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Gaussian wave formalism model for propagation of charged-particle beam through a first-order optical system*

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An elliptical Gaussian wave formalism model of a charged-particle beam is proposed by analogy with an elliptical Gaussian light beam. In the paraxial approximation, the charged-particle beam can be described as a whole by a complex radius of curvature in the real space domains. Therefore, the propagation and transform of charged-particle beam passing through a first-order optical system is represented by the *ABCD*-like law. As an example of the application of this model, the relation between the beam waist and the minimum beam spot at a fixed target is discussed. The result, well matches that from conventional phase space model, and proves that the Gaussian wave formalism model is highly effective and reasonable.

Keywords: complex radius of curvature, transfer matrix, *ABCD*-like law, charged-particle beam
PACC: 0365C, 9265K, 4180

1. Introduction

Charged-particle beam is assumed to have a statistical non-wave property in classical theory, but the charged-particle beam transport through a generic optical device has been well described by the quantum-like approach and the so-called thermal wave model in the past few years,[1−9] which suggests that the wave-like approach may be a powerful tool in the study of the propagation and transformation characteristics of charged-particle beam, and the importance of using the quantum formalism to describe classical systems has been recognized in beam physics. One of the major issues of this trend in frontier beam physics, as well as the future challenge, is whether or not the high-energy charged-particle beam can be converged and if so, under what condition it happens. To this end, comparing it with a laser beam should be helpful. Motivated by looking for a possible coherent charged-particle beam analogous to a laser beam, recently we have studied a paraxial and dilute charged-particle beam passing through a first-order optical system by using the similarity (in terms of formalism and common methodologies) between the charged-particle motion and the behaviour of light rays in laser optics.[10] This investigation can show us how to apply the knowledge in laser optics branch, such as algorithms and many solutions of laser optics, to charged-particle beam optics, and how the paraxial approximation where slopes of the electronic rays are considered to be very small with respect to the propagation coordinate can be equivalent to the so-called slowly-varying amplitude approximation that is widely used in laser optics. This procedure allows us to transform the classical statistical non-wave formulation of the charged-particle beam to another formulation in terms of the Gaussian wave-like, and to obtain the *ABCD*-like law for a complex radius of curvature in real space domains. In this paper, we deliberate an elliptical Gaussian wave formalism model with the wave-like beam propagation parameters of charged-particle beam to investigate the propagation and transformation behaviour of charged-particle beam passing through a first-order optical system. In this context, charged-particle beam can be described as a whole by the complex curvature radius in real space domains, and the electronic ray envelope transformation equation in phase space domains, which takes into account the

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spreading (diffusion) among the electronic rays and is widely used in charged-particle beam linear transport, can reduce to the *ABCD*-like law in real space domains. In particular, this wave-like model allows one to interpret the charged-particle beam spot spreading to be due to the finite emittance that is responsible for the electronic ray's diffusion from a diffraction-like behaviour point of view, in which the beam emittance plays a role of formal wavelength. Finally, the practical application of this wave-like model is deliberated with a thin lens system. The result is in agreement with that of conventional phase space model. This opens the possibility of studying transverse beam optics in a novel and promising approach. For simplicity, we specialize our analysis hereafter in a dilute and paraxial charged-particle beam with separable variables x and y only. However, the results can be readily applied to the axially symmetric charged-particle beam propagation.

2. Wave-like model for the charged-particle beam

Although the charged-particle beam propagation actually is a 3-D problem (two transversal dimensions x, y, and one axial dimension z), its treatment can begin with the analysis of a 2-D charged-particle beam (one transversal dimension x, and one axial dimension z). In a monoenergetic charged-particle beam without time structure, only the transversal properties are of importance. By analogy with the elliptical Gaussian light beam description given in laser optics, we introduce the functions $\omega(z)$, $R(z)$ and $\Theta(z)$ to evaluate the transversal characteristics of the charged-particle beam in the framework of the Gaussian wave formalism model.

