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# Chaotic dynamics of the fractional-order Ikeda delay system and its synchronization*

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In this paper we numerically investigate the chaotic behaviours of the fractional-order Ikeda delay system. The results show that chaos exists in the fractional-order Ikeda delay system with order less than 1. The lowest order for chaos to be able to appear in this system is found to be 0.1. Master–slave synchronization of chaotic fractional-order Ikeda delay systems with linear coupling is also studied.

**Keywords:** chaos, Ikeda delay system, fractional-order system, fractional calculus
**PACC:** 0545

## 1. Introduction

Fractional calculus is a 300-year-old mathematical topic. Although it has a long history, for many years it was not used in physics and engineering. However, during the last decades, fractional calculus has attracted much attention of physicists and engineers from an application point of view.[1,2] It is found that many systems in interdisciplinary fields can be elegantly described with the help of fractional derivatives. Many systems are known to display fractional-order dynamics, such as viscoelastic systems,[3] dielectric polarization,[4] electrode-electrolyte polarization,[5] electromagnetic waves,[6] quantitative finance,[7] and quantum evolution of complex systems.[8] More recently, many investigations are devoted to chaotic dynamics of fractional order dynamical systems.[9−20]

However, up to now, there have been no reports on chaotic behaviours of the fractional-order delay systems. In this paper, we numerically investigate the chaotic behaviours of the fractional-order Ikeda delay system. We find that chaos exists in the fractional-order Ikeda delay system with order less than 1. The lowest order for chaos to be able to appear in this system is found to be 0.1. Our results are validated by the existence of a positive Lyapunov exponent. A chaotic synchronization approach is also presented for the chaotic fractional-order Ikeda delay systems. Like many other studies of dynamics in fractional-order systems, we investigate the dynamics of the fractional-order Ikeda delay system through numerical simulations.

## 2. Fractional derivative and its approximation

Fractional calculus is a generalization of integration and differentiation to a non-integer-order integro-differential operator ${}_aD_t^\alpha$ defined by

$$ {}_aD_t^\alpha = \begin{cases} \dfrac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}, & \alpha > 0, \\ 1, & \alpha = 0, \\ \displaystyle\int_a^t (\mathrm{d}\tau)^{-\alpha}, & \alpha < 0. \end{cases} \quad (1) $$

There are many definitions of fractional derivatives.[1] Perhaps the best-known one is the Riemann–Liouville definition, which is given by

$$ \frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} \mathrm{d}\tau, \quad (2) $$

for $n-1 \le \alpha < n$, where $\Gamma(\cdot)$ is the Gamma function. The geometric and the physical interpretation of the fractional derivatives are given in Ref.[21].

The Laplace transform of the Riemann–Liouville

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fractional derivative (2) is

$$\int_0^\infty \mathrm{e}^{-st}\,{}_0D_t^\alpha f(t)\mathrm{d}t$$
$$= s^\alpha L\{f(t)\} - \sum_{k=0}^{n-1} s^k\,{}_0D_t^{\alpha-k-1} f(t)\,|_{t=0}\,, \qquad (3)$$

for $n-1 \le \alpha < n$, where $s \equiv \mathrm{j}\omega$ denotes the Laplace variable.[1] Upon considering the initial conditions to be zero, this formula reduces to

$$L\left\{\frac{\mathrm{d}^\alpha f(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha L\{f(t)\}\,. \qquad (4)$$

On the other hand, the fractional integral operator of order "$\alpha$" can be represented by the transfer function $F(s) = \dfrac{1}{s^\alpha}$ in the frequency domain.

