Vol 15 No 2, February 2006
1009-1963/2006/15(02)/0347-06

Chinese Physics

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Image recovery from double amplitudes in fractional Fourier domain

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(Received 24 August 2005; revised manuscript received 7 November 2005)

The classical Gerchberg–Saxton algorithm is introduced into the image recovery in fractional Fourier domain after adaptation. When this algorithm is applied directly, its performance is good for smoothed image, but bad for unsmoothed image. Based on the diversity of fractional Fourier transform on its orders, this paper suggests a novel iterative algorithm, which extracts the information of the original image from amplitudes of its fractional Fourier transform at two orders. This new algorithm consists of two independent Gerchberg–Saxton procedures and an averaging operation in each circle. Numerical simulations are carried out to show its validity for both smoothed and unsmoothed images with most pairs of orders in the interval [0, 1].

Keywords: fractional Fourier transform, image recovery, phase retrieval, iterative algorithm
PACC: 4230V, 4230N

1. Background and introduction

Phase retrieval is a problem of great practical interest and thus has been extensively studied (see Ref.[1] and references therein). Because of the dominant status of classical Fourier transform in optical field propagation,[2] most of these related studies are performed in classical Fourier domain. The introduction of fractional Fourier transform (FRFT) is a prominent progress in optics and signal processing[3−6] in recent years. It has been proven that FRFT is equivalent to the propagation of optical field in quadratic gradient index media,[5] and that this transform is also closely related to optical Fresnel transform.[7] So considerable attention has been paid to phase retrieval problem in fractional Fourier domain.[8−12] Similar to the case in classical Fourier domain, this problem can also have two variants. In the first, only the magnitude of FRFT of a function is known, but additional assumption regarding the function, such as its support and non-negativity are made. In the second, the magnitudes of both the function and its FRFT at some orders are known. The present paper will consider the problem of the first variant (or in other words, image recovery).

The classical Gerchberg–Saxton (GS) algorithm has been introduced into fractional Fourier domain in Refs.[8] and [10] for beam shaping and complex signal recovery, respectively. In particular, the dependence of recovery error on the order of FRFT has been studied in a great detail in Ref.[10]. The main result there is that in general, better results are obtained when the order is close to unity and poor results are obtained when the order is close to zero. As will be shown, when GS algorithm is applied directly to image recovery in fractional Fourier domain, its performance is not satisfactory. The main purpose of this paper is to deal with the problem of how to obtain reconstructed images of high quality. The basic principle here is to recover the image from the magnitudes of its FRFT at two orders. The bad performance of the GS-like algorithm may come from the weak constraint which one magnitude of FRFT can offer. It is possible that two weak constraints can be combined into a strong one, which drives the iterative process to converge more quickly and efficiently.

Because our work deals with the recovery of two-dimensional (2D) digital image, the definition of discrete fractional Fourier transform (DFRFT) is discussed firstly.[13] For a one-dimensional (1D) discrete signal of length N, the transform kernel of DFRFT at order a is computed as

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$$F^a = VD^aV^{\mathrm{T}} = \begin{cases} \sum_{k=0}^{N-1} \exp(-\mathrm{j}k\alpha) v_k v_k^{\mathrm{T}}, \\ \text{when} \quad N = 4m+1, 4m+3, \\ \sum_{k=0}^{N-2} \exp(-\mathrm{j}k\alpha) v_k v_k^{\mathrm{T}} + \exp(-jN) v_{N-1} v_{N-1}^{\mathrm{T}}, \\ \text{when} \quad N = 4m, 4m+2, \end{cases} \tag{1}$$

where v_k is the kth-order discrete Fourier transform Hermite eigenvector,[14] $\alpha = a\pi/2$, $V = [v_0|v_1|\ldots|v_{N-1}]$. Matrix D is a diagonal matrix in which the diagonal entries have the same eigenvalues corresponding to the column eigenvectors of matrix V in its diagonal entries. Then the transform kernel of the 2D DFRFT can be defined as

$$F^{a,b} = F^a \otimes F^b, \tag{2}$$

where $\otimes$ denotes tensor product, a and b denote the individual fractional orders in two dimensions. For this separable kernel, 2D DFRFT of a $M \times N$-point image $f(m, n)$ is computed as

$$f_{a,b}(m,n) = \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} f(p,q) F^{a,b}(p,q,m,n). \tag{3}$$

In the present paper only 2D FRFT is considered, and its two individual fractional orders are equal to each other. So this transform will be denoted by F^a, and $f_{a,a}$ will be replaced by f_a for simplicity.