2.1. Beam width $\omega(z)$

The electronic rays are affected by diffusion, as a consequence, they will spread out while the beam propagates. This tendency of "dispersion" among the electronic rays is in competition with the effect of focusing that forces them to be "ordered". Consequently, even if the electronic rays are straight in vacuum, they can be mixed up due to the diffusion, producing the beam envelope. This transversal extent can be characterized by the width parameter. In other words, the beam width is the beam envelope width. The beam width's evolution along the direction of propagation z in free space is hyperbolic and given by[11,12]

$$\frac{\omega^2(z)}{\omega_0^2} - \frac{(z-z_0)^2}{\lambda^2} = 1, \tag{1}$$

where $\lambda = \omega_0^2/\varepsilon$ and ε represent the characteristic length and the emittance of the charged-particle beam, respectively. It is evident that $\omega(x)$ reaches a minimum at $z = z_0$, which is the minimum value of ω_0. This parameter governs the evolution of the beam, and, is referred to as the beam waist width. In each perpendicular plane, the z beam shows a Gaussian profile. The width reaches the minimum at the waist and then the beam expands.

2.2. Radius of curvature of wave front $R(z)$

This parameter characterize the convergence and divergence of a charged-particle beam. In the paraxial approximation, its dependence on z is as follows:

$$\begin{aligned} R(z) =& \frac{\omega(z)}{\mathrm{d}\omega(z)/\mathrm{d}z} = (z-z_0)\left[1+\left(\frac{\lambda}{z-z_0}\right)^2\right] \\ =& \lambda\left(\frac{z-z_0}{\lambda} + \frac{\lambda}{z-z_0}\right). \end{aligned} \tag{2}$$

It is the paraxial approach to a spherical wavefront with a radius $R(z)$. Therefore this function is known as the radius of curvature of the wavefront for the charged-particle beam. When z approaches infinity, it shows a linear variation with z, which is typical of a spherical wavefront taking its source at $z = z_0$, i.e. coming from a point source. However, the radius of curvature is infinity at the beam waist position. This means that at the beam waist, the wavefront is plane. The beam reaches a minimum of the absolute value of the radius at a distance of $+\lambda$ and $-\lambda$ from the beam waist. The parameter λ in Eqs.(1) and (2) is related to the behaviour of the beam along the propagation distance. It is can be said that the beam waist dimension along z is λ. The width at $z - z_0 = \lambda$ is $\sqrt{2}$ larger than that in the waist. The radius of curvature shows its minimum value at $z - z_0 = \lambda$. From Eq.(1), we notice that the axial size of the waist becomes bigger as the width turns large (with quadratic form).

2.3. Beam divergence angle $\Theta(z)$

Equation (1) has a very interesting behaviour when z tends to ∞ (or $-\infty$). This width dependence shows an oblique asymptote with a slope of

$$\Theta_0 \cong \tan\Theta_0 = \frac{\varepsilon}{\omega_0}, \tag{3}$$

where the paraxial approximation is used. This parameter is named divergence angle of the charged-particle beam. It describes the spreading of the beam when propagating towards infinity. From this it is clear that the divergence angle and the width are in reciprocal relationship, which means that the larger values of the width corresponds to lower values of the divergence angle, and vice versa. Using this relation, we can conclude that a good collimation (very low value of the divergence angle) will be obtained when the beam is wide. On the contrary, a highly focused beam can be obtained by allowing a large divergence angle. In particular, Eq.(3) shows explicitly that the beam divergence angle is the same as the diffraction-like angle of beam formal wave with width ω_0 in value, in which the beam emittance ε plays a role of formal wavelength. A very useful relationship can be obtained when Eq.(1) is combined with Eq.(2):

$$\Theta(z) = \sqrt{\left[\frac{\omega(z)}{R(z)}\right]^2 + \left[\frac{\varepsilon}{\omega(z)}\right]^2}. \tag{4}$$

In fact, Eq.(3) holds true for a beam passing through free space, while Eq.(4) for a beam propagating in an arbitrary linear optical system. Obviously, when a beam is waisted at z, i.e. $R(z) \to \infty$, Eq.(4) reduces to Eq.(3). Combining radius of curvature of wave front and the relation between the width and the divergence angle, we can find that as the collimation becomes better, the region of collimation turns even larger because the axial extension of the beam waist is longer.

