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Linear optics implementation for quantum game under quantum noise*

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It has recently been shown that linear optics alone would suffice to implement efficient quantum computation. Quantum computation circuits using coherent states as the logical qubits can be constructed from very simple linear networks, conditional measurements and coherent superposition resource states. We present the quantum game under quantum noise and a proposal for implementing the noisy quantum game using only linear optics.

Keywords: entanglement, quantum game, linear optics
PACC: 0365, 4250

1. Introduction

Quantum information processing provides a variety of new capabilities in potentially significant performance over the conventional technique. Due to the powerful property of superposition states to execute large parallel processing, quantum computation has become a significant subject within quantum information theory. Motivated in part by potential applications in quantum computing,[1−4] a rapid growth of interest appears in the investigation of extending classical games to the quantum realm. Eisert *et al* first introduce quantum entanglement to the two-player prisoner's dilemma, and show that the game ceases to pose a dilemma if quantum strategies are allowed for.[5] The properties of the prisoner's dilemma game in different degrees of entanglement also have been presented in Ref.[6], and the investigations on multi-player quantum games,[7,8] multi-choice quantum games,[9] continuous-variable quantum game[10] are presented. Recently, quantum game with decoherence have been discussed.[11,12]

The prisoner's dilemma game has first been realized on a nuclear magnetic resonance (NMR) quantum computer.[13] However, if the players are separated far distant from each other, the NMR quantum computer is not the best proposal. As is known, optical realization is one of the most effective methods for quantum information processing. The important experimental implementations of quantum cryptography,[14] quantum teleportation,[15] quantum dense coding[16] are completed in quantum systems. And two proposals for optically realizing quantum game[17,18] are presented. In Ref.[17], the nonlinear Kerr medium is used in their proposal. In Ref.[18], the single photon's spatial and polarization degrees of freedom is utilized to act as qubits. However, the Kerr medium and the technology of single photon are all a bit difficult in practical experiment. But, recently Ralph *et al*[19] and Knill *et al*[20] showed that only linear optics can implement effective quantum computation. In this paper, we present a proposal for implementing the quantum prisoner's dilemma under quantum noise using only linear optics.

2. Quantum game with quantum noise

Let us briefly recall the scheme of the quantum game presented in Ref.[5]. Classically, there are two players and each player can choose two possible strategies: co-operate ($\widehat{C}$) and defect ($\hat{D}$). The payoff table for the players is shown in Table 1.

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Table 1. Payoff matrix for the prisoner's dilemma. The first entry in the parentheses denotes the payoff of Alice and the second that of Bob.

	Bob: $\hat{C}$	Bob: $\hat{D}$
Alice: $\hat{C}$	$(3,3)$	$(0,5)$
Alice: $\hat{D}$	$(5,0)$	$(1,1)$

From the table one can see that defecting gives a better payoff regardless of the other player's strategy, so it is a dominant strategy, and mutual defection is the Nash equilibrium. The well-known dilemma arises from the fact that both players would be better off with mutual co-operation, if this could be engineered. In the quantum version, each of the two players possesses a qubit $|C\rangle\,(|D\rangle)$. The qubits firstly are prepared in the initial state $|C\rangle\,|C\rangle$, then the entangling gate $\hat{J}$ acts on it. The initial state of the quantum game is $|\psi_i\rangle = \widehat{J}\,|CC\rangle$, here $\hat{J}$ is $\exp(i\gamma D \otimes D/2)$, with $\gamma \in [0, \pi/2]$ measuring the entanglement of the state $|\psi_i\rangle$:

$$|\psi_i\rangle = \cos\left(\frac{\gamma}{2}\right)|C\rangle|C\rangle + \mathrm{i}\sin\left(\frac{\gamma}{2}\right)|D\rangle|D\rangle. \quad (1)$$