The digital computation of FRFT has been summarized in detail and developed in Ref.[15], and the computation in this paper is based on the algorithms proposed there.

The rest of this paper is organized as follows. In Section 2 two alternative algorithms for image recovery in fractional Fourier domain are designed, and their performance is commented briefly. In Section 3 we compare these two algorithms by numerical experiments, and the superiority of our suggested algorithm is revealed. A note of the suggested algorithm is given in Section 4, and Section 5 concludes the whole paper.

2. New iterative algorithm for image recovery

We assume that the amplitude at the ath-order FRFT of an image f is known and will be denoted by $|f_a|$. As mentioned in Section 1, the support of the image and its non-negativity are also assumed to be known. The support and its indicator will be denoted by D and χ_D , respectively. With these assumptions the block diagram of GS-like algorithm for image recovery is designed and displayed in Fig.1(a). Initialized by a random estimate, the assumptions of non-negativity and its support are applied consequently. Then the ath-order FRFT is taken. Leave the phase intact but the amplitude replaced by $|f_a|$, an estimate of f_a is obtained. After an inverse ath-order FRFT of this estimate, a new estimate of the image is obtained and a new circle begins. The points below will be validated by numerical simulations in Section 3:

i) The operation of fetching modulus plays an important role in convergence.

ii) This GS-like algorithm works well for smoothed image, but badly for unsmoothed image.

Following the idea mentioned in Section 1, a new iterative algorithm is designed and shown in Fig.1(b) by its block diagram. Compared with the GS-like algorithm, this algorithm is based on the amplitudes of the image at the ath order and bth order. After two independent GS-like procedures, an averaging operation is taken for the two new estimates:

$$f^{k+1} = (f_1^{k+1} + f_2^{k+1})/2, \tag{4}$$

and f^{k+1} is the input of the next circle. The iterative process in this algorithm is restricted by two constraints from amplitudes at two orders simultaneously, so we expect it to converge more quickly and efficiently. The points below will be validated by numerical examples in Section. 3:

iii) This algorithm works well for both smoothed and unsmoothed images.

iv) This suggested algorithm works well for most pairs of orders in the interval [0, 1].

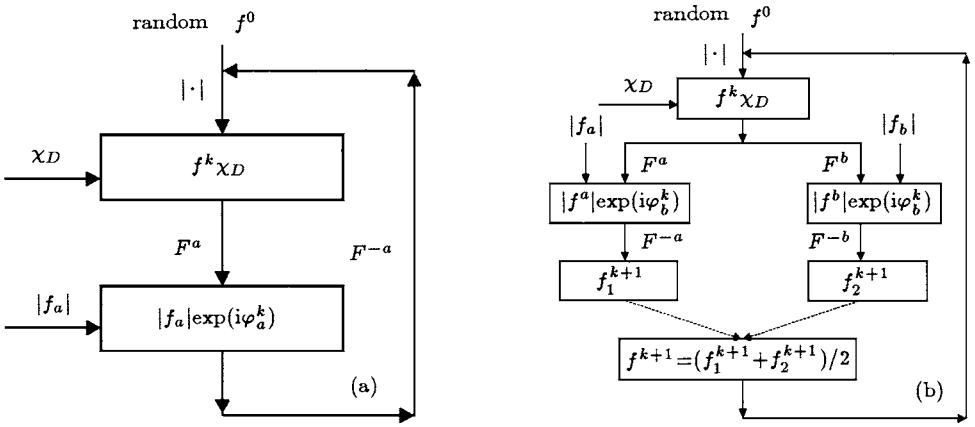


Fig.1. Block diagrams of (a) GS-like algorithm and (b) algorithm suggested in this paper. In order to apply the non-negativity assumption the operation of obtaining modulus Abs is performed in the beginning of each circle. In this figure f^k is the kth estimate of the original image f, and ϕ_a^k (ϕ_b^k) is the estimate of the phase of f_a (f_b) obtained from single magnitude $|f_a|$ ($|f_b|$) in the kth circle.