The above results allow us to improve the conventional way that the dilute and paraxial charged-particle beam, regarded as a whole, is thought of as a single elliptical Gaussian wave-like light beam with diffraction-like spreading due to the emittance responsible for the electronic rays diffusion. From this wave-like description technique, it is possible to define a radius of curvature describing both the curvature of the wavefront and the transversal size of the charged-particle beam. This radius of curvature is complex and given by[13]

$$\frac{1}{P(z)} = \frac{1}{R(z)} - \mathrm{i}\frac{\varepsilon}{\omega(z)^2}. \tag{5}$$

Here the complex radius of curvature fully depicts the behaviour of charged-particle beam. Once the complex radius of curvature is calculated along the optical system, this, together with Eq.(4), allows us to obtain the beam width and the beam divergence angle by appropriately taking the real and imaginary parts of the complex radius of curvature.

3. *ABCD*-like law in real space domains

The traditional treatment of charged-particle beam transformed by linear optical device in the paraxial approximation is performed by Liouville theorem such as the phase space ellipse transformation equation,[11,12] i.e. σ-matrix transformation equation, given by

$$\sigma_{\mathrm{o}} = M\sigma_{\mathrm{i}}M^{\mathrm{T}}, \tag{6}$$

where $\sigma_{\mathrm{i}} = \begin{bmatrix} \sigma_{11\mathrm{i}} & \sigma_{12\mathrm{i}} \\ \sigma_{12\mathrm{i}} & \sigma_{22\mathrm{i}} \end{bmatrix}$ and $\sigma_{\mathrm{o}} = \begin{bmatrix} \sigma_{11\mathrm{o}} & \sigma_{12\mathrm{o}} \\ \sigma_{12\mathrm{o}} & \sigma_{22\mathrm{o}} \end{bmatrix}$ represent the charged-particle beam matrices at the entrance and at the exit of the first-order optical device, respectively. The 2×2 matrix $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ characterizes the transfer matrix of the optical device; the superscript T means transposition. The relations hips[10] between wave-like parameters and the matrix elements of σ-matrix can be presented as follows:

$$\omega = \sqrt{\sigma_{11}}, \quad R^{-1} = \sigma_{12}/\sigma_{11}, \quad \Theta = \sqrt{\sigma_{22}}. \tag{7}$$

With a little more work done, we can solve explicitly for the width, the divergence angle and the radius of curvature of charged-particle beam under consideration at the exit of the optical device associated with 2×2 transfer matrix M from Eqs.(6) and (7):

$$\omega_{\mathrm{o}}^2 = (A + BR_{\mathrm{i}}^{-1})^2\omega_{\mathrm{i}}^2 + B^2\varepsilon_{\mathrm{i}}^2\omega_{\mathrm{i}}^{-2}, \tag{8a}$$

$$\Theta_{\mathrm{o}}^2 = (C + DR_{\mathrm{i}}^{-1})^2\omega_{\mathrm{i}}^2 + D\varepsilon_{\mathrm{i}}^2\omega_{\mathrm{i}}^{-2}, \tag{8b}$$

$$R_{\mathrm{o}}^{-1}\omega_{\mathrm{o}}^2 = (A + BR_{\mathrm{i}}^{-1})(C + DR_{\mathrm{i}}^{-1})\omega_{\mathrm{i}}^2 + BD\varepsilon_{\mathrm{i}}^2\omega_{\mathrm{i}}^{-2}. \tag{8c}$$