Then the unitary $\hat{U}_A$ and $\hat{U}_B$ are chosen from 2-parameter set of unitary $2 \otimes 2$ matrices :

$$\hat{U}\,(\theta,\phi) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi}\cos\theta/2 & \sin\theta/2 \\ -\sin\theta/2 & \mathrm{e}^{-\mathrm{i}\phi}\cos\theta/2 \end{pmatrix} \quad (2)$$

for all $0 \le \theta \le \pi$ and $0 \le \phi \le \pi/2$; indeed, the strategies C and D can be associated with $U(0,0)$ and $U(\pi,0)$ respectively. Finally, the disentangling gate $\widehat{J}^{+}$ is carried out and the system is measured in the computational basis, giving rise to one of the four outcomes $|CC\rangle$, $|CD\rangle$, $|DC\rangle$, $|DD\rangle$. Alice's and Bob's payoffs are $\$_A = 3P_{CC} + 5P_{DC} + P_{DD}$ and $\$_B = 3P_{CC} + 5P_{CD} + P_{DD}$ respectively, where $P_{\sigma\sigma'} = \langle\sigma\sigma'|\,\rho\,|\sigma\sigma'\rangle$ is the probability of finding the state $|\sigma\sigma'\rangle$ (with $\sigma, \sigma' \in \{C, D\}\,; P_{\sigma\sigma'} = 1$).

In order to consider the effect of noise in the quantum game, the noise is introduced before and after the adoption of the appropriate strategies by the two players, as shown in Fig.1.

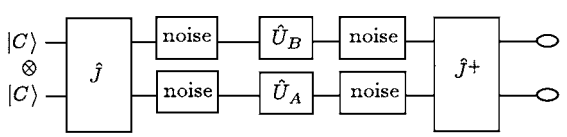


Fig.1. Quantum noise introduced before and after the adoption of the appropriate strategies by the quantum players, based on Eisert *et al*'s model.

In Ref.[11], Chen *et al* considered the situation of maximal entanglement. They showed that the Nash equilibria are not affected by the introduction of the noise, but Alice's payoff eventually decreases with increasing noise. Because the second decoherence channel occurs after the players have selected their strategies while the first channel acts on the state before the players have chosen their strategies, they have shown that the second channel seems to dominate the behaviour of the game. While, in the restricted quantum game in which one player is restricted to classical strategies, Flitney and Abbott[12] showed that under quantum noise, the loss of advantage to the quantum player is very similar to that occurring when the level of entanglement between the player's qubits is reduced.

In order to consider the quantum noise, the general way to understand the behaviour of one part of a principal system is to extend the system to a larger one (in which the environment is incorporated). So the evolution of the whole system becomes unitary. Then we perform a partial trace over the environment to obtain the reduced state of the principal system alone. The quantum operator-sum representation is a general tool to describe the process. Amplitude damping is one of the most subtle and important processes in the study of the quantum computation and quantum information. In terms of the operator-sum representation, we have

$$\rho' = \sum_{j=0}^{1} \mathcal{E}_j\,\rho\,\mathcal{E}_j^{\dagger}, \quad (3)$$

where

$$\mathcal{E}_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix},\ \mathcal{E}_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}, \quad (4)$$

and $p = \sin^2\theta$, ρ is the density matrix of the system. The circuit model for amplitude damping is shown in Fig.2. In next section, we will give the optical realization of noisy quantum game.

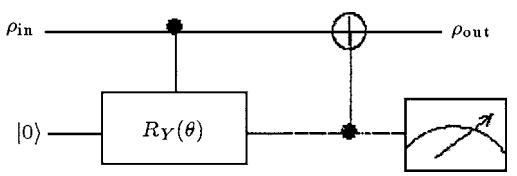


Fig.2. Circuit model for amplitude damping.

3. Proposal for optical realization of the noisy quantum game

We encode the qubits $|C\rangle$ ($|D\rangle$) in the form $|C\rangle = |0\rangle$, and $|D\rangle = |\alpha\rangle$, where α is real. The advantage of using such a state is that the detection of it is relatively easy, requiring only efficient homodyne detection. These qubits are approximately orthogonal, assuming that $\alpha \gg 1$. Ralph *et al* [19] have shown that quantum computation circuits using coherent states as the logical qubits can be constructed from very simple linear networks, conditional measurements and coherent superposition resource states. A beam splitter can be described by a unitary operator in the form

