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Tunnelling effect from a Vaidya–de Sitter black hole*

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In this paper, we extend Parikh' recent work to the Vaidya–de Sitter black hole which is non-stationary. We view Hawking radiation as a tunnelling process across the event horizon and calculate the tunnelling probability when the particle crosses the event horizon. From the tunnelling probability we also find a leading correction to the semiclassical emission rate.

Keywords: tunnelling effect, Bondi mass, event horizon
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1. Introduction

Steven Hawking made a striking discovery that basic principles of quantum field theory lead to the emission of thermal radiation from a classical black hole[1] in 1973, which gives rise to a famous paradox – the information loss paradox of black hole physics. When Hawking first proved the existence of black hole radiation, he described it as the tunnelling triggered by vacuum fluctuations near the horizon. But, actual derivation of Hawking radiation did not proceed in this way at all and there seemed not to be any barrier.

Recently, a method describing the Hawking radiation as a tunnelling process, where a particle moves in dynamic geometry, has been developed by Kraus and Wilczek[2] and elaborated by Parikh and Wilczek.[3−5] They have taken the self-interaction effect into account and given a leading correction to the emission rate arising from loss of mass of the black hole corresponding to the energy carried by the radiated quantum. Following this method, Hemming and Keski-Vakkuri have investigated the radiation from AdS black holes,[6] and Medved has studied those from a de Sitter cosmology.[7] In all these investigations, however, all the black holes are static and spherically symmetric. In this paper, we will extend the investigation to a non-stationary black hole: the Vaidya–de Sitter black hole. The Vaidya–de Sitter black hole radiates energy continuously and does not have a time-like Killing vector field. So there is a problem how to use the condition of the energy conservation. In quantum mechanics, particle tunnelling a barrier is an instantaneous process, so, the metric in the coordinates should satisfy Landau's coordinate clock synchronization condition.[8] Although not all the non-stationary black holes satisfy Landau's condition, the Vaidya–de Sitter black hole does satisfy it.[9] It is the Landau's coordinate clock synchronization condition that makes it possible to generalize Parikh's work to the non-stationary cases. It is not very important whether the spacetime has a time-like Killing vector field or not. Of course, if the time-like Killing vector field exists, one will find a coordinate system which satisfies the Landau's coordinate clock synchronization condition at least. So, we could fix a certain Bondi time v' and at this moment of v' the Bondi energy of the Vaidya–de Sitter black hole is conserved in the tunnelling process. The result is successful and we can conclude that the tunnelling rate is the exponent of the difference in Bekenstein–Hawking entropies, ΔS, before and after emission, which is in agreement with the stationary cases. We believe that our result is reasonable. We emphasize that although the calculation has been carried out at the moment v', v' can be chosen arbitrarily. Hence our results reveal the dynamical change of the non-stationary black hole. Throughout the paper, the units $G = c = \hbar = k_{\rm B} = 1$ are used.

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2. New coordinates

The Vaidya–de Sitter metric is given as

$$\begin{aligned} \mathrm{d}s^2 =& -\left[1-\frac{2m(v)}{r}-\frac{1}{3}\Lambda r^2\right]\mathrm{d}v^2 \\ &+2\mathrm{d}v\mathrm{d}r+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\right), \end{aligned} \tag{1}$$

where Λ is the cosmological constant. In order to obtain the equation of the event horizon, we use the null surface equation

$$g^{\mu\upsilon}\frac{\partial G}{\partial x^\mu}\frac{\partial G}{\partial x^\upsilon}=0, \tag{2}$$

where $G=G(r,v)$ because of the symmetry of the spacetime. From Eq.(2), the equation of the event horizon r_{H} satisfies

$$1-2\dot{r}_{\mathrm{H}}-\frac{2m(v)}{r_{\mathrm{H}}}-\frac{1}{3}\Lambda r_{\mathrm{H}}^2=0, \tag{3}$$

where $\dot{r}_{\mathrm{H}}=\dfrac{\mathrm{d}r_{\mathrm{H}}}{\mathrm{d}v}$. One solution of Eq.(3) is the event horizon

$$r_{\mathrm{H}}=2\sqrt{\frac{1-2\dot{r}_{\mathrm{H}}}{\Lambda}}\cos\left(\alpha_1+\frac{\pi}{3}\right); \tag{4}$$

another is the universal horizon

$$r_{\mathrm{C}}=2\sqrt{\frac{1-2\dot{r}_C}{\Lambda}}\cos\left(\alpha_2-\frac{\pi}{3}\right). \tag{5}$$

