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Lie symmetries, perturbation to symmetries and adiabatic invariants of Poincaré equations*

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Based on the invariance of differential equations under infinitesimal transformations, Lie symmetry, laws of conservations, perturbation to the symmetries and adiabatic invariants of Poincaré equations are presented. The concepts of Lie symmetry and higher order adiabatic invariants of Poincaré equations are proposed. The conditions for existence of the exact invariants and adiabatic invariants are proved, and their forms are also given. In addition, an example is presented to illustrate these results.

Keywords: Poincaré equations, perturbation to symmetry, exact invariant, adiabatic invariant
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1. Introduction

Poincaré constructed a new type of differential equations of motion of holonomic mechanical systems by use of the method of Lie movable group of infinitesimal transformation in 1901.[1] Chetaev extended the theory to the situation in which the transformation group was non-movable, constraints were non-steady and variables were non-independent, in the 1920s.[2] The equations constructed by the two scholars are called Poincaré equations, which are of significance in the modern theory of Hamiltonian systems. Mei has studied the Poincaré equations and obtained some important results in recent years.[3,4]

The symmetrical principle plays an important role in the study of mathematics and physics. The research on conserved quantities of mechanical systems not only has important theoretic significance, but also can reveal the physical essence involved in them.[5,6] The approach of Lie symmetries was proposed in the 19th century,[7] but it was not applied to mechanics until 1979.[8] In recent years, many results have sprung up in the studies of Lie symmetries.[9−35]

It is well known that even tiny changes in symmetry, called perturbation to symmetry, are of great importance for physical systems.[36−38] Burgers pioneered in this area by proposing adiabatic invariants for a special kind of Hamiltonian system.[39] The adiabatic invariants are almost unchanged when the parameter varies very slowly, and thus play a very important role in the research on the quasi-integrability of mechanical systems. Ibragimov,[40] Kruskal[41] and Djukic[42] also made great contributions to the research in this respect and obtained a series of important results separately. Hereafter, studies of adiabatic invariants are very active in many areas such as mechanics, atomic and molecular physics, spherical physics and engineering.[43−49]

In this paper, we discuss Lie symmetry, perturbation to symmetry and adiabatic invariants of Poincaré equations. Firstly, we give the Poincaré equations, and propose the concepts of Lie symmetry and higher order adiabatic invariants of the systems. Then, we prove the conditions for existence of the exact invariants and adiabatic invariants, and give their forms. Finally, we present an example to illustrate these results.

2. Poincaré equations

The position of an n degree-of-freedom holonomic mechanical system is determined by m variables $x_i (i = 1, \cdots, m; m \geq n)$. If $m = n$, then x_i are inde-

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pendent generalized coordinates; If $m > n$ then x_i are remainder coordinates of the system.

Suppose that the generalized velocities possess the following forms:

$$\dot{x}_i = \zeta_i^s(\boldsymbol{x})\eta_s \quad (i = 1, \cdots, m;\ s = 1, \cdots, n),$$
$$\text{rank}\ (\zeta_i^s) = n. \tag{1}$$

Let

$$\tilde{L}(t, x_i, \eta_s) = L(t, x_i, \zeta_i^s \eta_s), \tag{2}$$

which is the Lagrangian function expressed by the parameters $\boldsymbol{\eta}$. When $m = n$, Poincaré equations can be expressed as[4,50]

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \eta_s} = C_{rs}^k \eta_r \frac{\partial \tilde{L}}{\partial \eta_k} + \mathrm{X}_s \tilde{L} \quad (s, k, r = 1, \cdots, n), \tag{3}$$

where

$$\mathrm{X}_s = \zeta_i^s \frac{\partial}{\partial x_i}, \tag{4}$$

and C_{rs}^k are structure parameters, which are the functions of $\boldsymbol{x}$, and are determined by the following equations:

$$[\mathrm{X}_s, \mathrm{X}_k] = \mathrm{X}_s\mathrm{X}_k - \mathrm{X}_k\mathrm{X}_s = C_{sk}^r \mathrm{X}_r = -C_{ks}^r \mathrm{X}_r. \tag{5}$$

From Eqs.(4) and (5), one can obtain

$$C_{sk}^r \zeta_i^r = \zeta_j^s \frac{\partial \zeta_i^k}{\partial x_j} - \zeta_j^k \frac{\partial \zeta_i^s}{\partial x_j}. \tag{6}$$

Suppose that Eqs.(3) are not singular, i.e.