$$U_{BS} = \exp[\mathrm{i}\theta(ab^{\dagger} + a^{\dagger}b)], \tag{5}$$

where a and b are the annihilation operators corresponding to two coherent state qubits $|\gamma\rangle_a$ and $|\beta\rangle_b$, respectively, with γ and β taking values of α or 0. It is well known that the output state produced by such an interaction is

$$U_{BS}|\gamma\rangle_a|\beta\rangle_b = |\cos\theta\gamma + \mathrm{i}\sin\theta\beta\rangle_a|\cos\theta\beta + \mathrm{i}\sin\theta\gamma\rangle_b, \tag{6}$$

where $\cos^2\theta$ ($\sin^2\theta$) is the reflectivity (transmissivity) of the beam splitter. Under condition $\alpha \gg 1$, $\theta^2\alpha^2 << 1$, and $\theta\alpha^2 = \pi/2$, the BS can produce a controlled sign shift gate:

$$U_{BS}|\gamma\rangle_a|\beta\rangle_b \approx \exp[2\mathrm{i}\theta\gamma\beta]|\gamma\rangle_a|\beta\rangle_b. \tag{7}$$

That is, if one or both of the qubits are in the logical zero state, the transformation produces no effect on the state. However, if both modes are initially in the logical one state, then a sign change is produced. A bit flip gate, $X = R_Z(\pi)D(-\alpha)$, is physically just a halfwave length delay and a displacement. The latter can be implemented by mixing a very strong coherence field with the qubit on a highly reflective beam splitter. The phase rotation is just an operation of $R_Z(\beta)$. It can be implemented to a good approximation by imposing a small phase shift on the qubit. The Hadamard gate H induces the following transformation for the logical state:

$$\begin{aligned} H|C\rangle &= \frac{1}{\sqrt{2}}(|C\rangle + |D\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |\alpha\rangle), \\ H|D\rangle &= \frac{1}{\sqrt{2}}(|C\rangle - |D\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |\alpha\rangle). \end{aligned} \tag{8}$$

The arrangement for realizing the Hadamard gate is shown in Fig.3. The state we wish to transform is inserted into port 1 of the BS gate, while a resource cat state $\frac{1}{\sqrt{2}}(|0\rangle + |\alpha\rangle)$ is inserted into port 2. Then a cat-measurement on the output port 1 is made. If the result is the same cat state as the one inserted into port 1, then the state of output port 2 is the required Hadamard transformation. On the other hand, if it is "minus state" $\frac{1}{\sqrt{2}}(|0\rangle - |\alpha\rangle)$, a bit flip on the output port 1 is made.

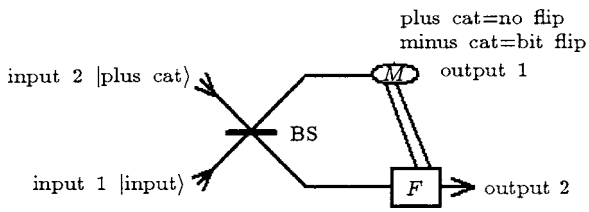


Fig.3. Schematic of Hadamard gate based on the two-qubit beam splitter (BS) gate and conditional implementation of the bit flip gate (F). If an even number of photons is counted, then the output is in the desired state. If an odd number of photons is counted, a bit flip operation is needed to place the output in the correct state.

The CNOT gate can be implemented as shown in Fig.4. Firstly, a Hadamard gate is applied to the target bit, then the BS gate is applied to the control bit and target bit. Finally, another Hadamard gate is applied to the target bit.

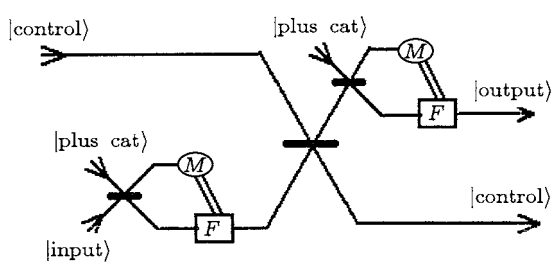


Fig.4. Schematic of CNOT gate based on two-qubit BS gates and conditional implementation of bit flip gates (F).