The standard definitions of fractional differintegral do not allow direct implementation of the fractional operators in time-domain simulations. An effective method to circumvent this problem is to approximate fractional operators by using standard integer-order operators. In Ref.[22], an effective algorithm is developed to approximate fractional-order transfer functions. Basically, the idea is to approximate the system behaviour in the frequency domain, using a specified error in decibels and a bandwidth to generate a continuous sequence of pole-zero pairs for the system with a single fractional power pole. This approximation is created by choosing an initial break point; the line with slope of $-20\alpha$dB/decade is approximated by a number of zigzag lines connected together with slopes alternating between 0 dB/decade and $-20$ dB/decade.[22] By utilizing frequency-domain techniques based on Bode diagrams, one can obtain a linear approximation of a fractional-order integrator with any desired accuracy over any frequency band, the order of which depends on the desired bandwidth and discrepancy between the actual and the approximate magnitudes of the corresponding Bode diagrams. This approximation has been adopted in Refs.[9–12, 17–20] and has sufficient accuracy for time-domain implementations. In Table 1 of Ref.[9], approximations for $1/s^\alpha$ with $\alpha = 0.1 - 0.9$ in step size 0.1 are given with errors of approximately 2dB. We will use these approximations in the following simulations.

## 3. Chaotic dynamics of the fractional-order Ikeda delay system

The mathematical model developed by Ikeda and Matsumoto[23] is a one-order, autonomous, ordinary, and differential-delay equation in the form:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -bx(t) + a\sin[cx(t-\tau)], \qquad (5)$$

where $(a,\ b,\ c) \in R^3$ are system parameters, and $\tau$ is the time delay. The equation contains a linear decay term that results in an exponential relaxation of the system state, and a nonlinear delay production term that results in complicated dynamics. When $(a,\ b,\ c) = (24,\ 3,\ 1)$ and $\tau = 0.2$, in the system (5) there exists a chaotic attractor shown in Fig.1.

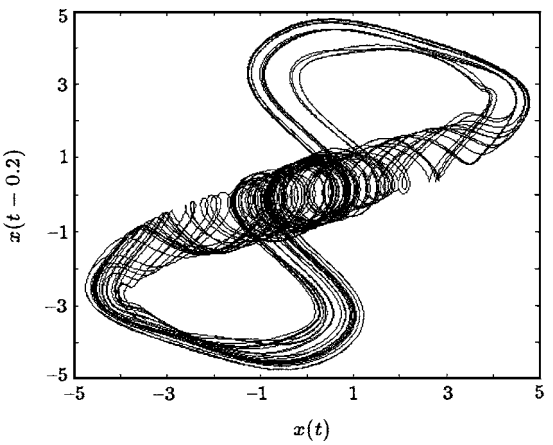



**Fig.1.** Chaotic attractor of the integer-order Ikeda delay system (5).

Now, we consider the fractional-order Ikeda delay system

$$\frac{\mathrm{d}^q x(t)}{\mathrm{d}t^q} = -bx(t) + a\sin[cx(t-\tau)], \qquad (6)$$

where $q$ is the fractional order satisfying $0 < q \le 1$. In the following simulations, we only vary the derivative order $q$ and parameter $b$, the other system parameters are fixed.

Simulations are performed for $q = 0.9 - 0.1$ in step size 0.1. The simulation results demonstrate that chaos indeed exists in the fractional-order Ikeda delay system (6) with order less than 1. When $q = 0.9 - 0.1$ in step size 0.1, chaotic attractors are found and the phase portraits are shown in Figs.2–10, separately. The largest Lyapunov exponents $\lambda_{\max}$ of the simulation time series are calculated and shown in Table 1.

When $q = 0$, no chaotic behaviour is found, which indicates that the lowest limit of the fractional order $q$ for this fractional-order Ikeda delay system (6) to be chaotic is $q = 0 - 0.1$. Through careful simulations, the lowest order for this fractional-order Ikeda delay system to yield chaos is found to be 0.1.

**Table 1.** The largest Lyapunov exponent $\lambda_{\max}$.

| $q$ | $b$ | $\lambda_{\max}$ |
|---|---|---|
| 0.9 | 3 | 0.0011 |
| 0.8 | 3 | 0.0017 |
| 0.7 | 3 | 0.002 |
| 0.6 | 3 | 0.0027 |
| 0.5 | 3 | 0.0030 |
| 0.4 | 1 | 0.022 |
| 0.3 | 1 | 0.032 |
| 0.2 | 1 | 0.041 |
| 0.1 | 1 | 0.32 |