$$\det\left(\frac{\partial^2 \tilde{L}}{\partial \eta_s \partial \eta_k}\right) \neq 0, \tag{7}$$

then all $\dot{\eta}_s$ can be solved from Eqs.(3)

$$\dot{\eta}_s = \alpha_s(t, x_i, \eta_k). \tag{8}$$

3. Infinitesimal transformations and exact invariants

Introducing the infinitesimal transformations of groups

$$t^* = t + \Delta t, \pi_s^*(t^*) = \pi_s(t) + \Delta\pi_s, (s = 1, \cdots, n), \tag{9}$$

and their expansions

$$t^* = t + \varepsilon\tau^0(t, \boldsymbol{x}, \boldsymbol{\eta}),$$
$$\pi_s^*(t^*) = \pi_s(t) + \varepsilon\xi_s^0(t, \boldsymbol{x}, \boldsymbol{\eta}),\ \ (s = 1, \cdots, n), \tag{10}$$

where ε is an infinitesimal parameter, τ^0 and ξ_s^0 are infinitesimal generators, from Eqs.(1) one can obtain

$$\delta x_i = \zeta_i^s \delta\pi_s, \tag{11}$$

hence

$$\begin{aligned}\Delta x_i =& \delta x_i + \dot{x}_i \Delta t \\ =& \zeta_i^s(\varepsilon\xi_s^0 - \varepsilon\eta_s\tau^0) + \varepsilon\zeta_i^s\eta_s\tau^0 \\ =& \varepsilon\zeta_i^s\xi_s^0,\end{aligned} \tag{12}$$

and

$$\begin{aligned}\Delta\eta_s =& \delta\eta_s + \dot{\eta}_s\Delta t \\ =& \frac{\mathrm{d}}{\mathrm{d}t}\delta\pi_s + C_{rk}^s\eta_r\delta\pi_k + \dot{\eta}_s\Delta t \\ =& \varepsilon(\dot{\xi}_s^0 - \eta_s\dot{\tau}^0 + C_{rk}^s\eta_r\xi_k^0).\end{aligned} \tag{13}$$

Introduce an infinitesimal generator vector

$$\mathrm{X}^{(0)} = \tau^0\frac{\partial}{\partial t} + \xi_s^0\zeta_i^s\frac{\partial}{\partial x_i}, \tag{14}$$

and its first extension

$$\mathrm{X}^{(1)} = \mathrm{X}^{(0)} + (\dot{\xi}_s^0 - \eta_s\dot{\tau}^0 + C_{rk}^s\eta_r\xi_k^0)\frac{\partial}{\partial\eta_s}, \tag{15}$$

and its second extension

$$\mathrm{X}^{(2)} = \mathrm{X}^{(1)} + [(\dot{\xi}_s^0 - \eta_s\dot{\tau}^0 + C_{rk}^s\eta_r\xi_k^0)\dot{} - \dot{\eta}_s\dot{\tau}^0]\frac{\partial}{\partial\dot{\eta}_s}, \tag{16}$$

then the invariance of Eqs.(3) under the infinitesimal transformations (10) will lead to the following equations:

$$\begin{aligned}&\mathrm{X}^{(2)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\eta_s}\right) - \mathrm{X}^{(1)}\left(C_{rs}^k\eta_r\frac{\partial\tilde{L}}{\partial\eta_k}\right) \\ &- \mathrm{X}^{(1)}\{\mathrm{X}_s\tilde{L}\} \underset{(3)}{\Big|} = 0,\end{aligned} \tag{17}$$

which is called the Lie symmetrical determining equations, and its equivalent form is

$$\mathrm{X}^{(2)}(\dot{\eta}_s) - \mathrm{X}^{(1)}(\alpha_s) \underset{(8)}{\Big|} = 0. \tag{18}$$

Definition 1[4] If the generators τ^0 and ξ_s^0 satisfy the determining equation (17), then the corresponding transformations are called the Lie symmetrical transformations, and the corresponding symmetries are called the Lie symmetries.

The Lie symmetries do not always imply exact invariants. The following theorem gives the conditions of the existence of exact invariants and their forms.