The rotation around the Y-axis by θ can be realized in the following form:

$$R_Y(\theta) = HR_Z(\pi/2)HR_Z(\theta)HR_Z\left(\frac{7\pi}{2}\right)H, \tag{9}$$

which is shown in Fig.5.

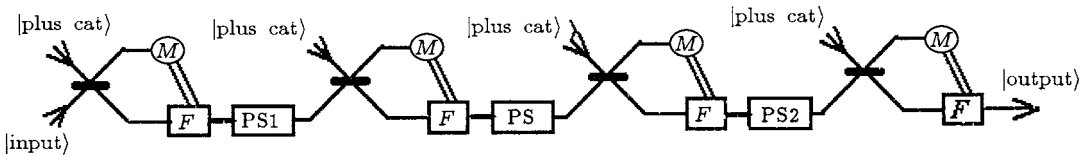


Fig.5. Schematic of rotations around the y axis by θ. PS1 is $R_Z\left(\frac{7\pi}{2}\right)$. PS2 is $R_Z(\pi/2)$. PS is $R_Z(\theta)$.

So the unitary operator of the player's strategy shown in Eq.(4) can be realized using the following identity:

$$\begin{aligned}\hat{U}(\theta,\phi) =& R_Z(\phi)R_Y(\theta)R_Z(\phi)\\ =& R_Z(\phi)HR_Z(\pi/2)HR_Z(\theta)\\ &\times HR_Z\left(\frac{7\pi}{2}\right)HR_Z(\phi). \qquad (10)\end{aligned}$$

In order to consider the effect of noise caused by the environment, we suppose an ancillary qubit $|0\rangle$ or $|\alpha\rangle$ to model the environment. So the circuit can also be decomposed into a CNOT operation and a rotation operation, as shown in Fig.6.

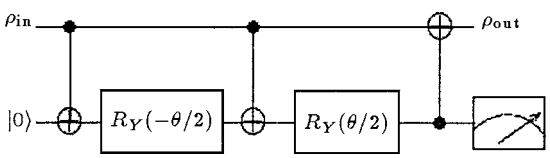


Fig.6. Circuit model for amplitude damping, which is decomposed into a CNOT operation and a rotation operation.

The entangling $\hat{J}$ gate is the most important operation in the quantum game since it introduces quantum entanglement. The transformation to realize the $\hat{J}$ gate can be expressed as the product of H gate transformation, the rotation around Z axis, controlled-Z gate (CZ gate) transformation and controlled-NOT gate transformation, which is shown in Fig.7.

The phase shift and the H gate operation, CZ gate and the CNOT gate have been discussed as mentioned above. As a direct test of the entangling $\hat{J}$ gate, let us consider the setup. First, the state $|CC\rangle$ passes through, when the first CZ gate and CNOT gate is applied, and the state is unchanged. Then H gate, phase shift and another H gate act on the first qubit. The phase shift represents the rotation around Z axis by $(-\gamma)$. Finally, CZ gate and CNOT gate are applied again. So the state evolves into a superposition state: $\cos\left(\frac{\gamma}{2}\right)|C\rangle|C\rangle + \mathrm{i}\sin\left(\frac{\gamma}{2}\right)|D\rangle|D\rangle$, where the parameter γ measures the entanglement in the game's initial state. With the realization of $\hat{J}^+$ gate, the only difference in Fig.7 is the phase shift which is a rotation around Z by γ.

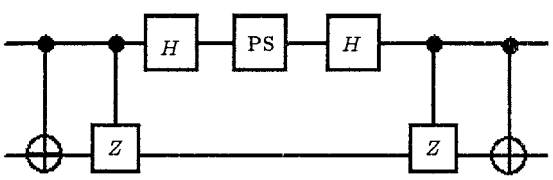


Fig.7. A circuit notation for $\hat{J}$ gate. PS is $R_Z(\gamma)$.

4. Conclusion

In this paper we have presented a quantum game with quantum noise and the linear optical scheme to realize the game based on encoding qubits as vacuum and coherent states, and their superposition. The operations we have used in this paper are all basic quantum gate operation. So this scheme is not only for linear optical but also for any setup that can be a quantum computer. With the current technology, our scheme is at the reach of current experiments. We also believe that the practical realization of quantum game may provide further insight into the study of quantum networks.

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