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Non-Noether symmetries and Lutzky conservative quantities of nonholonomic nonconservative dynamical systems*

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Non-Noether symmetries and conservative quantities of nonholonomic nonconservative dynamical systems are investigated in this paper. Based on the relationships among motion, nonconservative forces, nonholonomic constrained forces and Lagrangian, non-Noether symmetries and Lutzky conservative quantities are presented for nonholonomic nonconservative dynamical systems. The relation between non-Noether symmetry and Noether symmetry is discussed and it is further shown that non-Noether conservative quantities can be obtained by a complete set of Noether invariants. Finally, an example is given to illustrate these results.

Keywords: conserved quantity, non-Noether symmetry, nonholonomic nonconservative system, infinitesimal transformation

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1. Introduction

It is well known from the Noether theorem[1] that the motion constant for a dynamical system is associated with a continuous transformation of coordinates and time that keeps the action integral invariant. Transformation that can keep the equation of motion invariant is referred to as symmetry. Lutzky gave non-Noether conservative quantities of Lie point symmetries and velocity-dependent symmetries for Lagrangian systems, and pointed out that a given symmetry, if keeping the action integral invariant, leads to Noether symmetry.[2–4] Cicogna and Gaeta made some remarks on the existence and properties of Lie point symmetries in mechanics, but they did not obtain conservative quantity corresponding to these systems.[5] Without using either Lagrangians or Hamiltonians, Hojman presented a theorem to obtain directly conservative quantities with Lie symmetries.[6] Mei made major progresses in Lie symmetries for constraint mechanical systems.[7,8] Many investigators have given important results.[9–19] Recently the authors have worked on the non-Noether symmetries of the nonconservative dynamical systems and the velocity-dependent symmetries of the nonholonomic dynamical systems, and pointed out that the non-Noether symmetries may lead to non-Noether conservative quantities.[20,21] However, their investigations are limited to holonomic dynamical systems. In this paper, the authors give a relation among the motion, nonholonomic constrained forces, non-potential generalized forces and the Lagrangian for a nonholonomic dynamical system. Then non-Noether symmetries are extended from Lagrangian systems to nonholonomic nonconservative dynamical systems. A formula is derived for the non-Noether conservative quantities of nonholonomic nonconservative dynamical systems. The authors further give a condition under which Noether symmetries can be obtained from non-Noether symmetries of nonholonomic dy-

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namical systems. Finally, the conservative quantities of Noether symmetries are shown to form a complete set of conservative quantities for nonholonomic dynamical systems.

2. The non-Noether symmetries of nonholonomic nonconservative systems

Consider a dynamical system with Lagrangian $L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$, subject to non-potential generalized forces Q_s and g ideal nonlinear nonholonomic constraints in the form

$$f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \quad (\beta = 1, \cdots, g). \tag{1}$$

Its equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^s} - \frac{\partial L}{\partial q^s} = Q'_s + \Lambda_s, \ \ (s = 1, \cdots, n; \ n = N - g), \tag{2}$$

where $\Lambda_s = \lambda_\beta \partial f_\beta / \partial \dot{q}^s$ is called nonholonomic constrained force. From Eqs.(1) and (2), one can obtain all generalized accelerations of a nonholonomic system as follows:

$$\ddot{q}^s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}). \tag{3}$$

Introduce the infinitesimal transformations of time and coordinates

$$t^* = t + \varepsilon \xi_0(t, \boldsymbol{q}), \quad q^{s^*} = q^s + \varepsilon \xi_s(t, \boldsymbol{q}), \tag{4}$$

where ε is a small parameter, and ξ_0 and ξ_s are infinitesimal generators.