Proposition 1[4] For the infinitesimal generators τ^0 and ξ_s^0 satisfying Eq.(17), if there exists a gauge function $G^0 = G^0(t, \boldsymbol{x}, \boldsymbol{\eta})$ satisfying the following structure equation:

$$\tilde{L}\dot{\tau}^0 + \mathrm{X}^{(1)}(\tilde{L}) + \dot{G}^0 = 0, \tag{19}$$

then the Poincaré equation has the following exact invariant of Lie symmetries:

$$I_0 = \tilde{L}\tau^0 + \frac{\partial \tilde{L}}{\partial \eta_s}(\xi_s^0 - \eta_s\tau^0) + G^0 = \text{const.} \tag{20}$$

Proof

$$\begin{aligned}\frac{\mathrm{d}I_0}{\mathrm{d}t} =&\dot{\tilde{L}}\dot{\tau}^0 + \left(\frac{\partial \tilde{L}}{\partial t} + \frac{\partial \tilde{L}}{\partial x_i}\dot{x}_i + \frac{\partial \tilde{L}}{\partial \eta_s}\dot{\eta}_s\right)\tau^0 \\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \eta_s}(\xi_s^0 - \eta_s\tau^0) \\ &+ \frac{\partial \tilde{L}}{\partial \eta_s}(\dot{\xi}_s^0 - \dot{\eta}_s\tau^0 - \eta_s\dot{\tau}^0) + \dot{G}^0 \\ =&\dot{\tilde{L}}\dot{\tau}^0 + \left(\frac{\partial \tilde{L}}{\partial t} + \frac{\partial \tilde{L}}{\partial x_i}\dot{x}_i + \frac{\partial \tilde{L}}{\partial \eta_s}\dot{\eta}_s\right)\tau^0 \\ &+ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \eta_s}(\xi_s^0 - \eta_s\tau^0) \\ &+ \frac{\partial \tilde{L}}{\partial \eta_s}(\dot{\xi}_s^0 - \dot{\eta}_s\tau^0 - \eta_s\dot{\tau}^0) \\ &- \tilde{L}\dot{\tau}^0 - \tau^0\frac{\partial \tilde{L}}{\partial t} - \xi_s^0\zeta_i^s\frac{\partial \tilde{L}}{\partial x_i} \\ &- (\dot{\xi}_s^0 - \eta_s\dot{\tau}^0 + C_{rs}^k\eta_r\xi_s^0)\frac{\partial \tilde{L}}{\partial \eta_k} \\ =&(\xi_s^0 - \eta_s\tau^0)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \eta_s} - C_{rs}^k\eta_r\frac{\partial \tilde{L}}{\partial \eta_k} - \mathrm{X}_s\tilde{L}\right) = 0.\end{aligned}$$

4. Perturbation to the symmetries and adiabatic invariants

Firstly we give the concept of high-order adiabatic invariants.

Definition 2 If $I_z(t, \boldsymbol{x}, \boldsymbol{\eta}, \varepsilon)$ is a physical quantity including ε which has the highest power of z in a mechanical system, and its derivative with respect to time t is directly proportional to ε^{z+1}, then I_z is called a zth order adiabatic invariant of the mechanical system.

Suppose that the Poincaré equations corresponding to Eqs.(3) is perturbed by small quantities εW_s, then the equations of motion of the system become

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \tilde{L}}{\partial \eta_s} =&C_{rs}^k\eta_r\frac{\partial \tilde{L}}{\partial \eta_k} + \mathrm{X}_s\tilde{L} + \varepsilon W_s, \\ &(s, k, r = 1, \cdots, n).\end{aligned} \tag{21}$$

Due to the action of εW_s, the primary symmetries and invariants of the system may be changed. Suppose that the change is a small perturbation based on the symmetrical transformation of the system without perturbation, and $\tau(t, \boldsymbol{x}, \boldsymbol{\eta})$ and $\xi_s(t, \boldsymbol{x}, \boldsymbol{\eta})$ express the generators of time and space respectively after being perturbed, then

$$\begin{aligned}\tau &= \tau^0 + \varepsilon\tau^1 + \varepsilon^2\tau^2 + \cdots, \\ \xi_s &= \xi_s^0 + \varepsilon\xi_s^1 + \varepsilon^2\xi_s^2 + \cdots,\end{aligned} \tag{22}$$

and they satisfy

$$\dot{\tilde{L}}\dot{\tau} + \mathrm{X}^{(1)'}(\tilde{L}) + \varepsilon W_s(\xi_s - \eta_s\tau) + \dot{G} = 0, \tag{23}$$

where

$$\begin{aligned}\mathrm{X}^{(1)'} =&\mathrm{X}^{(0)'} + (\dot{\xi}_s - \eta_s\dot{\tau} + C_{rk}^s\eta_r\xi_k)\frac{\partial}{\partial \eta_s} \\ =&\tau\frac{\partial}{\partial t} + \xi_s\zeta_i^s\frac{\partial}{\partial x_i} + (\dot{\xi}_s - \eta_s\dot{\tau} + C_{rk}^s\eta_r\xi_k)\frac{\partial}{\partial \eta_s}.\end{aligned}$$

