty}\|e(t)\| \le \sigma$, and the two systems of (4) achieve robust unified synchronization.

3.2. Robust synchronization with nonlinear uncertain terms

Now we study robust synchronization of chaotic systems satisfying (2) with nonlinear uncertain terms

$$\begin{cases} \dot{x} = Ax + f(x(t)) + g_1(x(t)),\\ \dot{y} = Ay + f(y(t)) + g_2(y(t)) + U(x, y, t). \end{cases} \tag{14}$$

The functions $g_1, g_2 : R^n \to R^n$ represent structural uncertainty or uncertain perturbation characterized by

$$g_1(x) = \mu_{g_1}(x)\delta_{g_1}(x),\ \|\delta_{g_1}(x)\| \le \|m_{g_1}(x)\|, \tag{15}$$

$$g_2(y) = \mu_{g_2}(y)\delta_{g_2}(y),\ \|\delta_{g_2}(y)\| \le \|m_{g_2}(y)\|, \tag{16}$$

where $\mu_{g_1}, \mu_{g_2} \in R^{n\times m}$ are known matrix functions whose entries are smooth functions of the state, and $\delta_{g_1}(x), \delta_{g_2}(y)$ are unknown vector valued functions whose norm are bounded, respectively, by the norm of the vector valued functions $m_{g_1}(x), m_{g_2}(y)$. Here, $m_{g_1}, m_{g_2} : R^n \to R^m$ are given smooth functions.

Let $e(t) = y(t) - x(t-\tau)$, thus

$$\begin{aligned}\dot{e}(t) =& Ae(t) + f(y(t)) - f(x(t-\tau))\\&+ U(x(t-\tau), y(t), t) + g_2(y(t)) - g_1(x(t-\tau))\\=&(A + M_{y(t),x(t-\tau)})e(t) + U(x(t-\tau), y(t), t)\\&+ g_2(y(t)) - g_1(x(t-\tau)). \end{aligned} \tag{17}$$

Theorem 3 Suppose that $\mu \triangleq \mu_{g_1}(x(t-\tau)) = \mu_{g_2}(y(t))$. Given a constant $\alpha > 0$ and the controller is chosen as $U(x(t-\tau), y(t), t) = Ke(t)$. Assume there is a Lyapunov function $V(e(t), t)$ and a vector valued function $m : R^n \to R^m$ such that $\|m\| \ge 2\max(\|m_{g_1}\|, \|m_{g_2}\|)$. Thus the two systems of (14) achieve unified synchronization if the following Riccati inequality holds:

$$\begin{aligned}&\frac{\partial V}{\partial t} + \frac{\partial V}{\partial e}(A + M_{y(t),x(t-\tau)} + K)e(t)\\&+ \frac{\alpha^2}{2}\frac{\partial V}{\partial e}\mu\mu^{\mathrm{T}}\frac{\partial V^{\mathrm{T}}}{\partial e} + \frac{1}{2\alpha^2}m^{\mathrm{T}}m \le 0. \end{aligned} \tag{18}$$

Proof Let $V = V(e(t), t) \ge 0$, $\delta \triangleq \delta_{g_2}(y(t)) - \delta_{g_1}(x(t-\tau))$, according to theorem 3, the derivative of V with respect to time is

$$\begin{aligned}\dot{V} =&\frac{\partial V}{\partial t} + \frac{\partial V}{\partial e}[(A + M_{y(t),x(t-\tau)} + K)e(t)\\&+ g_2(y(t)) - g_1(x(t-\tau))]\\=&\frac{\partial V}{\partial t} + \frac{\partial V}{\partial e}(A + M_{y(t),x(t-\tau)} + K)e(t) + \frac{\partial V}{\partial e}\mu\delta\\=&\frac{\partial V}{\partial t} + \frac{\partial V}{\partial e}(A + M_{y(t),x(t-\tau)} + K)e(t)\\&+ \frac{\alpha^2}{2}\frac{\partial V}{\partial e}\mu\mu^{\mathrm{T}}\frac{\partial V^{\mathrm{T}}}{\partial e} + \frac{1}{2\alpha^2}m^{\mathrm{T}}m\\&- \frac{1}{2}\left(\alpha\frac{\partial V}{\partial e}\mu - \frac{1}{\alpha}\delta^{\mathrm{T}}\right)\left(\alpha\mu^{\mathrm{T}}\frac{\partial V^{\mathrm{T}}}{\partial e} - \frac{1}{\alpha}\delta\right)\\&- \frac{1}{2\alpha^2}(m^{\mathrm{T}}m - \delta^{\mathrm{T}}\delta), \end{aligned}$$

because

$$\begin{aligned}\|\delta\| =&\|\delta_{g_2}(y(t)) - \delta_{g_1}(x(t-\tau))\|\\\le&\|\delta_{g_1}(x(t-\tau))\| + \|\delta_{g_2}(y(t))\|\\\le&\|m_{g_1}(x(t-\tau))\| + \|m_{g_2}(y(t))\|\\\le&2\max(\|m_{g_1}(x(t-\tau))\|, \|m_{g_2}(y(t))\|)\\\le&\|m\|, \end{aligned}$$