From metric (1), we find that the event horizon and the time-like limit surface are not located at the same place, which causes difficulties of calculating the tunnelling rate. In order to study the tunnelling effect, we define new coordinates, $R=r-r_{\mathrm{H}}(v)$ and $V=v$. Then we have

$$\begin{aligned} \mathrm{d}R =& \mathrm{d}r-\dot{r}_{\mathrm{H}}\mathrm{d}v, \mathrm{d}V=\mathrm{d}v, \\ \frac{\partial}{\partial R} =& \frac{\partial}{\partial r}, \frac{\partial}{\partial V}=\frac{\partial}{\partial v}+\dot{r}_{\mathrm{H}}\frac{\partial}{\partial r}, \end{aligned} \tag{6}$$

where $\dot{r}_{\mathrm{H}}=\dfrac{\mathrm{d}r_{\mathrm{H}}}{\mathrm{d}v}$. The essentiality and the merit of the new coordinates will be explained in Sec.4. So the line element (1) can be rewritten as

$$\begin{aligned} \mathrm{d}s^2 =& -\left[1-\frac{2m(v)}{r}-2\dot{r}_{\mathrm{H}}-\frac{1}{3}\Lambda r^2\right]\mathrm{d}V^2 \\ &+2\mathrm{d}V\mathrm{d}R+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\right) \\ =& \tilde{g}_{00}\mathrm{d}V^2+2\tilde{g}_{01}\mathrm{d}V\mathrm{d}R+\tilde{g}_{22}\mathrm{d}\theta^2+\tilde{g}_{33}\mathrm{d}\varphi^2. \end{aligned} \tag{7}$$

In the new coordinates, from $\tilde{g}_{00}=0$ we could obtain the equation of event horizon.

3. Landau's coordinate clock synchronization condition

Before calculating the tunnelling probability, we will verify that the metric in line element (7) satisfies Landau's coordinate clock synchronization condition in the following because particle tunnelling a barrier is an instantaneous process.

According to Landau's coordinate clock synchronization theory, in a spacetime decomposed into $3+1$, the difference between coordinate times of two events taking place simultaneously in different places is[8]

$$\Delta T=-\int\frac{\tilde{g}_{0i}}{\tilde{g}_{00}}\mathrm{d}x^i, \tag{8}$$

where $i=1,2,3$. If the simultaneity of coordinate clocks could be transmitted from one place to another and has nothing to do with the integration path, the components of the metric should satisfy[9]

$$\begin{aligned} \frac{\partial}{\partial x^j}\left(\frac{\tilde{g}_{0i}}{\tilde{g}_{00}}\right) =& \frac{\partial}{\partial x^i}\left(\frac{\tilde{g}_{0j}}{\tilde{g}_{00}}\right), \\ i,j =& 1,2,3. \end{aligned} \tag{9}$$

In this case, $x^1=R, x^2=\theta, x^3=\varphi$, thus we obtain

$$\begin{aligned} \frac{\partial}{\partial x^2}\left(\frac{\tilde{g}_{01}}{\tilde{g}_{00}}\right) =& \frac{\partial}{\partial x^1}\left(\frac{\tilde{g}_{02}}{\tilde{g}_{00}}\right)=\frac{\partial}{\partial x^3}\left(\frac{\tilde{g}_{01}}{\tilde{g}_{00}}\right) \\ =& \frac{\partial}{\partial x^1}\left(\frac{\tilde{g}_{03}}{\tilde{g}_{00}}\right)=0. \end{aligned} \tag{10}$$

That is to say, we could define the coordinate clock synchronization although the spacetime is non-stationary. This feature of the coordinates is very important for us to discuss the tunnelling process.

4. Tunnelling effect

To describe the across-horizon phenomena, it is necessary to choose coordinates which are not singular at the horizon; fortunately the line element (7) is not singular at the horizon. From $\mathrm{d}s^2=0$ in Eq.(7), the radial null geodesics are given by

$$\dot{R}\equiv\frac{\mathrm{d}R}{\mathrm{d}V}=\frac{1}{2}\left[1-\frac{2m(v)}{r}-\frac{1}{3}\Lambda r^2-2\dot{r}_{\mathrm{H}}\right], \tag{11}$$

corresponding to outgoing geodesics. These equations are modified when the particle's self-gravitation is taken into account. We could consider the particle as a shell of energy. We fix the total mass (Bondi mass) and allow the hole mass to fluctuate. When the shell of energy ω travels along the geodesics, we should replace $m(v')$ with $m(v')-\omega$ in the geodesic equation (11) to describe the movement of the shell.[5] However, there is a problem that in the non-stationary

spacetime the energy of a particle travelling along the geodesics is not conserved because the spacetime does not have a time-like Killing vector field. So the energy of the shell ω will vary when v increases towards the future. The crucial point is that the particle tunnelling through a barrier is an instantaneous process. Therefore we could fix a certain moment v' and the energy of the black hole is $m(v')$ at the moment v'. When the virtual particle with positive energy ω just inside the event horizon tunnels just outside the event horizon where it materializes as a real particle, its energy is still ω. At the same time the total energy of the black hole changes to $m(v')-\omega$ and the black hole shrinks a little. The line element (7) should be rewritten as

$$\begin{aligned} \mathrm{d}s^2 =& -\left\{1-\frac{2\left[m\left(v'\right)-\omega\right]}{r}-\frac{1}{3}\Lambda r^2-2\dot{r}_{\mathrm{H}}\right\}\mathrm{d}V^2 \\ &+2\mathrm{d}V\mathrm{d}R+r^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2). \end{aligned} \tag{12}$$