Paying regard to Eq.(8), and multiplying P_{o}^{-1} with $\omega_{\mathrm{o}}^2\omega_{\mathrm{i}}^{-2}$, one can obtain

$$\begin{aligned} &P_{\mathrm{o}}^{-1}[(A + BR_{\mathrm{i}}^{-1})^2 + B^2\varepsilon_{\mathrm{i}}^2\omega_{\mathrm{o}}^{-4}] \\ =&(A + BR_{\mathrm{i}}^{-1})(C + DR_{\mathrm{i}}^{-1}) + BD\varepsilon_{\mathrm{i}}^2\omega_{\mathrm{i}}^{-4} - \mathrm{i}\varepsilon_{\mathrm{o}}\omega_{\mathrm{i}}^{-2} \end{aligned} \tag{9}$$

and

$$\begin{aligned} &(A + BP_{\mathrm{i}}^{*-1})(C + DP_{\mathrm{i}}^{-1}) \\ =&(A + BR_{\mathrm{i}}^{-1})(C + DR_{\mathrm{i}}^{-1}) \\ &+ BD\varepsilon_{\mathrm{i}}^2\omega_{\mathrm{i}}^{-4} - \mathrm{i}\varepsilon_{\mathrm{i}}\omega_{\mathrm{i}}^{-2}(AD - BC), \end{aligned} \tag{10}$$

where

$$P_{\mathrm{o}}^{-1} = R_{\mathrm{o}}^{-1} - \mathrm{i}\varepsilon_{\mathrm{o}}\omega_{\mathrm{o}}^{-2}, \quad P_{\mathrm{i}}^{-1} = R_{\mathrm{i}}^{-1} - \mathrm{i}\varepsilon_{\mathrm{i}}\omega_{\mathrm{i}}^{-2} \tag{11}$$

are the complex radii of curvature of charged-particle beam at the exit and at the entrance of the optical device, respectively, the superscript $*$ denotes the complex conjugation of corresponding function. Moreover, it is easy to prove that the complex radii of curvature $P_{\rm i}$ and $P_{\rm o}$ satisfy the following equation:

$$(A+BP_{\rm i}^{*-1})[P_{\rm o}^{-1}(A+BP_{\rm i}^{-1})-(C+DP_{\rm i}^{-1})] = {\rm i}\omega_{\rm i}^{-2}[(AD-BC)\varepsilon_{\rm i}-\varepsilon_{\rm o}]. \quad (12)$$

Considering the fact that the determination of transfer matrix is unity and the beam emittance is a propagation invariant,[11,12] i.e. $AD-BC=1$ and $\varepsilon_{\rm i}=\varepsilon_{\rm o}$, one can have

$$P_{\rm o}^{-1}=(C+DP_{\rm i}^{-1})(A+BP_{\rm i}^{-1})^{-1}. \quad (13)$$

Formula (13) may be called the *ABCD*-like law for a charged-particle beam passing through a first-order optical device. The above discussion shows that the conservative beam emittance is the necessary and sufficient condition of the *ABCD*-like law. This is an important conclusion of this paper.

For an optical system composed of N devices, the *ABCD*-like law is in the form

$$P_{\rm o}^{-1}=(\tilde{C}+\tilde{D}P_{\rm i}^{-1})(\tilde{A}+\tilde{B}P_{\rm i}^{-1})^{-1}, \quad (14)$$

which is the same as Eq.(13), given $M=\begin{bmatrix}\tilde{A} & \tilde{B}\\ \tilde{C} & \tilde{D}\end{bmatrix}=M_NM_{N-1}\cdots M_2M_1$ as the assembly matrix of the system.

In fact, a general review involving the *ABCD*-like law in classical mechanics and the *ABCD* law in optics can be found in the recent paper.[14] It is shown that the relation ship $AD-BC=1$ is satisfied in the case of the most universal and elementary forms of motions including those of classical mechanical particles and the light rays in geometrical optics. As a consequence, for a charged-particle beam, non-proper wave system, it is appropriate to say wave-like description instead of the proper wave one, because beam physics involved, which basically is a problem about a classical Hamiltonian system, can be described by formally replacing the wavelength with the beam emittance. In this context, *ABCD*-like law of charged-particle beam passing through a linear optical system holds true in general.