Based on the invariance of differential equations under the infinitesimal transformations, which will lead to Lie symmetries, the invariance of dynamical equation (3) and constraint equation (1) under the infinitesimal transformation (4) leads to the determining equations

$$\ddot{\xi}_s - \dot{q}^s \ddot{\xi}_0 - 2\alpha_s \dot{\xi}_0 = X^{(1)}(\alpha_s), \tag{5}$$

and the constraint equations

$$X^{(1)}(f_\beta(t, \boldsymbol{q}, \dot{\boldsymbol{q}})) = 0, \tag{6}$$

where $X^{(1)}$ is the first extended operator of the $X^{(0)}$ given by

$$X^{(1)} = \xi_0 \frac{\partial}{\partial t} + \xi_s \frac{\partial}{\partial q^s} + \left(\dot{\xi}_s - \dot{q}^s \dot{\xi}_0\right) \frac{\partial}{\partial \dot{q}^s}. \tag{7}$$

If Eqs.(3) and (1) remain invariant under the infinitesimal transformations (4) but the transformations (4) do not conserve the action integral of the system, the invariance is called the non-Noether symmetry of non-holonomic nonconservative dynamical systems.

Equations (5) and (6) can be regarded as a criterion of non-Noether symmetries.

Criterion 1 The symmetries are non-Noether symmetries of the holonomic nonconservative dynamical system (3), which is associated with nonholonomic nonconservative systems (1) and (2), if the infinitesimal generators ξ_s and ξ_0 satisfy the determining equation (5) but do not conserve the action integral of the system.

Criterion 2 The symmetries are non-Noether symmetries corresponding to the nonholonomic nonconservative dynamical systems governed by Eqs.(1) and (3), if the infinitesimal generators ξ_s and ξ_0 satisfy the determining equation (5) and constraint equation (6) but do not conserve the action integral of the system.

To derive non-Noether conserved quantities, two relationships for nonholonomic nonconservative dynamical systems are needed. Firstly, the n-dimensional nonholonomic nonconservative dynamical system (2) is written in the form

$$\frac{\partial^2 L}{\partial \dot{q}^s \partial \dot{q}^k}\ddot{q}^k = \frac{\partial L}{\partial q^s} - \frac{\partial^2 L}{\partial \dot{q}^s \partial t} - \frac{\partial^2 L}{\partial \dot{q}^s \partial q^k}\dot{q}^k + Q'_s + \Lambda_s. \tag{8}$$

And we have

$$\frac{\partial \alpha_s}{\partial \dot{q}^s} = -\frac{M_{ks}}{D}\left[\frac{\partial^2 L}{\partial q^k \partial \dot{q}^s} - \frac{\partial}{\partial t}\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^s} - \frac{\partial}{\partial q^k}\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^s}\dot{q}^k - \frac{\partial^2 L}{\partial q^k \partial \dot{q}^s} + \frac{\partial}{\partial \dot{q}^s}(Q'_k + \Lambda_k)\right], \tag{9}$$

in which

$$D = \det\left[\frac{\partial^2 L}{\partial \dot{q}^k \partial \dot{q}^s}\right] = \det T_{ij}, \tag{10}$$

where M_{ks} is the cofactor of the matrix element $T_{ij} = \partial^2 L / \partial \dot{q}^k \partial \dot{q}^s$, α_s, $Q'_s \Lambda_s$ and L are related as

$$\frac{\partial \alpha_s}{\partial \dot{q}^s} - \frac{\partial}{\partial \dot{q}^s}\left(\frac{M_{ks}}{D}(Q'_k + \Lambda_k)\right) + \frac{\mathrm{d}}{\mathrm{d}t}(\ln D) = 0. \tag{11}$$

Secondly, if ξ_s and ξ_0 satisfy Eq.(5), it follows that[3]

$$\dot{X}^{(1)}(\phi) = X^{(1)}\left(\dot{\phi}\right) + \dot{\xi}_0 \dot{\phi} \tag{12}$$

is satisfied for all functions $\phi(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$.

The main result can be proved.

Theorem 1 The nonholonomical nonconservative dynamical system (3) possesses a conservative quantity in the form

$$\begin{aligned}\Phi =&2\left(\frac{\partial\xi_s}{\partial q^s}-\dot{q}^s\frac{\partial\xi_0}{\partial q^s}\right)-N\dot{\xi}_0+X^{(1)}\left(\ln D\right)\\&-\int\left[X^{(1)}\left(\frac{\partial}{\partial\dot{q}^s}\left(\frac{M_{ks}}{D}\left(Q'_k+\Lambda_k\right)\right)\right)\right.\\&\left.-\dot{\xi}_0\frac{\partial}{\partial\dot{q}^s}\left(\frac{M_{ks}}{D}\left(Q'_k+\Lambda_k\right)\right)\right]\mathrm{d}t,\end{aligned}\tag{13}$$

If infinitesimal transformation generators ξ_0 and ξ_s satisfy Eqs.(5) and (6).