In our picture, a point particle description is appropriate. Because of the infinite blue shift near the horizon, the characteristic wavelength of any wave packet is always arbitrarily small there, so that the geometrical optics limit becomes an especially reliable approximation. Since we fix a certain moment v' to study the tunnelling process, the Wentzel–Kramers–Brillouin formula could be applied as in the static cases. We address that v describes the movement of the particle, while v' is fixed. The emission rate, Γ, is the square of the tunnelling amplitude.[4]

$$\Gamma\sim\exp\left(-2\mathrm{Im}S\right). \tag{13}$$

The imaginary part of the action for an outgoing positive energy particle which crosses the horizon outwards from R_{in} to R_{out} can be expressed as

$$\begin{aligned} \mathrm{Im}S =& \mathrm{Im}\int_{R_{\mathrm{in}}}^{R_{\mathrm{out}}}p_{\mathrm{R}}\mathrm{d}R \\ =& \mathrm{Im}\int_{R_{\mathrm{in}}}^{R_{\mathrm{out}}}\int_0^{p_{\mathrm{R}}}\mathrm{d}p'_{\mathrm{R}}\mathrm{d}R, \end{aligned} \tag{14}$$

where p_{R} is the canonical momentum conjugate to R. In the coordinates (V,R,θ,φ), $R_{\mathrm{in}}=r\mid_{r_{\mathrm{H}}}-r_{\mathrm{H}}=0$ is the initial radius of the black hole, and $R_{\mathrm{out}}=\tilde{r}_{\mathrm{H}}-r_{\mathrm{H}}=-\delta$ is the final radius of the hole, where $\tilde{r}_{\mathrm{H}}=r_{\mathrm{H}}\left[m\left(v'\right)-\omega\right]$. We substitute the Hamilton's equation $\dot{R}=\dfrac{\mathrm{d}H}{\mathrm{d}p_{\mathrm{R}}}\mid_{\mathrm{R}}$ into Eq.(14), change variable from momentum to energy, and switch the order of integration to obtain

$$\begin{aligned} \mathrm{Im}S =& \mathrm{Im}\int_m^{(m-\omega)}\int_{R_{\mathrm{in}}}^{R_{\mathrm{out}}}\frac{\mathrm{d}R}{\dot{R}}\mathrm{d}H \\ =& \mathrm{Im}\int_0^{\omega}\int_{R_{\mathrm{in}}}^{R_{\mathrm{out}}}\frac{2\mathrm{d}R}{1-\dfrac{2\left[m(v')-\omega'\right]}{r}-\dfrac{1}{3}\Lambda r^2-2\dot{r}_{\mathrm{H}}}\left(-\mathrm{d}\omega'\right). \end{aligned} \tag{15}$$

We have used the modified Eq.(11) and H is the Bondi energy of the black hole.[4] The radiative field does not radiate energy in the tunnelling process because a certain moment v' is fixed to study the tunnelling process, while the difference between the Bondi energies of the black hole at two different moments v_1 and v_2 is the energy radiated from the black hole during v_2-v_1. So we have $\mathrm{d}H=\mathrm{d}\left[m\left(v'\right)-\omega\right]=-\mathrm{d}\omega$ and the minus sign appears. Now let us explain why we must adopt the coordinate system (R,V,θ,φ). If we do not adopt the new coordinates, then Eq.(15) will be

$$\mathrm{Im}S=\mathrm{Im}\int_0^{\omega}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\frac{2\mathrm{d}r}{1-\dfrac{2\left[m(v')-\omega'\right]}{r}-\dfrac{1}{3}\Lambda r^2}\left(-\mathrm{d}\omega'\right). \tag{16}$$

The first-order pole is at $r_{\mathrm{TLS}}=2\Lambda^{-1/2}\cos\left(\alpha+\dfrac{\pi}{3}\right)$ which is the time-like limit surface, not the event horizon. This means that the particle tunnels out of the time-like limit surface, not the event horizon. It is well known that the Hawking radiation comes from the event horizon, not from the time-like limit surface.