so we have $m^{\mathrm{T}}m \ge \delta^{\mathrm{T}}\delta$, and

$$\begin{aligned}\dot{V} \le&\frac{\partial V}{\partial t} + \frac{\partial V}{\partial e}(A + M_{y(t),x(t-\tau)} + K)e(t)\\&+ \frac{\alpha^2}{2}\frac{\partial V}{\partial e}\mu\mu^{\mathrm{T}}\frac{\partial V^{\mathrm{T}}}{\partial e} + \frac{1}{2\alpha^2}m^{\mathrm{T}}m. \end{aligned}$$

so if (18) holds, $\dot{V} \le 0$ and the two systems of (14) achieve unified synchronization.

Theorem 4 Given a constant $\gamma > 0$, let $U(x(t-\tau), y(t), t) = Ke(t)$. Thus the two systems of (14) achieve robust unified synchronization with tolerance σ, if there exists a positive definite symmetric matrix

P such that for all possible $g_1(x(t-\tau)), g_2(y(t))$, the following inequality holds:

$$(A + M_{y(t),x(t-\tau)} + K)^{\mathrm{T}}P + P(A + M_{y(t),x(t-\tau)} + K) + \gamma^{-2}P^2 \leq -I. \quad (19)$$

Proof Consider $g_s(y(t)) - g_1(x(t-\tau))$ as the external noise $w(t)$ of Eq.(17), let $B = I$ and $e(t)$ is the output vector, i.e., $C = I$. According to lemma 3, we have $\lim\limits_{t\to+\infty} \|e(t)\| \leq \gamma \lim\limits_{t\to+\infty} \|g_2(y(t)) - g_1(x(t-\tau))\|$. For any functions g_1 and g_2 satisfying Eqs.(15) and (16), if only $g_2(y(t)) - g_1(x(t-\tau))$ is bounded, there always exists a small positive number γ such that $\gamma \lim\limits_{t\to+\infty} \|g_2(y(t)) - g_1(x(t-\tau))\| \leq \sigma$. So $\lim\limits_{t\to+\infty} \|e(t)\| \leq \sigma$, and the two systems of (14) achieve robust unified synchronization.

4. Simulations

To illustrate the scheme developed in Section 3, the famous Lorenz system is used.

$$\begin{cases} \dot{x}_1 = a_1(x_2 - x_1), \\ \dot{x}_2 = c_1 x_1 - x_2 - x_1 x_3, \\ \dot{x}_3 = -b_1 x_3 + x_1 x_2, \end{cases} \quad (20)$$

where $x = (x_1, x_2, x_3)^{\mathrm{T}}$, the parameter set is chosen as $a_1 = 10$, $b_1 = 8/3$, $c_1 = 28$, such that the system shows chaotic behaviour.

System (20) can be rewritten as

$$\dot{x} = Ax + f(x), \quad (21)$$

where

$$A = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix}, \quad f(x) = \begin{pmatrix} 0 \\ -x_1 x_3 \\ x_1 x_2 \end{pmatrix}. \quad (22)$$

Suppose that the manifold is $y(t) = x(t-\tau)$, $\tau \in R$. We take the drive system and response system in the form of (3) with A and $f(\cdot)$ as defined in (22).

Define the error state $e(t) = (e_1(t), e_2(t), e_3(t))^{\mathrm{T}} = (y_1(t) - x_1(t-\tau), y_2(t) - x_2(t-\tau), y_3(t) - x_3(t-\tau))^{\mathrm{T}}$ and let $\|e(t)\| = \sqrt{e_1^2(t) + e_2^2(t) + e_3^2(t)}$. Suppose the controller to be $U(x, y, t) = (k_1 e_1(t), k_2 e_2(t), k_3 e_3(t))^{\mathrm{T}}$.

Case 1 The uncertainty is a linear term and $A_1(t) = A_2(t)$. In this case, we set $A_1(t) = A_2(t) = \mathrm{diag}\{\sin t, \sin t, \cos t\}$, and we know $A_1(t) = A_2(t - 2k\pi)$ (k is an arbitrary integer). According to theorem 1, we have $y(t) = x(t-\tau)$ only when $\tau = 2k\pi$, i.e., the drive and the response achieve LS (when k is an arbitrary positive integer), CS (when $k = 0$) or AS (when k is an arbitrary negative integer). Without loss of generality, Figure 1 shows complete synchronization when $\tau = 0$ and $k_1 = k_2 = k_3 = -3$.

Case 2 The uncertainty is a linear term and $A_1(t) \neq A_2(t-\tau)$. According to theorem 2, the drive and the response can only achieve robust lag, complete or anticipated synchronization with a given degree of disturbed decay. Here, we set $A_1(t) = \begin{pmatrix} -0.1 & 0.1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $A_1(t) = \begin{pmatrix} 0.1 & -0.1 & 0 \\ -0.1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ Figure 2 shows the robust synchronization when $\tau = 0$ and $k_1 = k_2 = k_3 = -5$.