Note That the difference of the two orders in this suggested algorithm is not very small (not less than 1/20 in our numerical simulations). The direct method in Refs.[10, 11] may be a better choice when this difference is very small, which is beyond the interest here.

3. Two numerical simulations

This section is devoted to validating the four points in Section 2. To measure the convergence behaviour of the iterative process and the quality of reconstructed image, we define the mean square error (MSE) as

$$E(k) = \frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N}[f^k(i,j) - f(i,j)]^2, \qquad (5)$$

where f^k is the kth estimate of the original image f.

Two images for numerical simulations are shown in Fig.2. The size of the image in Fig.2(a) is 128×128-point, and the central 64×64 part (also the support of this image) is an image of a woman (normalized by the largest pixel value) from MATLAB archives. The size of the image in Fig.2(b) is 128×128-point, and the central 54×54 part (also the support of this image) is a Sierpinski carpet at the second level of construction,[16] in which the white area corresponds to 1 and dark area corresponds to 0. As pointed out in Ref.[17], the strategy of enlarging the size of the original image by zero-part is quite useful for iterative algorithms of phase retrieval. In fact, the enlarged size should be large enough to include the main non-zero part of the image's FRFT.

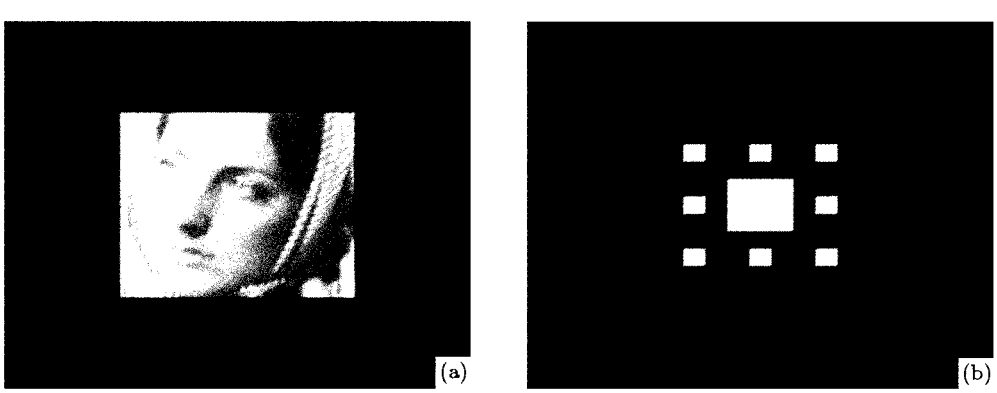


Fig.2. Two images for numerical simulations.

Firstly we will test the GS-like algorithm in Fig.1(a). In order to reveal the importance of fetching modulus operation, we can cancel it from the block diagram in Fig.1(a) and compare the performance of these two versions of GS-like algorithm. Based on the amplitudes at order 1/2 of the images in Fig.2, the reconstructed ones by these two algorithms after 100 iterations are shown in the left and middle column of Fig.3. The importance of this operation is validated immediately by visual inspection. The reconstructed image of woman indicates that GS-like algorithm in Fig.1(a) works well for such a smooth image, but the reconstructed Sierpinski carpet implies that it performs badly for such an unsmooth image. Simulations with more FRFT orders (1/20, 2/20, ..., 19/20) are performed, and the related final error after 100 iterations are displayed in Fig.4. It can be found that there is no monotone relationship between the recovery error and the order of FRFT, which is quite different from the cases discussed in Refs.[8, 10]. Another conclusion obtained from Fig.4 is that this GS-like algorithm works well for smoothed image (such as the one in Fig.2(a)) with many orders of FRFT, but badly for unsmoothed image (such as the one in Fig.2(b)) with most orders of FRFT.