G in Eq.(23) is a gauge function. Let

$$G = G^0 + \varepsilon G^1 + \varepsilon^2 G^2 + \cdots, \tag{24}$$

$$\begin{aligned}\mathrm{X}^{(1)m} =&\tau^m\frac{\partial}{\partial t} + \xi_s^m\zeta_i^s\frac{\partial}{\partial x_i} \\ &+ (\dot{\xi}_s^m - \eta_s\dot{\tau}^m + C_{rk}^s\eta_r\xi_k^m)\frac{\partial}{\partial \eta_s}.\end{aligned} \tag{25}$$

Substituting Eqs.(22) and Eq.(24) into Eq.(23), we obtain

$$\begin{aligned}&\dot{\tilde{L}}\dot{\tau}^m + X^{(1)m}(\tilde{L}) + W_s(\xi_s^{m-1} - \eta_s\tau^{m-1}) + \dot{G}^m = 0, \\ &(m = 0, 1, 2, \cdots, z),\end{aligned} \tag{26}$$

when $m = 0$, the condition $W_s = 0$ holds.

Proposition 2 For the Poincaré equations perturbed by small quantities εW_s, if the generators $\tau^m(t, \boldsymbol{x}, \boldsymbol{\eta})$ and $\xi_s^m(t, \boldsymbol{x}, \boldsymbol{\eta})$ under infinitesimal transformations satisfy the structure equation (26), and $G^m(t, \boldsymbol{x}, \boldsymbol{\eta})$ is a gauge function, then

$$\begin{aligned}I_z =&\varepsilon^m\left[\tilde{L}\tau^m + \frac{\partial \tilde{L}}{\partial \eta_s}(\xi_s^m - \eta_s\tau^m) + G^m\right], \\ &(m = 0, 1, \cdots, z),\end{aligned} \tag{27}$$

is a zth order adiabatic invariant of the mechanical system.

Proof

$$\begin{aligned}\frac{\mathrm{d}I_z}{\mathrm{d}t} =&\varepsilon^m\Big[\dot{\tilde{L}}\tau^m+\tilde{L}\dot{\tau}^m+\frac{\partial\tilde{L}}{\partial\eta_s}(\dot{\xi}_s^m-\dot{\eta}_s\tau^m-\eta_s\dot{\tau}^m)\\&+(\xi_s^m-\eta_s\tau^m)\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\eta_s}-\tilde{L}\dot{\tau}^m-X^{(1)m}(\tilde{L})\\&-W_s(\xi_s^{m-1}-\eta_s\tau^{m-1})\Big]\\=&\varepsilon^k\Big[\Big(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\tilde{L}}{\partial\eta_s}-C_{rs}^k\eta_r\frac{\partial\tilde{L}}{\partial\eta_k}-\mathrm{X}_s\tilde{L}\Big)(\xi_s^m-\eta_s\tau^m)\\&-W_s(\xi_s^{m-1}-\eta_s\tau^{m-1})\Big].\end{aligned}$$

Using Eq.(21), we have

$$\frac{\mathrm{d}I_z}{\mathrm{d}t}=\varepsilon^m\left[\,\varepsilon W_s(\xi_s^m-\eta_s\tau^m)-W_s(\xi_s^{m-1}-\eta_s\tau^{m-1})\,\right].$$

Expanding the above formula and performing summation, we have

$$\frac{\mathrm{d}I_z}{\mathrm{d}t}=\varepsilon^{z+1}W_s(\xi_s^z-\eta_s\tau^z). \tag{28}$$

This shows that $\mathrm{d}I_z/\mathrm{d}t$ is directly proportional to ε^{z+1}, so I_z is a zth order adiabatic invariant of the mechanical system.

5. An illustrative example

Now we come to study the exact invariants and adiabatic invariants of a particle with mass M moving on the smooth surface of a taper with semi-vertex angle θ in gravitational field.