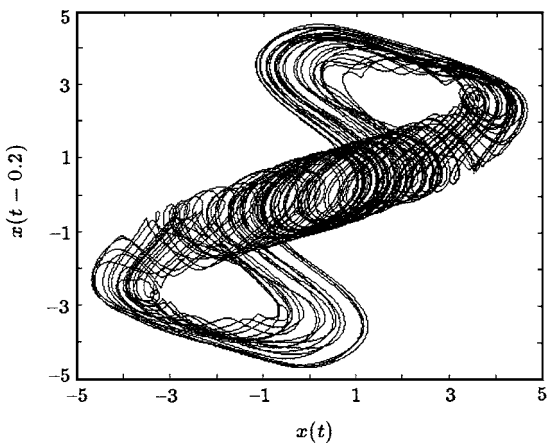



**Fig.2.** Chaotic attractor of the fractional-order Ikeda delay system (6) with $q = 0.9$, $b = 3$.

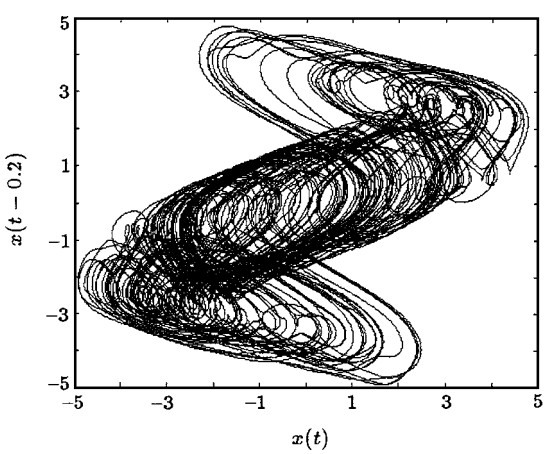



**Fig.3.** Chaotic attractor of the fractional-order Ikeda delay system (6) with $q = 0.8$, $b = 3$.

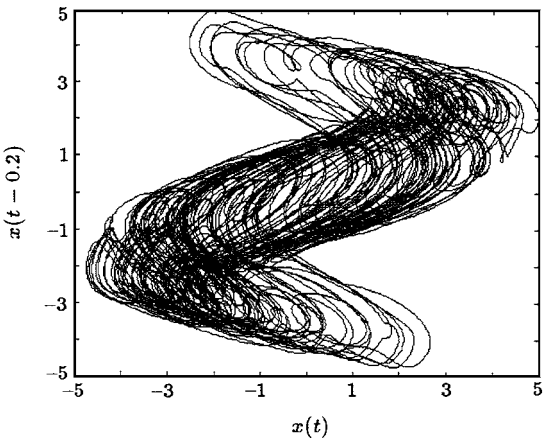



**Fig.4.** Chaotic attractor of the fractional-order Ikeda delay system (6) with $q = 0.7$, $b = 3$.

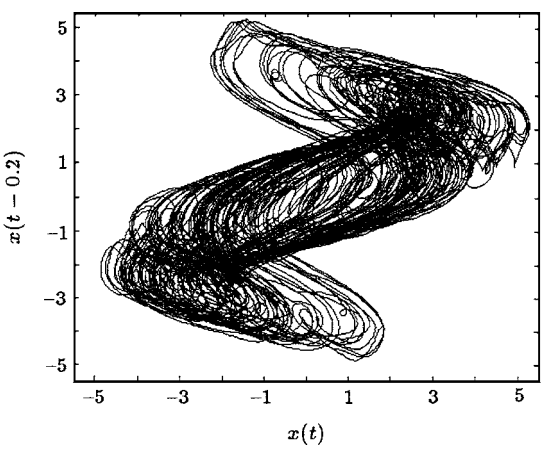



**Fig.5.** Chaotic attractor of the fractional-order Ikeda delay system (6) with $q = 0.6$, $b = 3$.

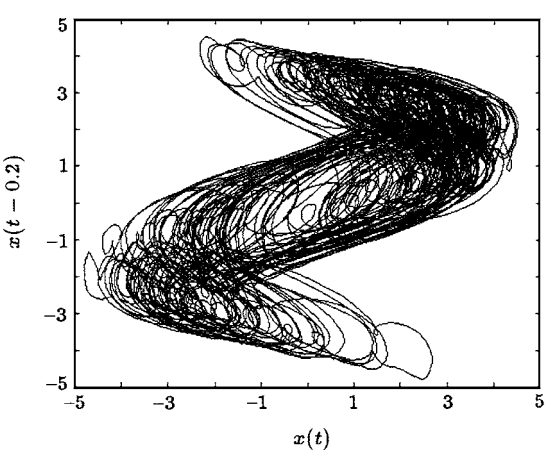



**Fig.6.** Chaotic attractor of the fractional-order Ikeda delay system (6) with $q = 0.5$, $b = 3$.

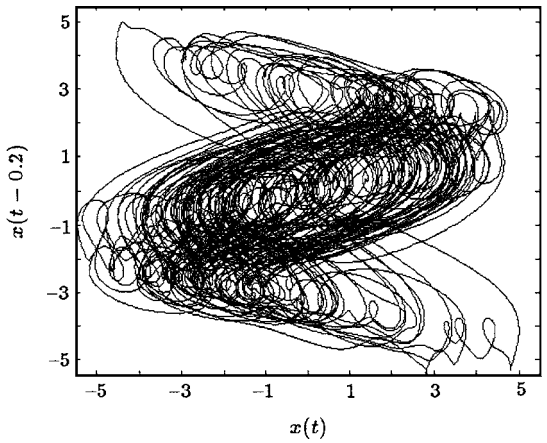



**Fig.7.** Chaotic attractor of the fractional-order Ikeda delay system (6) with $q = 0.4$, $b = 1$.

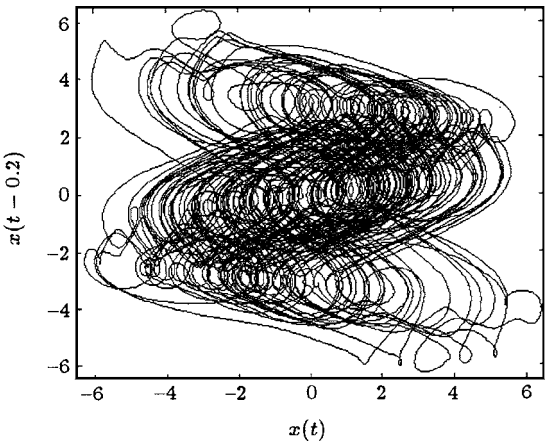



**Fig.9.** Chaotic attractor of the fractional-order Ikeda delay system (6) with $q = 0.2$, $b = 1$.

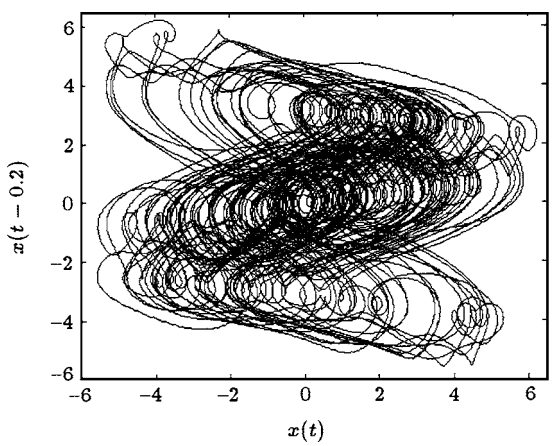



**Fig.8.** Chaotic attractor of the fractional-order Ikeda delay system (6) with $q = 0.3$, $b = 1$.

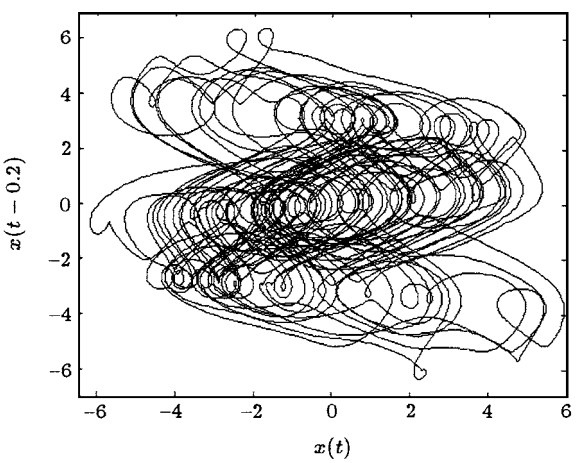