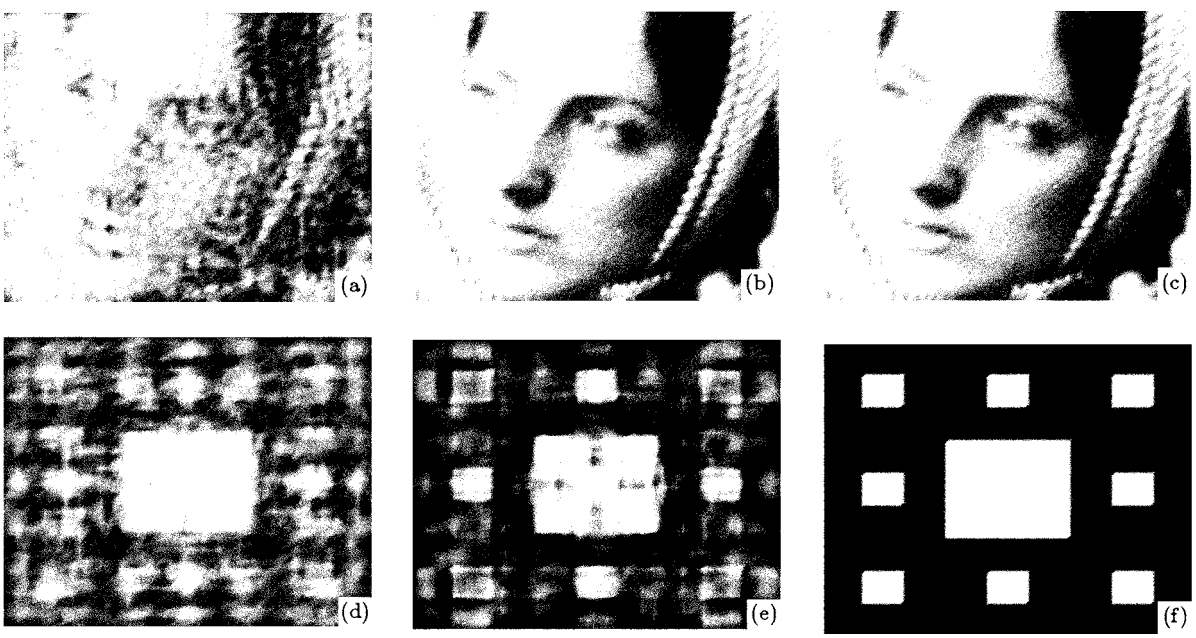


Fig.3. Reconstructed images by three algorithms mentioned in the present paper. The left two images are from the GS-like algorithm in which the Abs operation is cancelled, the middle two are from the GS-like algorithm in Fig.1(a), and the right two are from the algorithm in Fig.1(b) (more details can be found in the text).

Next we will show the superiority of the algorithm in Fig.1(b). In each circle the algorithm in Fig.1(a) needs two times FRFT, the algorithm in Fig.1(b) needs four times, so the images reconstructed by the algorithm in Fig.1(b) after 50 iterations are compared with former one from the consideration of computational complexity. Based on the amplitudes at orders 1/20 and 2/20, the latter algorithm offers images of good qualities, which are displayed in the right column of Fig.3. It is difficult to compare the qualities of Fig.3(b) and 3(c) by visual inspection, but the quality of Fig.3(f) is obviously much better than Fig.3(e). In order to compare the qualities of these images and the performance of these two algorithms quantitatively, more pairs of orders are tried and the recovery errors after 50 iterations are displayed in Fig.5. The pairs of orders in Fig.5 include the typical combinations of small and large orders in the scope [0, 1]. It can be found that except for the case in which the two orders of FRFT are 18/20 and 19/20, our suggested algorithm works well for both smoothed and unsmoothed images with almost all of the pairs of orders.