4. Application example

The application of wave-like model is to recover the relation between the charged-particle beam waist and the minimum beam spot size at a given target. We treat a simple thin lens system with a free drift distance L from a fixed target T, which is commonly used in practice, as shown Fig.1.

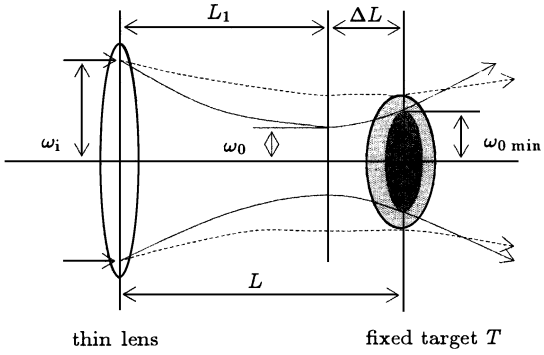


Fig.1. A thin lens system following a fixed target T.

For a given charged-particle beam, the beam waist is the minimum beam width, and the radius of curvature is infinity at the beam waist position. The possible minimum beam spot size at the fixed target T varies as the focusing strength of thin lens, and this minimum beam spot position generally is not the beam waist position. The incident beam with width $\omega_{\rm i}$ exhibits a waist at the fixed target T (dashed line) by varying the focusing strength of thin lens, and this beam waist is not the minimum beam spot. In order to obtain the minimum beam spot size $\omega_{0\,\rm min}$ at the fixed target T (solid line), it is necessary to adjust the focusing strength of this lens, meantime the beam waist with width ω_0 is formed in front of the target T. We consider an axially symmetric charged-particle beam with a complex radius of curvature at the lens plane given by

$$P_{\rm i}^{-1}=R_{\rm i}^{-1}-{\rm i}\varepsilon\omega_{\rm i}^{-2} \quad (15)$$

passing through the thin lens system. The transfer matrix of the thin lens system $\boldsymbol{M}$ reads as follows:

$$\boldsymbol{M}=\begin{bmatrix}1 & L\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -f^{-1} & 1\end{bmatrix}=\begin{bmatrix}1-Lf^{-1} & L\\ -f^{-1} & 1\end{bmatrix},$$

where f is the variable of the focal length of thin lens. According to Eq.(14) one has the complex radius of curvature of the beam at the target T

$$P_{\rm o}^{-1} = (-f^{-1} + P_{\rm i}^{-1})(1 - Lf^{-1} + LP_{\rm i}^{-1})^{-1}$$
$$= \frac{[(-f^{-1} + R_{\rm i}^{-1})(1 - Lf^{-1} + LR_{\rm i}^{-1}) + \varepsilon^2\omega_{\rm i}^{-4}L] - {\rm i}\varepsilon\omega_{\rm i}^{-2}}{(1 - Lf^{-1} + LR_{\rm i}^{-1})^2 + \varepsilon^2L^2\omega_{\rm i}^{-4}}.$$

And, consequently, the beam width and radius of curvature at the target T are

$$\omega_{\rm o}^2 = (1 - Lf^{-1} + LR_{\rm i}^{-1})^2\omega_{\rm i}^2 + \varepsilon^2L^2\omega_{\rm i}^{-2} \tag{16}$$

and

$$R_{\rm o}^{-1} = \frac{(-f^{-1} + R_{\rm i}^{-1})(1 - Lf^{-1} + LR_{\rm i}^{-1}) + \varepsilon^2\omega_{\rm i}^{-4}L}{(1 - Lf^{-1} + LR_{\rm i}^{-1})^2 + \varepsilon^2L^2\omega_{\rm i}^{-4}}, \tag{17}$$