Proof Move the terms on the right-hand side of Eq.(5) to the left side and represent the resulting terms on the left-hand side by Π_s, which has the partial derivative

$$\begin{aligned}\frac{\partial\Pi_s}{\partial\dot{q}^s}=&\frac{\mathrm{d}}{\mathrm{d}t}\left[2\left(\frac{\partial\xi_s}{\partial q^s}-\dot{q}^s\frac{\partial\xi_0}{\partial q^s}\right)-N\dot{\xi}_0\right]\\&-X^{(1)}\left(\frac{\partial\alpha_s}{\partial\dot{q}^s}\right)-\frac{\partial\alpha_s}{\partial\dot{q}^s}\dot{\xi}_0.\end{aligned}\tag{14}$$

We now should show that it implies $\Pi_s=0$. If ξ_0 and ξ_s satisfy Eq.(5), one can obtain from Eqs.(11) and (12)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}X^{(1)}\left(\ln D\right)\\=&-X^{(1)}\left(\frac{\partial\alpha_s}{\partial\dot{q}^s}-\frac{\partial}{\partial\dot{q}_s}\left(\frac{M_{ks}}{D}\left(Q'_k+\Lambda_k\right)\right)\right)\\&-\dot{\xi}_0\frac{\partial\alpha_s}{\partial\dot{q}^s}-\dot{\xi}_0\frac{\partial}{\partial\dot{q}^s}\left(\frac{M_{ks}}{D}\left(Q'_k+\Lambda_k\right)\right)\end{aligned}\tag{15}$$

which allows Eq.(14) to be rewritten in the form

$$\begin{aligned}\frac{\partial\Pi_s}{\partial\dot{q}^s}=&\frac{\mathrm{d}}{\mathrm{d}t}\left[2\left(\frac{\partial\xi_s}{\partial q^s}-\dot{q}^s\frac{\partial\xi_0}{\partial q^s}\right)-N\dot{\xi}_0+X^{(1)}\left(\ln D\right)\right]\\&-X^{(1)}\left(\frac{\partial}{\partial\dot{q}^s}\left(\frac{M_{ks}}{D}\left(Q'_k+\Lambda_k\right)\right)\right)\\&-\dot{\xi}_0\frac{\partial}{\partial\dot{q}^s}\left(\frac{M_{ks}}{D}\left(Q'_k+\Lambda_k\right)\right).\end{aligned}\tag{16}$$

Substituting Eq.(13) into Eq.(16) yields

$$\begin{aligned}\frac{\partial\Pi_s}{\partial\dot{q}^s}=&\frac{\mathrm{d}}{\mathrm{d}t}\left[2\left(\frac{\partial\xi_s}{\partial q^s}-\dot{q}^s\frac{\partial\xi_0}{\partial q^s}\right)-N\dot{\xi}_0+X^{(1)}\left(\ln D\right)\right]\\&-X^{(1)}\left(\frac{\partial}{\partial\dot{q}^s}\left(\frac{M_{ks}}{D}\left(Q'_k+\Lambda_k\right)\right)\right)\\&-\dot{\xi}_0\frac{\partial}{\partial\dot{q}^s}\left(\frac{M_{ks}}{D}\left(Q'_k+\Lambda_k\right)\right).\end{aligned}\tag{17}$$

Furthermore, one assumes $\dfrac{\partial\Pi_s}{\partial\dot{q}^s}=0$, then it follows form Eq.(13) that Φ is a conservative quantity. This is the main result in the present paper. It should be noted that, in order to obtain this conservative quantity, it is necessary to postulate that the equation of motion is derived from a Lagrangian, but the action integral invariance is not essential.

Theorem 1 can be regarded as a criterion for obtaining non-Noether conservative quantities associated with nonholonomic nonconservative dynamical systems, where Eq.(6) is a constraint condition of the nonconservative and constrained forces, and Eq.(13) gives non-Noether conservative quantity of the nonholonomic nonconservative dynamical systems. Fu and Chen[17] considered holonomic nonconservative dynamical systems, which can be regarded as a particular case.