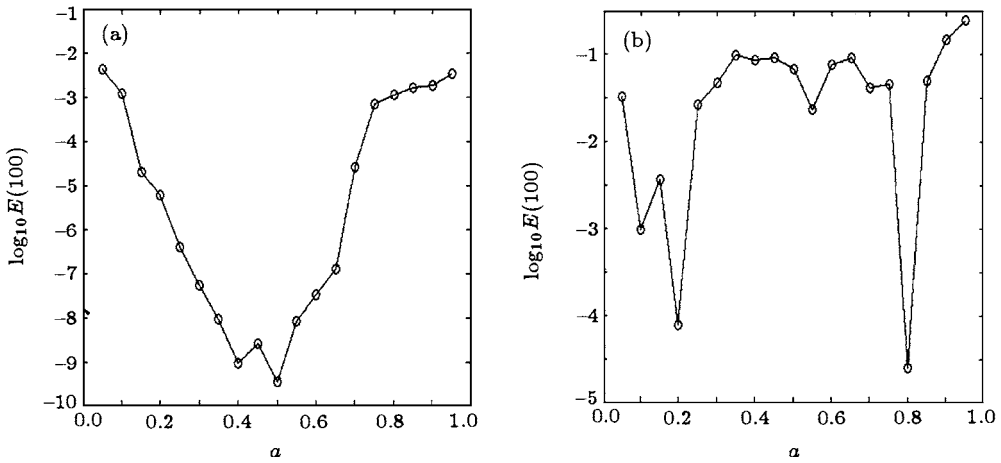


Fig.4. MSEs of reconstructed images by algorithm in Fig.1(a) after 100 iterations with (a) the image in Fig.2(a) and (b) the image in Fig.2(b).

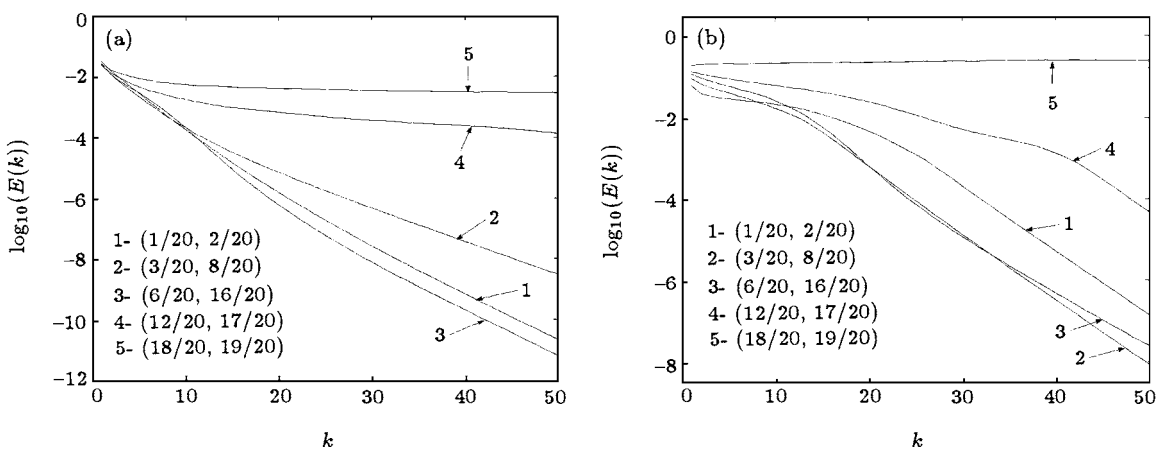


Fig.5. Convergence behaviours of the algorithm in Fig.1(b) with (a) the image in Fig.2(a) and (b) the image in Fig.2(b). The corresponding relationship between curves and pairs of orders is shown in the figures.

Numerical experiments with more smoothed and unsmoothed images also validate the above discussion.

