Vol 15 No 2, February 2006
1009-1963/2006/15(02)/0249-04

Chinese Physics

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Variational principle and dynamical equations of discrete nonconservative holonomic systems*

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(Received 19 November 2004; revised manuscript received 1 August 2005)

By analogue with the methods and processes in continuous mechanics, a Lagrangian formulation and a Hamiltonian formulation of discrete mechanics are obtained. The dynamical equations including Euler–Lagrange equations and Hamilton's canonical equations of the discrete nonconservative holonomic systems are derived on a discrete variational principle. Some illustrative examples are also given.

Keywords: discrete mechanics, variational principle, dynamical equation
PACC: 0320, 0220

1. Introduction

There are various circumstances in mechanics where it is convenient or necessary to replace a continuous time (temporal evolution) parameter with a discrete parameter. Computer simulation of waves is an example where the configuration of a system at time $t+h$ is calculated from a knowledge of its configuration at times t and $t-h$. At some unimaginably small scale, time is really discrete. For example, in modern theories such as string theory and quantum gravity theory, the Planck time of 10^{-43} seconds sets a scale at which conventional notions of space and time break down. There have been various attempts to construct classical and quantum mechanical theories of discrete time systems based on this notion. Cadzow[1] has derived the Euler–Lagrange equations of discrete conservative systems by using a variational principle. Marsden and West[2] have given a review of discrete variational principle and integration algorithms for conservative systems. Logan[3] has discussed the first integrals of discrete conservative systems. Ikeda and Maeda[4] have described the symmetries of the discrete natural mechanical systems. Jaroszkiewicz[5,6] has discussed the principles in discrete classic and quantum mechanics.

In this paper, we mimic the methods and processes in continuous mechanics to obtain a Lagrangian formulation and a Hamiltonian formulation of discrete mechanics based on a discrete variational principle. The main topics are the discrete variational principle, Euler–Lagrange equations and canonical equations of the discrete nonconservative holonomic systems. In Section 2, we begin by introducing the discrete variants, discrete Lagrangian, discrete action, and then obtain the discrete variational principle from the continuous one. In Section 3, we apply the obtained principle to the discrete action to derive the Euler–Lagrange equations of nonconservative holonomic systems. By introducing discrete Legendre transformations, we can transform the Lagrangian formulation into the Hamiltonian formulation.

2. The discrete variational principle of nonconservative holonomic systems

Consider a nonconservative holonomic system with Lagrangian being $L(q^i, \dot{q}^i)$. We work in R^n or in generalized coordinates and will use vector notion for simplicity, so $q^i = (q^1, q^2, ..., q^n)$. In the continuous case, the Hamilton's principle of the nonconservative holonomic system is written in the form[7]

$$\int_{t_0}^{t_\mathrm{f}} (\delta L + Q_i' \delta q^i) \mathrm{d}t = 0, \tag{1}$$

*Project supported by the National Natural Science Foundation of China (Grant No 19572018).

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where $L = T - V, Q_i = Q_i'' + Q_i'$, and $Q_i'' = -\dfrac{\partial V}{\partial q^i}$ are the generalized conservative forces and Q_i' are the generalized nonconservative forces. From Eq.(1) the Euler–Lagrange equations can be derived

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} = Q_i'. \tag{2}$$

In the following, we will see how to make an analogue with the above process for the discrete system. In our version of discrete mechanics we postulate that the dynamical variables $q^i(t)$ are observed or sampled at a finite number of times $t_k, k = 0, 1, \cdots, N$, such that the intervals are all equal to some fundamental interval h. For convenience we write $q_k^i \equiv q^i(t_k)$ and assume $h = 1$.

Now, consider two positions q_k^i and q_{k+1}^i rather than a position $q_k^i$and velocity $\dot{q}_k^i$. These positions should be thought of as being two points on a curve at time h apart ($h = 1$).

We now introduce a discrete Lagrangian $L_{d,k}(q_k^i, q_{k+1}^i)$, which we think of as approximating the action integral along the curve segment between q_k^i and q_{k+1}^i. Specifically, consider a very simple approximation to the integral $\int_{t_0}^{t_f} L\mathrm{d}t$ given by using the rectangle rule (the length h of the interval by the value of the integrand with the velocity vector replaced by $q_{k+1}^i - q_k^i$).