3. The Noether symmetries derived from a non-Noether symmetry

Equation (13) gives non-Noether conservative quantities. However, it may reduce to Noether conservative quantities under the following condition.

Theorem 2 If the non-Noether symmetries possess non-Noether conservative quantity of the form

$$\begin{aligned}\tilde{\Phi}=&-\int\left[X^{(1)}\left(\frac{\partial}{\partial\dot{q}^s}\left(\frac{M_{ks}}{D}\left(Q'_k+\Lambda_k\right)\right)\right)\right.\\&\left.-\dot{\xi}_0\frac{\partial}{\partial\dot{q}^s}\left(\frac{M_{ks}}{D}\left(Q'_k+\Lambda_k\right)\right)\right]\mathrm{d}t\\&-\frac{\partial^2}{\partial\dot{q}^l\partial\dot{q}^s}\left[\frac{M_{ks}}{D}\left(\dot{\xi}_l-\dot{q}^l\dot{\xi}_0\right)\left(Q'_l+\Lambda_l\right)\right],\end{aligned}\tag{18}$$

and the symmetry group preserves the action integral invariant, the non-Noether symmetry leads to a Noether symmetry and possesses the Noether conservative quantity

$$I=\xi_0L+\left(\xi_s-\dot{q}^s\xi_0\right)\frac{\partial L}{\partial\dot{q}^s}+G_N=\mathrm{const}\tag{19}$$

for a gauge function G_N.

Using the means in Ref.[9], one can verify Theorem 2.

Thus it is demonstrated that a conservative quantity can be associated with any symmetry of a nonholonomic nonconservative dynamical system. For a holonomic nonconservative dynamical system, there are no constrained forces; the results here reduce to

those given by

$$\tilde{\Phi} = -\frac{\partial^2}{\partial\dot{q}^k\partial\dot{q}^s}\left[\frac{M_{ks}}{D}\left(\dot{\xi}_l - \dot{q}^l\dot{\xi}_0\right)Q_l\right] - \int\left[X^{(1)}\left(\frac{\partial}{\partial\dot{q}^s}\left(\frac{M_{ks}}{D}Q'_k\right)\right) - \dot{\xi}_0\frac{\partial}{\partial\dot{q}^s}\left(\frac{M_{ks}}{D}Q'_k\right)\right]\mathrm{d}t. \tag{20}$$

The present study may lead to the non-Noether symmetries of the Lagrangian systems when both the nonpotential generalized force and the constrained forces of the systems vanish. The conservative quantity associated with a Lagrangian systems is[2]

$$\Phi = 2\left(\frac{\partial\xi_s}{\partial q^s} - \dot{q}^s\frac{\partial\xi_0}{\partial q^s}\right) - N\dot{\xi}_0 + X^{(1)}\left(\ln D\right), \tag{21}$$

which is the same as the result in Ref.[3]. Furthermore, if the symmetry keeps action integral invariant and Φ vanishes, Noether symmetry is obtained, and the system possesses a classical Noether conservative quantity (19).

It should be pointed out that the conservative quantity (13) may be expressible in terms of the Noether invariants of the system. In Ref.[22] the Noether invariants form a complete set of conservative quantities for a holonomic system. It is conjectured that for systems whose Noether invariants do not span a complete set of independent invariants, it might be possible to find additional non-Noether invariants of the conservative quantities (13).

4. An example

Consider a dynamical system with Lagrangian

$$L = \frac{1}{2}((\dot{q}^1)^2 + (\dot{q}^2)^2 + (\dot{q}^3)^2) \tag{22}$$

subject to a nonholonomic constraint

$$f = \dot{q}^3 - q^2\dot{q}^1 = 0. \tag{23}$$

Substituting Eqs.(22) and (23) into Eq.(3), one can obtain the equations of motion of the system as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}^1} - \frac{\partial L}{\partial q^1} = -\frac{q^2\dot{q}^1\dot{q}^2}{1+(q^2)^2}, \tag{24}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}^2} - \frac{\partial L}{\partial q^2} = 0, \tag{25}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial\dot{q}^3} - \frac{\partial L}{\partial q^3} = \frac{\dot{q}^1\dot{q}^2}{1+(q^2)^2}. \tag{26}$$