respectively. By setting $\frac{{\rm d}\omega_{\rm o}^2}{{\rm d}f} = 0$, i.e. $f^{-1} = (1 + LR_{\rm i}^{-1})L^{-1}$ in Eqs.(16) and (17), the minimum beam spot is found at the target, with the width and radius of curvature expressed by

$$\omega_{0\,\min}^2 = L^2\varepsilon^2\omega_{\rm i}^{-2} \tag{18}$$

and

$$R_{0\,\min}^{-1} = L^{-1}, \tag{19}$$

respectively. Equation (19) shows explicitly that the minimum beam spot position is not at the location of beam waist. To calculate ΔL, firstly, we consider that an incident beam with a complex radius of curvature $P_1^{-1} = R_1^{-1} - {\rm i}\varepsilon\omega_1^{-2}$ sets up the beam waist through free drift distance ΔL. Hence, the complex radius of curvature P_2 at this beam waist position is

$$P_2^{-1} = P_1^{-1}(1 + \Delta LP_1^{-1})^{-1}$$
$$= \frac{R_1^{-1} - {\rm i}\varepsilon\omega_1^{-2}}{1 + \Delta LR_1^{-1} - {\rm i}\varepsilon\Delta L\omega_1^{-2}} = -{\rm i}\varepsilon\omega_2^{-2}. \tag{20}$$

Consequently, we obtain

$$\Delta L = -\frac{R_1^{-1}}{R_1^{-2} + \varepsilon^2\omega_1^{-4}}. \tag{21}$$

For the problem under consideration, on the right-hand side of Eq.(21) we replace R_1^{-1} and ω_1 with $R_{0\,\min}^{-1}$ and $\omega_{0\,\min}$, respectively, and obtain

$$\Delta L = -\frac{L}{1 + \omega_{\rm i}^4L^{-2}\varepsilon^{-2}}. \tag{22}$$

Then (see the Appendix)

$$\Delta L = -L\omega_0^2\omega_{\rm i}^{-2}, \tag{23}$$

where ω_0 is the beam waist width, which is consistent with the result in Ref.[12]. It is shown that this wave-like model leads to the same result such as the phase space ellipse transform derived from Liouville theorem.

5. Conclusion and discussion

In this paper, the charged-particle beam passing through a first-order optical system has been investigated with the Gaussian wave formalism model. By starting from wave-like concepts defined for a charged-particle beam in the paraxial approximation, we have given the wave formalism describing the charged-particle beam evolution, which allows us to obtain an elliptical Gaussian wave-like picture of the charged-particle beam passing in the real space domains, where a sort of diffraction-like relation (see Eq.(3)) holds for the spread of particle position distribution and the spread of particle momentum (divergence angle). Within the framework of the Gaussian wave-like picture, the charged-particle beam evolution is governed by the *ABCD*-like law, showing that the above Gaussian wave formalism model could be a useful tool for the investigation of transverse charged-particle beam optics and particle accelerator physics. It is worth mentioning that this description is in agreement with phase space analysis by comparing the results of the Gaussian wave-like charged-particle beam passing through first-order optical systems without aberration with the results of the standard σ-transform.

However, the following question naturally arises: whether or not the charged-particle beam passing through an arbitrary electrostatically accelerated or decelerated optical device (the beam emittance changes during the charged-particle beam propagation) obeys the *ABCD*-like law and if so, under what condition it happens. Liouville theorem states that the area in the phase space $(x, \sqrt{V}x')$ now remains unchanged in this case, where $\sqrt{V}$ stands for the normalized potential where the beam propagates, the effect of an accelerated (or a decelerated) optical device represented by an associated 2×2 transfer matrix $\bar{\boldsymbol{M}} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix}$ is to introduce the vector $[\ x \quad \sqrt{V}x'\ ]_{\rm i}^{\rm T}$

into a new vector

$$\begin{bmatrix} x \\ \sqrt{V}x' \end{bmatrix}_{\rm o} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \begin{bmatrix} x \\ \sqrt{V}x' \end{bmatrix}_{\rm i} \tag{24}$$