The kinetic energy and potential energy of the particle are respectively

$$\begin{aligned}T=&\frac{1}{2}M(\dot{r}^2+r^2\dot{\varphi}^2\sin^2\theta),\\V=&Mgr\cos\theta,\end{aligned} \tag{29}$$

where r is the distance of the particle from the pole of taper, and φ is the angle between the projection of r on horizontal plane and the fixed line. Let

$$x_1=r,\quad x_2=\varphi,\quad \dot{x}_1=\eta_1,\quad \dot{x}_2=\frac{\eta_2}{x_1}, \tag{30}$$

then we will have

$$\begin{aligned}&\tilde{L}=\frac{1}{2}M(\eta_1^2+\eta_2^2\sin^2\theta)-Mgx_1\cos\theta,\\&\zeta_1^1=1,\quad \zeta_1^2=\zeta_2^1=0,\quad \zeta_2^2=\frac{1}{x_1},\\&C_{12}^2=-C_{21}^2=-\frac{1}{x_1}.\end{aligned} \tag{31}$$

So we can obtain the following Poincaré equations:

$$\begin{aligned}&\dot{\eta}_1=-g\cos\theta+\frac{\eta_2^2}{x_1}\sin^2\theta,\\&\dot{\eta}_2=-\frac{1}{x_1}\eta_1\eta_2.\end{aligned} \tag{32}$$

The Lie symmetrical determining equation (17) gives

$$\begin{aligned}&(\dot{\xi}_1^0-\eta_1\dot{\tau}^0)^{\cdot}-\dot{\tau}^0\Big(-g\cos\theta+\frac{\eta_2^2}{x_1}\sin^2\theta\Big)\\=&\frac{2\eta_2}{x_1}\sin^2\theta(\dot{\xi}_2^0-\eta_2\dot{\tau}^0+C_{rt}^2\eta_r\xi_t^0)-\frac{\eta_2^2}{x_1^2}\xi_1^0\sin^2\theta\\&(\dot{\xi}_2^0-\eta_2\dot{\tau}^0+C_{rt}^2\eta_r\xi_t^0)^{\cdot}-\dot{\tau}^0\Big(-\frac{\eta_1\eta_2}{x_1}\Big)\\=&\frac{\eta_1\eta_2}{x_1^2}\xi_1^0-\frac{1}{x_1}\eta_2(\dot{\xi}_1^0-\eta_1\dot{\tau}^0)\\&-\frac{1}{x_1}\eta_1(\dot{\xi}_2^0-\eta_2\dot{\tau}^0+C_{rt}^2\eta_r\xi_t^0),\end{aligned} \tag{33}$$

with solutions

$$\tau^0=-1,\quad \xi_1^0=\xi_2^0=0. \tag{34}$$

Substituting Eqs.(34) into the structure equation (19), we obtain

$$G^0=0. \tag{35}$$

So we can obtain the following exact invariant by using Proposition 1:

$$I_0=\frac{1}{2}M(\eta_1^2+\eta_2^2\sin^2\theta)+Mgx_1\cos\theta=\text{const.} \tag{36}$$

Suppose that the system is perturbed by the small quantities as follows:

$$\varepsilon W_1=\varepsilon x_1,\quad \varepsilon W_2=\varepsilon\frac{x_2}{x_1}, \tag{37}$$

then the structure equation (26) will be

$$\begin{aligned}&\tilde{L}\dot{\tau}^1+\tau^1\frac{\partial\tilde{L}}{\partial t}+\xi_s^1\zeta_i^s\frac{\partial\tilde{L}}{\partial x_i}\\&+(\dot{\xi}_s^1-\eta_s\dot{\tau}^1+C_{rk}^s\eta_r\xi_k^1)\frac{\partial\tilde{L}}{\partial\eta_s}\\&+W_1(\xi_1^0-\eta_1\tau^0)+W_2(\xi_2^0-\eta_2\tau^0)+\dot{G}^1=0\,,\end{aligned} \tag{38}$$

which has solutions as

$$\tau^1=1,\quad \xi_1^1=\xi_2^1=0,\quad G^1=\frac{1}{2}(x_1^2+x_2^2). \tag{39}$$

So we can obtain the following first order adiabatic invariants by use of Theorem 2:

$$\begin{aligned}I_1=&\frac{1}{2}M(\eta_1^2+\eta_2^2\sin^2\theta)+Mgx_1\cos\theta\\&+\varepsilon\Big[-\frac{1}{2}M(\eta_1^2+\eta_2^2\sin^2\theta)-Mgx_1\cos\theta\\&+\frac{1}{2}(x_1^2+x_2^2)\Big].\end{aligned} \tag{40}$$

Further we can obtain more higher order adiabatic invariants.

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