From Eqs.(24)–(26), one can obtain all generalized accelerations of the nonholonomic system

$$\ddot{q}^1 = -\frac{q^2\dot{q}^1\dot{q}^2}{1+(q^2)^2} = \alpha_1, \tag{27}$$

$$\ddot{q}^2 = 0 = \alpha_2, \tag{28}$$

$$\ddot{q}^3 = \frac{\dot{q}^1\dot{q}^2}{1+(q^2)^2} = \alpha_3. \tag{29}$$

The determining equations of the non-Noether symmetry of Eqs.(27)–(29) under the infinitesimal transformations $\xi_s = \xi_s(t,\boldsymbol{q})$ and $\xi_0 = \xi_0(t,\boldsymbol{q})$ are

$$\ddot{\xi}_1 - \dot{q}^1\ddot{\xi}_0 + \frac{2q^2\dot{q}^2\dot{q}^1}{1+(q^2)^2}\dot{\xi}_0 = -\xi_2\frac{\dot{q}^2\dot{q}^1\left(1-(q^2)^2\right)}{\left(1+(q^2)^2\right)^2} - \left(\dot{\xi}_1 - \dot{q}^1\dot{\xi}_0\right)\frac{q^2\dot{q}^2}{1+(q^2)^2} - \left(\dot{\xi}_2 - \dot{q}^2\dot{\xi}_0\right)\frac{q^2\dot{q}^1}{1+(q^2)^2}, \tag{30}$$

$$\ddot{\xi}_2 - \dot{q}^2\ddot{\xi}_0 = 0, \tag{31}$$

$$\ddot{\xi}_3 - \dot{q}^3\ddot{\xi}_0 - \frac{2\dot{q}^2\dot{q}^1}{1+(q^2)^2}\dot{\xi}_0 = \left(\dot{\xi}_1 - \dot{q}^1\dot{\xi}_0\right)\frac{\dot{q}^2}{1+(q^2)^2} + \left(\dot{\xi}_2 - \dot{q}^2\dot{\xi}_0\right)\frac{\dot{q}^1}{1+(q^2)^2}. \tag{32}$$

Equations (30)–(32) result in

$$\dot{\xi}_k = \frac{\partial\xi_k}{\partial t} + \frac{\partial\xi_k}{\partial q^1}\dot{q}^1 + \frac{\partial\xi_k}{\partial q^2}\dot{q}^2 + \frac{\partial\xi_k}{\partial q^3}\dot{q}^3, \quad (k = 0, 1, \cdots, n), \tag{33}$$

$$\begin{aligned}\ddot{\xi}_k =& \frac{\partial^2\xi_k}{\partial t^2} + 2\frac{\partial^2\xi_k}{\partial t\partial q^1}\dot{q}^1 + 2\frac{\partial^2\xi_k}{\partial t\partial q^2}\dot{q}^2 + 2\frac{\partial^2\xi_k}{\partial t\partial q^3}\dot{q}^3 + 2\frac{\partial^2\xi_k}{\partial q^2\partial q^1}\dot{q}^1\dot{q}^2 \\ &+ 2\frac{\partial^2\xi_k}{\partial q^3\partial q^1}\dot{q}^1\dot{q}^3 + 2\frac{\partial^2\xi_k}{\partial q^3\partial q^2}\dot{q}^3\dot{q}^2 + \frac{\partial^2\xi_k}{\partial q^1\partial q^1}(\dot{q}^1)^2 \\ &+ \frac{\partial^2\xi_k}{\partial q_2\partial q_2}(\dot{q}^2)^2 + \frac{\partial^2\xi_k}{\partial q^3\partial q^3}(\dot{q}^3)^2.\end{aligned} \tag{34}$$