Case 3 The uncertainty is a nonlinear term. For convenience, we set $g_1(x) = (0, 0.01x_1x_2, 0.01x_1x_3)^{\mathrm{T}}$, $g_2(x) = (0, -0.01x_1x_3, -0.01x_1x_2)^{\mathrm{T}}$. Figure 3 shows robust synchronization when $\tau = 0$ and $k_1 = k_2 = k_3 = -10$.

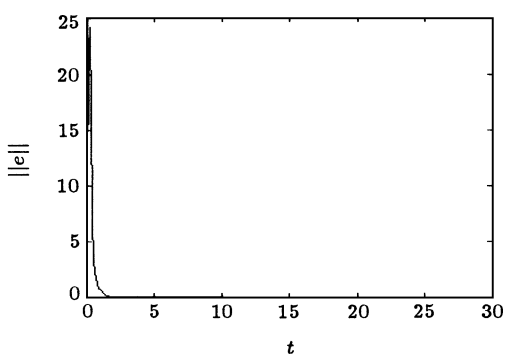


Fig.1. Synchronization error with linear uncertainty and $A_1(t) = A_2(t)$.

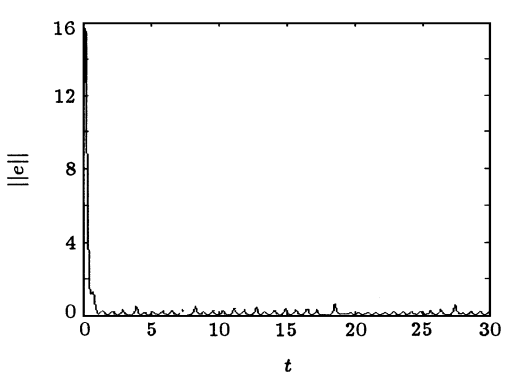


Fig.2. Synchronization error with linear uncertainty and $A_1(t) \neq A_2(t-\tau)$.

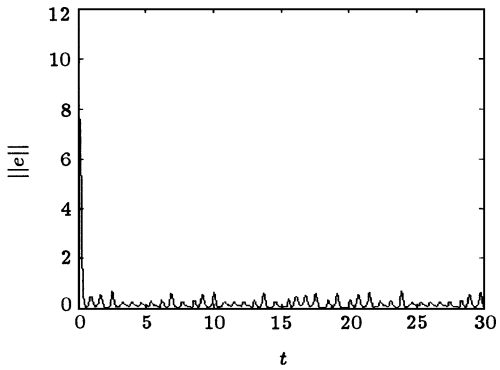


Fig.3. Synchronization error with nonlinear uncertainty.

Remark 4 As a matter of fact, the criteria given in this paper are suitable for $\tau \in R$. However, owing to the limitation of space and without loss of generality, all the simulation results displayed here are in the case of $\tau = 0$.

5. Conclusions

In this paper, based on the robust control theory, we investigate a type of unified (lag, anticipated, and identical) synchronization and robust unified synchronization. We discuss different cases of uncertainty and a systematic and powerful control scheme is presented.

References

[1] Feng X Q and Shen K 2005 *Chin. Phys.* **14** 1526
[2] Lu J G and Xi Y G 2005 *Chin. Phys.* **14** 274
[3] Ricàrdo F, Ramón J O and Gualberto S P 2001 *IEEE Trans. Circuit Syst.* I **48** 1161
[4] Li Zh and Chen G R 2004 *Phys. Lett.* A **324** 166
[5] Li Zh and Chen G R 2004 *Phys. Lett.* A **326** 333
[6] Tao C H, Lu J A and Lu J H 2002 *Acta Phys. Sin.* **51** 1497 (in Chinese)
[7] Guan X P, He Y H and Wu J 2003 *Acta Phys. Sin.* **52** 2718 (in Chinese)
[8] Li G H, Xu D M and Zhou S P 2004 *Acta Phys. Sin.* **53** 379 (in Chinese)
[9] Lu J G 2005 *Chin. Phys.* **14** 67
[10] Wei R and Wang X Y 2004 *Chin. Phys.* **13** 1210
[11] Brussler S L, Coppola R and Nakamura R 1991 *Nature* **366** 153
[12] Schack B, Vath N and Petsche H 2002 *Int. J. Psychophysiology* **44** 143
[13] Steinmetz P N, Roy A and Fitzgerald P J 2000 *Nature* **404** 187
[14] Patel A D and Balaban E 2000 *Nature* **404** 80
[15] Glass L 2001 *Nature* **410** 277
[16] Fires P, Reynold I H and Rorie A E 2001 *Science* **291** 1560
[17] Yan Z Y 2005 Chaos **15** 013101
[18] Kaining W, Michel A N and Derong L 1994 *IEEE Trans. Automat. Control* **39** 1251
[19] Isidori A 1994 *Int. J. Robust Nonlinear Control* **4** 553
[20] Lin W and Shen T 1999 *Automatica* **35** 35
[21] Yang Y S 2004 *Robust Control of Uncertain System and its Application* (Beijing: Science Press)