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Lie-form invariance of nonholonomic mechanical systems

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The Lie-form invariance of a nonholonomic mechanical system is studied. The definition and criterion of the Lie-form invariance of the nonholonomic mechanical system are given. The Hojman conserved quantity and a new type of conserved quantity are obtained from the Lie-form invariance. An example is given to illustrate the application of the results.

Keywords: nonholonomic mechanical systems, the Lie-form invariance, conserved quantity
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1. Introduction

It is significant to study the symmetries and conserved quantities of dynamic systems not only in mathematics but also in physics. There exist three main kinds of symmetries: the Noether symmetry,[1−4] the Lie symmetry[5−8] and the form invariance,[9−13] and three main kinds of conserved quantities: the Noether conserved quantity,[1−4] the Hojman conserved quantity[14] and a new type of conserved quantity.[15]

Recently, Mei *et al* gave a unified symmetry[16,17] that could lead to the three kinds of conserved quantities mentioned above. Mei gave the Noether–Lie symmetry[18] of the Lagrange system. Wu firstly defined the Lie-form invariance[19] of a holonomic mechanical system, from which the Hojman conserved quantity and a new type of conserved quantity are obtained. In this paper, based on the total time derivative along the trajectory of the system, we generalize Wu's results to a nonholonomic mechanical system and obtain the Hojman conserved quantity and a new type of conserved quantity at the same time.

2. The definition and criterion of the Lie-form invariance

Here we consider a mechanical system whose configuration is determined by the n generalized coordinates $q_s(s = 1, \cdots, n)$. Let its motion be subjected to the g ideal nonlinear nonholonomic constraints of Chetaev type

$$\varphi_\beta(t, q_s, \dot{q}_s) = 0, \quad (\beta = 1, \cdots, g), \tag{1}$$

$$\sum_{\beta=1}^{g} \frac{\partial \varphi_\beta}{\partial \dot{q}_s} \delta q_s = 0. \tag{2}$$

Then its motion can be described by the differential equations as follows:

$$E_s(L) = Q_s + \sum_{\beta=1}^{g} \lambda_\beta \frac{\partial \varphi_\beta}{\partial \dot{q}_s},$$

$$(s = 1, \cdots, n; \beta = 1, \cdots, g). \tag{3}$$

In general, it is supposed that the system (3) is nonsingular. Before integrating Eqs.(3), we can find λ_β is a function of t, q_s and $\dot{q}_s$, thus we can obtain the equations of motion of the holonomic system corresponding to the nonholonomic system expressed by Eqs.(1) and (3):

$$E_s(L) = Q_s + \Lambda_s, \quad (s = 1, \cdots, n), \tag{4}$$

where $\Lambda_s = \Lambda_s(t, q_s, \dot{q}_s) = \sum_{\beta=1}^{g} \lambda_\beta \frac{\partial \varphi_\beta}{\partial \dot{q}_s}$ are the generalized nonholonomic reactive constraint forces, $L = L(t, q_s, \dot{q}_s)$ is the Lagrangian, $Q_s = Q_s(t, q_s, \dot{q}_s)$ are the nonpotential generalized forces, λ_β are the multipliers

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of constraint, and the repeated subscripts represent the summation. E_s are the Euler operators

$$E_s = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s}. \tag{5}$$

From Eqs.(4), we can find all the generalized accelerations are

$$\ddot{q}_s = \alpha_s(t, q_s, \dot{q}_s), \quad (s = 1, \cdots, n). \tag{6}$$

Definition[17,19] The symmetry that is both a Lie symmetry and a form invariance is called a Lie- form invariance.

For a nonholonomic system, the constrained Eqs.(1) retain the form invariance

$$\mathrm{X}^{(1)}(\varphi_\beta) = 0. \tag{7}$$

The determining equations of the Lie symmetry are Eqs. (7) and

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - 2\alpha_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 = \mathrm{X}^{(1)}(\alpha_s). \tag{8}$$

The criterion equations of the form invariance are Eqs. (7) and

$$E_s\left\{\mathrm{X}^{(1)}(L)\right\} - \mathrm{X}^{(1)}(Q_s) - \mathrm{X}^{(1)}(\Lambda_s) = 0, \tag{9}$$

based on the total time derivative along the trajectory of the system, where

$$\begin{cases} \mathrm{X}^{(1)} = \xi_0\dfrac{\partial}{\partial t} + \displaystyle\sum_{s=1}^{n}\xi_s\frac{\partial}{\partial q_s} + \sum_{s=1}^{n}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial \dot{q}_s}, \\ \mathrm{X}^{(2)} = \mathrm{X}^{(1)} + \displaystyle\sum_{s=1}^{n}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - 2\ddot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0\right)\frac{\partial}{\partial \ddot{q}_s}. \end{cases} \tag{10}$$

According to the above definition, the following result can be obtained easily.

Criterion If the generators of infinitesimal transform ξ_0 and ξ_s satisfy Eqs.(7)

$$\begin{aligned} &\left\{\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - 2\alpha_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - \mathrm{X}^{(1)}(\alpha_s)\right\}^2 \\ &+\left\{E_s\left\{\mathrm{X}^{(1)}(L)\right\} - \mathrm{X}^{(1)}(Q_s) - \mathrm{X}^{(1)}(\Lambda_s)\right\}^2 = 0, \end{aligned} \tag{11}$$

then the corresponding symmetry is a Lie-form invariance of nonholonomic mechanical system.