Next consider a discrete curve of points $(q_k^i)_{k=0}^N$ and calculate the discrete action in this sequence by summing the discrete Lagrangian on each adjacent pair. We obtain the discrete action sum[8]

$$J_d = \sum_{k=0}^{N-1} L_{d,k}(q_k^i, q_{k+1}^i). \tag{3}$$

Using the rectangle rule, the principle (1) can be written in the form[9−11]

$$\delta J_d + \sum_{k=0}^{N-1} Q_{i,k}'\delta q_k^i = 0. \tag{4}$$

Formula (4) is called the variational principle of the discrete nonconservative holonomic systems.

Remark The variational principle (4) is not a stable action principle because generally the left-hand side of Eq.(4) cannot be written as the variation of a function, but here we still can used it to derive the equations of motion of the systems.

3. Dynamical equations of discrete nonconservative holonomic systems

3.1. The discrete Euler–Lagrange equations

Proposition 1 For the discrete nonconservative holonomic systems, the discrete Euler–Lagrange equations can be written as follows:

$$\begin{aligned}&\frac{\partial}{\partial q_k^i}L_{d,k-1}(q_{k-1}^i, q_k^i) + \frac{\partial}{\partial q_k^i}L_{d,k}(q_k^i, q_{k+1}^i)\\ &+ Q_{i,k}' = 0, \quad (k = 1, ..., N-1),\end{aligned} \tag{5}$$

where $L_{d,k-1} = L_{d,k-1}(q_{k-1}^i, q_k^i)$.

Proof We compute the variation of the action sum with the boundary points q_0^i and q_N^i fixed, and obtain

$$\begin{aligned}\delta J_d =& \delta\sum_{k=0}^{N-1} L_{d,k}(q_k^i, q_{k+1}^i)\\ =& \sum_{k=0}^{N-1}\left[\frac{\partial}{\partial q_k^i}L_{d,k}(q_k^i, q_{k+1}^i)\delta q_k^i\right.\\ &\left.+ \frac{\partial}{\partial q_{k+1}^i}L_{d,k}(q_k^i, q_{k+1}^i)\delta q_{k+1}^i\right]\\ =& \sum_{k=1}^{N-1}\left[\frac{\partial}{\partial q_k^i}L_{d,k-1}(q_{k-1}^i, q_k^i)\right.\\ &\left.+ \frac{\partial}{\partial q_k^i}L_{d,k}(q_k^i, q_{k+1}^i)\right]\delta q_k^i,\end{aligned} \tag{6}$$

where we have used a discrete integration by part and the fact that $\delta q_0^i = \delta q_N^i = 0$. Using Eqs.(4) and (6) yields the discrete Euler–Lagrange equations (5).

If we define the discrete velocities in the form

$$v_k^i = q_{k+1}^i - q_k^i, \tag{7}$$

then the discrete Lagrangian $L_{d,k}$ and formulas (3) and (4) can be respectively rewritten as

$$L_{d,k} = L_{d,k}(q_k^i, v_k^i), \tag{8}$$

$$J_d = \sum_{k=0}^{N-1} L_{d,k}(q_k^i, v_k^i), \tag{9}$$

$$\delta \sum_{k=0}^{N-1} L_{d,k}(q_k^i, v_k^i) + \sum_{k=0}^{N-1} Q'_{i,k}\delta q_k^i = 0. \qquad (10)$$

Proposition 2 For the discrete nonconservative holonomic systems with Lagrangian shown in Eq.(8), the discrete Euler–Lagrange equations can be written in the form

$$\frac{\partial L_{d,k}}{\partial v_k^i} - \frac{\partial L_{d,k-1}}{\partial v_{k-1}^i} - \frac{\partial L_{d,k}}{\partial q_k^i} = Q'_{i,k}, \qquad (11)$$

where $L_{d,k-1} = L_{d,k-1}(q_k^i, v_{k-1}^i)$.