and

$$\bar{A}\bar{D} - \bar{B}\bar{C} = 1. \tag{25}$$

Consequently, we can find that the following effective beam emittance is conservative

$$\varepsilon_{\rm eff} = \sqrt{V}\varepsilon. \tag{26}$$

Moreover, it is easy to obtain the following relations:

$$A = \bar{A},\ B = \sqrt{V_{\rm i}}\bar{B},\ C = \bar{C}/\sqrt{V_{\rm o}},\ D = \sqrt{\frac{V_{\rm i}}{V_{\rm o}}}\bar{D}. \tag{27}$$

The effective complex radius of curvature of the beam can be defined as follows:

$$P^{-1} = R^{-1} - {\rm i}\varepsilon_{\rm eff}\omega^{-2}, \tag{28}$$

which leads to the effective $ABCD$-like law

$$P_{\rm o}^{-1} = (\bar{C} + \bar{D}P_{\rm i}^{-1})(\bar{A} + \bar{B}P_{\rm i}^{-1})^{-1}. \tag{29}$$

Thus, we can conclude that for a charged-particle beam: (a) in vacuum as well as in an isopotential space, the conservative beam emittance is the necessary and sufficient condition of the $ABCD$-like law; (b) in the case of non-isopotential space, the conservative effective beam emittance given by Eq.(26) is the necessary and sufficient condition of the effective $ABCD$-like law.

In conclusion, we have suggested an elliptical Gaussian wave-like model for the charged-particle beam propagation through a first-order optical system by use of wave-like parameters defined for a charged-particle beam, i.e. the width, which shows the transversal extension of the beam, the divergence angle, which describes the spread of the beam in the far field, and the radius of curvature, which explains the curvature of associated wavefront. In this context, we have introduced the complex radius of curvature describing both the curvature of the wavefront and the transversal size of the beam. It is shown that, for a dilute and paraxial beam, the complex radius of curvature can directly describe the optical properties of a charged-particle beam while the $ABCD$-like law can describe the propagation and transformation characteristics of the charged-particle beam through a first-order optical system.

Appendix: Derivation of Eq.(23)

As shown in Fig.1, supposing that the minimum beam spot with a complex radius of curvature given by $P_{0\,\rm min}^{-1} = R_{0\,\rm min}^{-1} - {\rm i}\varepsilon\omega_{0\,\rm min}^{-2}$ is at the target T, which is generally not the location of the beam waist. In addition, the beam waist with a width ω_0 passes through a free distance ΔL, forming the minimum beam spot at the target T.

According to Eqs.(3) and (4), we have

$$\omega_0^2 = \frac{\varepsilon^2}{\Theta_0^2} = \frac{\varepsilon^2}{\dfrac{\omega_{0\,\rm min}^2}{R_{0\,\rm min}^2} + \dfrac{\varepsilon^2}{\omega_{0\,\rm min}^2}}, \tag{a1}$$

By employing Eqs.(18) and (19), we reproduce

$$\frac{\omega_{0\,\rm min}^2}{R_{0\,\rm min}^2} + \frac{\varepsilon^2}{\omega_{0\,\rm min}^2} = \frac{\omega_{\rm i}^2}{L^2} + \frac{\varepsilon^2}{\omega_{\rm i}^2}. \tag{a2}$$

Substituting Eq.(a2) into Eq.(a1), the beam waist width ω_0 can be obtained as follows:

$$\omega_0^2 = \frac{\omega_{\rm i}^2}{1 + \omega_{\rm i}^4 L^{-2}\varepsilon^{-2}}. \tag{a3}$$

Consequently, we can have

$$\Delta L = -\frac{L}{1 + \omega_{\rm i}^4 L^{-2}\varepsilon^{-2}} = -L\omega_0^2\omega_{\rm i}^{-2}. \tag{a4}$$

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