3. The conserved quantity deduced from the Lie-form invariance

Under certain conditions, by using the Lie-form invariance, both the Hojman conserved quantity and a new type of conserved quantity can be obtained.

Proposition 1 Under the special infinitesimal transformation in which the time($\xi_0 = 0$) remains unchanged, if there is a function $\mu = \mu(t, q_s, \dot{q}_s)$ such that

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{12}$$

then the Lie-form invariance will lead to a Hojman conserved quantity

$$I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s) + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \mathrm{const}, \tag{13}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial \dot{q}_s}. \tag{14}$$

Proof According to the definition, a Lie-form invariance must be a Lie symmetry, that is to say, it satisfies

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial \alpha_s}{\partial q_k}\xi_k + \frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \tag{15}$$

From the expressions (12) and (15) , by using Hojman's proof method, proposition 1 can be proved.

Proposition 2 If the generators of infinitesimal transform ξ_0 and ξ_s are the generators of Lie-form invariance of a nonholonomic mechanical system and there exists a function $G_{\mathrm{F}} = G_{\mathrm{F}}(t, q_s, \dot{q}_s)$ satisfying

$$\begin{aligned} &\mathrm{X}^{(1)}(L)\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 + \mathrm{X}^{(1)}\left\{\mathrm{X}^{(1)}(L)\right\} \\ &+\mathrm{X}^{(1)}(Q_s + \Lambda_s)(\xi_s - \dot{q}_s\xi_0) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{F}} = 0, \end{aligned} \tag{16}$$

then the Lie-form invariance will lead to a new type of conserved quantity

$$\begin{aligned} I_{\mathrm{F}} =& \mathrm{X}^{(1)}(L)\xi_0 + \frac{\partial \mathrm{X}^{(1)}(L)}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\mathrm{F}} \\ =& \mathrm{const}. \end{aligned} \tag{17}$$

Proof According to the definition, a Lie-form invariance must be a form invariance, so the criterion Eqs.(9) hold. Using the relationships (9) and (16), we can prove that the proposition 2 is correct.

4. Illustrative examples

The Lagrangian, the nonpotential generalized forces and the constraint equation of a nonholonomic mechanical system are, respectively,

$$L = \frac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2) - \frac{1}{2}q_2^2, \quad Q_1 = 0, \quad Q_2 = q_2, \tag{18}$$

$$\varphi = \dot{q}_1 + t\dot{q}_2 - q_2 + t = 0. \tag{19}$$

Choose the infinitesimal generators as

$$\xi_0 = \xi_1 = 0, \quad \xi_2 = t. \tag{20}$$

After calculation, we have

$$\begin{cases} \mathrm{X}^{(1)}(L) = \dot{q}_2 - q_2 t, \\ \mathrm{X}^{(1)}\left\{\mathrm{X}^{(1)}(L)\right\} = 1 - t^2, \\ \mathrm{X}^{(1)}(Q_1) = 0, \\ \mathrm{X}^{(1)}(Q_2) = t, \\ \mathrm{X}^{(1)}(\Lambda_1) = \mathrm{X}^{(1)}(\Lambda_2) = 0, \\ \mathrm{X}^{(1)}(\varphi) = 0. \end{cases} \tag{21}$$

Then we have

$$\begin{aligned} &\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \dot{q}_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - 2\alpha_s\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_0 - \mathrm{X}^{(1)}(\alpha_s) \\ =&E_s\left\{\mathrm{X}^{(1)}(L)\right\} - \mathrm{X}^{(1)}(Q_s) - \mathrm{X}^{(1)}(\Lambda_s) = 0. \end{aligned} \tag{22}$$

So, the generators (20) are Lie-form invariance. The expression (12) gives

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{23}$$

from which we obtain

$$\mu = 1, \tag{24}$$

and

$$\mu = \dot{q}_2 - t\dot{q}_1 + q_1. \tag{25}$$

From the expressions (20) and (25), by using Proposition 1, the conserved quantity expression (13) yields

$$I_{\mathrm{H}} = \frac{1}{\dot{q}_2 - t\dot{q}_1 + q_1} = \mathrm{const.} \tag{26}$$

This is the Hojman conserved quantity deduced from the Lie-form invariance.

Substituting the relationships (20) and (21) into Eqs.(16), we obtain

$$G_{\mathrm{F}} = -t, \tag{27}$$

and

$$G_{\mathrm{F}} = -t + \mathrm{arctg}(t) + \dot{q}_1. \tag{28}$$

From the expressions (20) and (28), by using Proposition 2, the conserved quantity expression (17) yields

$$I_{\mathrm{F}} = \mathrm{arctg}(t) + \dot{q}_1 = \mathrm{const}, \tag{29}$$

which is a trivial conserved quantity deduced from the Lie-form invariance.

Notice that based on the total time derivative along the trajectory of the system the Noether conserved quantity deduced from the generators (20) is

$$I_{\mathrm{N}} = \dot{q}_2 t - q_2 = \mathrm{const.} \tag{30}$$

It is obvious that both the Hojman conserved quantity (26) and the trivial conserved quantity (29) are non-Noether conserved quantities.

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