Proof Computing the variation of the action sum gives

$$\begin{aligned}
&\sum_{k=0}^{N-1}\left(\frac{\partial L_{d,k}}{\partial q_k^i}\delta q_k^i + \frac{\partial L_{d,k}}{\partial v_k^i}\delta v_k^i\right)\\
&= \sum_{k=0}^{N-1}\frac{\partial L_{d,k}}{\partial q_k^i}\delta q_k^i + \sum_{k=0}^{N-1}\frac{\partial L_{d,k}}{\partial q_k^i}\delta q_{k+1}^i - \sum_{k=0}^{N-1}\frac{\partial L_{d,k}}{\partial q_k^i}\delta q_k^i\\
&= \sum_{k=1}^{N-1}\left(\frac{\partial L_{d,k}}{\partial q_k^i} + \frac{\partial L_{d,k-1}}{\partial v_k^i} - \frac{\partial L_{d,k}}{\partial v_k^i}\right)\delta q_k^i. \qquad (12)
\end{aligned}$$

The Euler–Lagrange equations (11) can be derived from formulas (10) and (12). We have used a discrete integration by part and the fact that $\delta q_0^i = \delta q_N^i = 0$.

Example 1 Consider a nonconservative holonomic system with the Lagrangian

$$\begin{cases} L_{d,k}(q_k^i, q_{k+1}^i) = \dfrac{1}{2}[(q_{k+1}^1 - q_k^1)^2 + (q_{k+1}^2 - q_k^2)^2] + \mu[(q_k^1)^2 + (q_k^2)^2]^{-1/2}, \\ L_{d,k-1}(q_{k-1}^i, q_k^i) = \dfrac{1}{2}[(q_k^1 - q_{k-1}^1)^2 + (q_k^2 - q_{k-1}^2)^2] + \mu[(q_{k-1}^1)^2 + (q_{k-1}^2)^2]^{-1/2}, \end{cases} \qquad (13)$$

$$\begin{cases} L_{d,k}(q_k^i, v_k^i) = \dfrac{1}{2}[(v_k^1)^2 + (v_k^2)^2] + \mu[(q_k^1)^2 + (q_k^2)^2]^{-1/2}, \\ L_{d,k-1}(q_k^i, v_{k-1}^i) = \dfrac{1}{2}[(v_{k-1}^1)^2 + (v_{k-1}^2)^2] + \mu[(q_k^1)^2 + (q_k^2)^2]^{-1/2}, \qquad [(q_k^1)^2 + (q_k^2)^2] \neq 0. \end{cases} \qquad (14)$$

The nonconservative forces are given by

$$Q_{1,k} = -c_1(q_{k+1}^1 - q_k^1), \quad Q_{2,k} = -c_2(q_{k+1}^2 - q_k^2). \qquad (15)$$

We have

$$\begin{cases} \dfrac{\partial}{\partial q_k^1}L_{d,k-1}(q_{k-1}^i, q_k^i) = (q_k^1 - q_{k-1}^1), \\ \dfrac{\partial}{\partial q_k^1}L_{d,k}(q_k^i, q_{k+1}^i) = -(q_{k+1}^1 - q_k^1) - \mu q_k^1[(q_k^1)^2 + (q_k^2)^2]^{-3/2}, \\ \dfrac{\partial}{\partial q_k^2}L_{d,k-1}(q_{k-1}^i, q_k^i) = (q_k^2 - q_{k-1}^2), \\ \dfrac{\partial}{\partial q_k^2}L_{d,k}(q_k^i, q_{k+1}^i) = -(q_{k+1}^2 - q_k^2) - \mu[(q_k^1)^2 + (q_k^2)^2]^{-3/2}, \end{cases} \qquad (16)$$

$$\begin{cases} \dfrac{\partial L_{d,k}}{\partial v_k^1} = v_k^1, \quad \dfrac{\partial L_{d,k-1}}{\partial v_{k-1}^1} = v_{k-1}^1, \\ \dfrac{\partial L_{d,k}}{\partial q_k^1} = -\mu q_k^1[(q_k^1)^2 + (q_k^2)^2]^{-3/2}, \\ \dfrac{\partial L_{d,k}}{\partial v_k^2} = v_k^2, \quad \dfrac{\partial L_{d,k-1}}{\partial v_{k-1}^2} = v_{k-1}^2, \\ \dfrac{\partial L_{d,k}}{\partial q_k^2} = -\mu q_k^2[(q_k^1)^2 + (q_k^2)^2]^{-3/2}. \end{cases} \qquad (17)$$