4. A remark about suggested algorithm

The algorithm we suggested extracts the information of the original image from the amplitudes of its FRFT at two orders in parallel in each circle, and its distinct feature is the averaging operation in Eq.(4). This simple averaging operation with equal weights implies that the importance of the two amplitudes for image recovery is supposed to be the same. One may recall the work of Alieva and Calvo in Ref.[18], which investigates the importance of the phase and amplitude in fractional Fourier domain. Their research shows that the amplitude at lower order can offer reconstructed image of better quality than amplitude at higher order. So the averaging operation below with unequal weights seems to be more reasonable:

$$f^{k+1} = pf_1^{k+1} + qf_2^{k+1}, p+q=1, p, q>0. \qquad (6)$$

We have tried different values of p (=1/20, 2/20, ..., 19/20) with the images in Fig.2 and displayed the final errors after 50 iterations in Fig.5. These curves are marked by different numbers, and the corresponding relationship between these numbers and the pairs of orders are the same as in Fig.6. It can be found that the smallest MSEs are always achieved when p or q is near 1/2, and that the optimal choice depends on the specific values of the two orders. So the averaging operation with equal weights is the best choice for its simplicity and efficiency. In fact the method in Ref.[18] is to approximate the image by a deterministic formula involving only the modulus or phase of

its FRFT. Our work here is to reconstruct the image iteratively by a certain form of combination of amplitudes at the two orders, and each one is indispensable for reconstruction of high-quality image. Then the well performance of our suggested algorithm is not in contradiction with the result in Ref.[18].

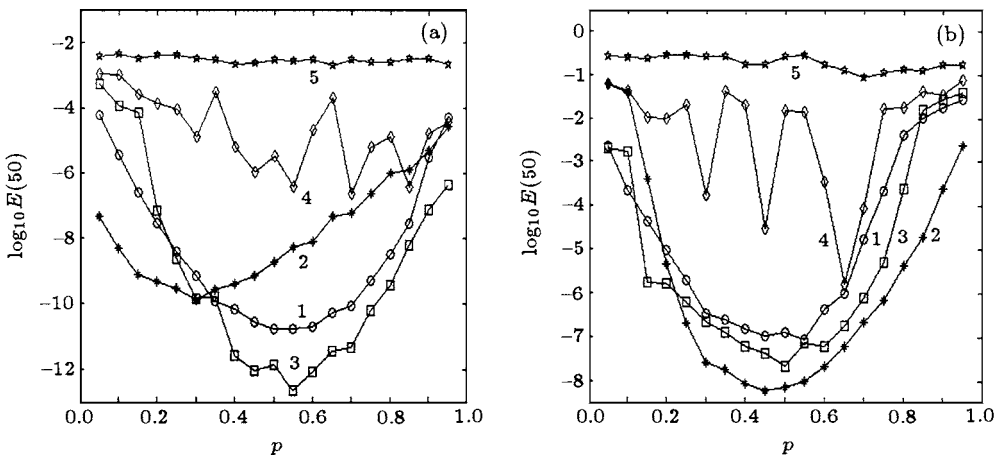


Fig.6. MSEs after 50 iterations when the averaging operation with equal weights in Fig.1(b) is replaced by unequal one with (a) the image in Fig.2(a), and (b) the image in Fig.2(b).

5. Conclusions

The problem of image recovery in fractional Fourier domain is studied and two alternative algorithms are discussed. The first one is based on the direct application of classical GS method named as GS-like algorithm. This algorithm works well for smoothed images, but badly for unsmoothed ones. The second one is based on amplitudes at two orders and suggested for its efficiency. This one consists of two GS procedures and an averaging operation with equal weights in each circle. Numerical simulations show its validity for both smoothed and unsmoothed images with most pairs of orders in the interval [0, 1]. The variant of the suggested algorithm with unequal weights is also tested, but the one with equal weights wins for its simplicity and efficiency. It is worth noting that this suggested algorithm can be generalized from two magnitudes to more than two if these magnitudes are available, and good performance is still expected.

Note After the submission of the present paper, we notice the work by Ertosun *et al.*[19] These authors study the problem of complex signal recovery from multiple FRFT magnitudes, and an algorithm with serial structure is suggested. So the application of our algorithm to complex signal recovery and application of the algorithm of Ertosun *et al* to image recovery may be an interesting and meaningful study.

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