**Fig.10.** Chaotic attractor of the fractional-order Ikeda delay system (6) with $q = 0.1$, $b = 1$.

# 4. Synchronization of chaotic fractional-order Ikeda delay systems

Chaos synchronization and its potential application to secure communications have attracted much attention from various disciplines in science and engineering[24,25] since the pioneering work of Pecora and Carroll.[26] In this paper, we briefly discuss the chaotic synchronization issue between the chaotic fractional-order Ikeda delay systems via master–slave configuration with linear coupling. Now, we write the master fractional-order Ikeda delay system as the following form:

$$\frac{\mathrm{d}^q x(t)}{\mathrm{d}t^q} = -bx(t) + a\sin[cx(t-\tau)]. \qquad (7)$$

And the coupled Slave fractional-order Ikeda delay system has the following form:

$$\frac{\mathrm{d}^q \bar{x}(t)}{\mathrm{d}t^q} = -b\bar{x}(t) + a\sin[c\bar{x}(t-\tau)] + k(x-\bar{x}). \qquad (8)$$

The constant parameter $k$ is the coupling strength. When $q = 0.9$ and $(a,\ b,\ c) = (24,\ 3,\ 1)$, the fractional-order Ikeda delay system (7) is chaotic as shown in Fig.2. To obtain a value of $k$ for making the two chaotic fractional-order Ikeda delay systems synchronized, we continuously increase the coupling strength $k$ from $k = 0$, in step size 1. When $k = 9$, the two coupled chaotic fractional-order Ikeda delay systems can be synchronized. Simulation results are shown in Fig.11 and the coupling starts at $t = 5$. As we can see from Fig.11, the designed chaotic synchronization method can effectively synchronize the chaotic fractional-order Ikeda delay systems. Similar chaotic synchronization results have also been found in the cases of $q = 0.1 - 0.8$, but they are omitted