Substituting Eqs.(15) and (16) into Eq.(5) or substituting Eqs.(15) and (17) into Eq.(11) we obtain the same equations of motion as follows

$$\begin{cases} (q_{k+1}^1 - 2q_k^1 + q_{k-1}^1) = -\mu q_k^1[(q_k^1)^2 + (q_k^2)^2]^{-3/2} + c_1(q_{k+1}^1 - q_k^1), \\ (q_{k+1}^2 - 2q_k^2 + q_{k-1}^2) = -\mu q_k^2[(q_k^1)^2 + (q_k^2)^2]^{-3/2} + c_2(q_{k+1}^2 - q_k^2). \end{cases} \qquad (18)$$

3.2. The discrete Legendre transformation and discrete canonical equations

Now we introduce the Legendre transformation

$$L_{d,k}(q_k^i, v_k^i) = p_{i,k}v_k^i - H_{d,k}(q_k^i, p_k^i), \tag{19}$$

where $H_{d,k}$ is called the discrete Hamiltonian and we assume $\det(\partial^2 L_{d,k}/\partial v_k^i \partial v_k^j) \neq 0$.

From Eq.(19), we can obtain

$$p_{i,k} = \frac{\partial L_{d,k}}{\partial v_k^i}. \tag{20}$$

Proposition 3 Under the Legendre transformation (19), the discrete Euler–Lagrange equations (11) can be transformed into the discrete canonical equations

$$\begin{cases} q_{k+1}^i - q_k^i = \dfrac{\partial H_{d,k}}{\partial p_{i,k}}, \\ p_{i,k} - p_{i,k-1} = -\dfrac{\partial H_{d,k}}{\partial q_k^i} + Q'_{i,k}. \end{cases} \tag{21}$$

Proof Computing the variation of the Hamiltonian gives

$$\begin{aligned} \delta H_{d,k} =& \sum_{k=0}^{N-1} \left(p_{i,k}\delta v_k^i + v_k^i \delta p_{i,k} \right. \\ & \left. - \frac{\partial L_{d,k}}{\partial q_k^i}\delta q_k^i - \frac{\partial L_{d,k}}{\partial v_k^i}\delta v_k^i \right) \\ =& \sum_{k=0}^{N-1} \left(v_k^i \delta p_{i,k} - \frac{\partial L_{d,k}}{\partial q_k^i}\delta q_k^i \right), \end{aligned} \tag{22}$$

where we have used Eq.(20).

Since $H_{d,k}$ are functions of q_k^i and $p_{i,k}$, we have

$$\delta H_{d,k} = \sum_{k=0}^{N-1} \left(\frac{\partial H_{d,k}}{\partial q_k^i}\delta q_k^i + \frac{\partial H_{d,k}}{\partial p_{i,k}}\delta p_{i,k} \right). \tag{23}$$

Using Eqs.(11) and (20), from Eqs.(22) and (23) we can obtain the canonical equations (21).

Example 2 Consider the same system as that shown in Example 1. From Eqs.(14) and (20) we can obtain

$$p_{1,k} = \frac{\partial L_{d,k}}{\partial v_k^1} = v_k^1, \quad p_{2,k} = \frac{\partial L_{d,k}}{\partial v_k^2} = v_k^2. \tag{24}$$

Then the discrete Hamiltonian of the system is given by

$$\begin{aligned} H_{d,k} =& p_{i,k}v_k^i - L_{d,k}(q_k^i, v_k^i) \\ =& \frac{1}{2}(p_{1,k}^2 + p_{2,k}^2) - \mu[(q_k^1)^2 + (q_k^2)^2]^{-1/2}. \end{aligned} \tag{25}$$

Substituting Eq.(25) into Eq.(21) we obtain the discrete canonical equations of motion of the system

$$\begin{cases} q_{k+1}^1 - q_k^1 = p_{1,k}, \\ q_{k+1}^2 - q_k^2 = p_{2,k}, \\ p_{1,k} - p_{1,k-1} = -\mu q_k^1[(q_k^1)^2 + (q_k^2)^2]^{-3/2} \\ \qquad\qquad -c_1(q_{k+1}^1 - q_k^1), \\ p_{2,k} - p_{2,k-1} = -\mu q_k^2[(q_k^1)^2 + (q_k^2)^2]^{-3/2} \\ \qquad\qquad -c_2(q_{k+1}^2 - q_k^2). \end{cases} \tag{26}$$

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