In Eq.(15), $1-\frac{2\left[m(v')-\omega'\right]}{r}-2\dot{r}_{\rm H}-\frac{1}{3}\Lambda r^2$ equals zero at the event horizon. So we adopt the new coordinate system in order that the first-order pole is at $r=r'_{\rm H}$, where $r'_{\rm H}=r_{\rm H}\left(m\left(v'\right)-\omega'\right)$. In order to evaluate the integral, we treat r as a constant and make the following change of variables:[7]

$$x=\frac{2\left[m(v')-\omega'\right]}{r}+\frac{1}{3}\Lambda r^2-2\dot{r}_{\rm H}. \tag{17}$$

When $\dot{r}_{\rm H}$ is not a function of $m\left(v'\right)$, it leads to

$$\mathrm{Im}S=-\mathrm{Im}\int_{R_{\rm in}}^{R_{\rm out}}(R+r_{\rm H})\mathrm{d}R\int_{x(0)}^{x(\omega)}\frac{\mathrm{d}x}{x-1}, \tag{18}$$

where $x=1$ is the first-order pole. We integrate in a counterclockwise direction in the upper half of the complex x plane and obtain

$$2\mathrm{Im}S=-\pi\left(R_{\rm out}^2-R_{\rm in}^2\right)=-\pi\left(\tilde{r}_{\rm H}^2-r_{\rm H}^2\right). \tag{19}$$

As a further check on our formalism, we consider the change in entropy during the process of emission. The entropies of the black hole[11−13] before and after the emission are, respectively,

$$S_{\rm BH_1}=\frac{1}{4}A_{\rm H_1}=\pi r_{\rm H}^2, \tag{20}$$

$$S_{\rm BH_2}=\frac{1}{4}A_{\rm H_2}=\pi\tilde{r}_{\rm H}^2. \tag{21}$$

The tunnelling rate is therefore

$$\begin{aligned}\Gamma\sim\exp\left(-2\mathrm{Im}S\right)=&\exp\left[\pi\left(\tilde{r}_{\rm H}^2-r_{\rm H}^2\right)\right]\\=&\exp\left(\triangle S_{BH}\right).\end{aligned} \tag{22}$$

5. Conclusion and discussion

The tunnelling rate obtained in this paper is a function of time v' which describes the black hole itself. Moreover, when m is a constant, the tunnelling rate could return to the result for the static Schwarzschild–de Sitter black hole. Therefore our results reflect the dynamical change of the Vaidya–de Sitter black hole.

ΔS can be also written as $\Delta S=\frac{\mathrm{d}S}{\mathrm{d}m}\Delta m+\frac{1}{2}\frac{\mathrm{d}^2S}{\mathrm{d}m^2}(\Delta m)^2+\cdots$ The derivative of the entropy S with respect to the mass m is

$$\frac{\mathrm{d}S}{\mathrm{d}m}=\frac{\mathrm{d}}{\mathrm{d}m}\left(\pi r_{\rm H}^2\right)=2\pi r_{\rm H}\frac{\mathrm{d}r_{\rm H}}{\mathrm{d}m}. \tag{23}$$

From the equation of the event horizon $1-2\dot{r}_{\rm H}-\frac{2m}{r_{\rm H}}-\frac{1}{3}\Lambda r_{\rm H}^2=0$, we obtain

$$\frac{\mathrm{d}r_{\rm H}}{\mathrm{d}m}=\frac{r_{\rm H}}{m-\frac{1}{3}\Lambda r_{\rm H}^2}. \tag{24}$$

In the tunnelling process, Δm is just $-\omega$. When we neglect the quadratic term of ω, ΔS reduces to

$$\Delta S\approx\frac{2\pi r_{\rm H}^2}{m-\frac{1}{3}\Lambda r_{\rm H}^2}(-\omega)=-\beta\omega, \tag{25}$$

where $\beta=\frac{2\pi r_{\rm H}^2}{m-\frac{1}{3}\Lambda r_{\rm H}^2}$.[14,15] We then have the familiar result that the rate Γ is a Boltzmann factor $\exp(-\beta\omega)$. But note that at higher energies the spectrum cannot be approximated as being thermal. The ω^2 correction arises from the law of energy conservation. Moreover, the emission spectrum (22) seems to be consistent with an underlying unitary theory. The above discussion is based on the fact that $\dot{r}_{\rm H}$ is not a function of $m\left(v'\right)-\omega'$. If $\dot{r}_{\rm H}$ is a function of $m\left(v'\right)-\omega'$, the differential of Eq.(17) will become $\mathrm{d}x=-\frac{\mathrm{d}\omega'}{r}-2\frac{\mathrm{d}\dot{r}_{\rm H}}{\mathrm{d}\omega'}$, which results in a different result. This is also an open question.

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