here.

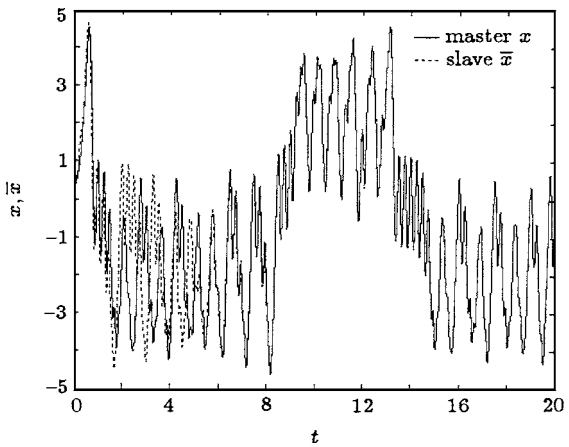



**Fig.11.** Synchronization of the chaotic fractional-order Ikeda delay systems (7) and (8) with $q = 0.9$, and $(a,\ b,\ c) = (24,\ 3,\ 1)$, the coupling starts at $t = 5$.

## 5. Conclusion

In this paper, we have numerically studied the chaotic behaviours in a fractional-order Ikeda delay system. We find that chaos exists in the fractional-order Ikeda delay dynamical system with order as low as 0.1. A simple, but effective, linear coupling Master-Slave synchronization method has also been designed to synchronize the chaotic fractional-order Ikeda delay systems.

It is important to systematically develop some analytical methods of studying bifurcation and chaos in fractional-order systems from a frequency domain approach,[27,28] because the time-domain fractional derivatives still lack physical and geometrical interpretations, despite some effort devoted to this attempt.[21,29] Future research topics include systematically developing some analytical methods of studying bifurcation and chaos in fractional-order systems from a frequency domain approach, as well as control and synchronization of general fractional-order systems.

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