Equations (30)–(32) have solutions as follows:

$$\xi_0 = c_1q^2 + c_2,$$

$$\xi_1 = c_3, \quad \xi_2 = 0,$$

and

$$\xi_3 = c_4q^2 + c_5t + c_6. \tag{35}$$

where c_1, c_2, c_3, c_4, c_5, and c_6 are arbitrary constants. Then the generator (35) is of non-Noether symmetry of the holonomic system (27)–(29), which is associated with the nonholonomic system (22) and (23). Substituting Eq.(23) into Eq.(6) yields

$$\begin{aligned} & X^{(1)}\left(\dot{q}^3 - q^2\dot{q}^1\right) \\ =& -\xi_2\dot{q}^1 - (\dot{\xi}_1 - \dot{q}^1\xi_0)q^2 + (\dot{\xi}_3 - \dot{q}^3\dot{\xi}_0) = 0. \end{aligned} \quad (36)$$

Substituting Eq.(35) into Eq.(36) gives

$$\xi_0 = c_2, \quad \xi_1 = c_3, \quad \xi_2 = 0, \quad \text{and} \quad \xi_3 = c_6. \quad (37)$$

The generator (36) is of non-Noether symmetry associated with the nonholonomic system (27)–(29) and (23).

Inserting Eq.(35) into Eq.(13) leads to the conservative quantity of non-Noether symmetry associated with Eqs.(27)–(29) as follows:

$$\begin{aligned} \Phi =& 2\left(\frac{\partial c_3}{\partial q^1} - \dot{q}^1\frac{\partial}{\partial q^1}(c_1q^2 + c_2)\right) \\ & + 2\left(\frac{\partial c_3}{\partial q^2} - \dot{q}^2\frac{\partial}{\partial q^2}(c_1q^2 + c_2)\right) \\ & + 2\left(\frac{\partial}{\partial q^3}(c_4q^2 + c_5) - \dot{q}^3\frac{\partial}{\partial q^3}(c_1q^2 + c_2)\right) \\ & - 3c_1\dot{q}^2 + X^{(1)}(\ln D) \\ & - \int\left[X^{(1)}\left(-\frac{q^2\dot{q}^2}{1+(q^2)^2} - \frac{q^2\dot{q}^1}{1+(q^2)^2}\right)\right. \\ & \left. - \dot{\xi}_0\left(-\frac{q^2\dot{q}^2}{1+(q^2)^2} - \frac{q^2\dot{q}^1}{1+(q^2)^2}\right)\right]\mathrm{d}t \\ =& -2c_1\dot{q}^2 - 3c_1\dot{q}^2 = \text{const.} \end{aligned} \quad (38)$$

Equation (38) indicates that the momentum is conservative in the direction of q^2 corresponding to Eqs.(27)–(29).

Substituting Eqs.(37) into Eq.(13) leads to the conservative quantity of non-Noether symmetry associated with the nonholonomic system (27)–(29) and (23)

$$\begin{aligned} \Phi =& 2\left(\frac{\partial c_3}{\partial q^1} - \dot{q}^1\frac{\partial \dot{c}_2}{\partial q^1}\right) + 2\left(\frac{\partial 0}{\partial q^2} - \dot{q}^2\frac{\partial c_2}{\partial q^2}\right) \\ & + 2\left(\frac{\partial c_6}{\partial q^3} - \dot{q}^3\frac{\partial c_2}{\partial q^3}\right) \\ & - \int\left[X^{(1)}\left(-\frac{q^2\dot{q}^2}{1+(q^2)^2} - \frac{q^2\dot{q}^1}{1+(q^2)^2}\right)\right. \\ & \left. - \dot{\xi}_0\left(-\frac{q^2\dot{q}^2}{1+(q^2)^2} - \frac{q^2\dot{q}^1}{1+(q^2)^2}\right)\right]\mathrm{d}t = 0. \end{aligned} \quad (39)$$

Therefore, the non-Noether symmetry (37) is trivial. However, Eq.(18) can be applied to obtain the following relation by non-Noether symmetry given by Eq.(37)

$$\begin{aligned} \tilde{\Phi} =& -\int\left[X^{(1)}\left(-\frac{q^2\dot{q}^2}{1+(q^2)^2} - \frac{q^2\dot{q}^1}{1+(q^2)^2}\right)\right. \\ & \left. - \dot{\xi}_0\left(-\frac{q^2\dot{q}^2}{1+(q^2)^2} - \frac{q^2\dot{q}^1}{1+(q^2)^2}\right)\right]\mathrm{d}t \\ & - \frac{\partial^2}{\partial\dot{q}^k\partial\dot{q}^s}\left[\frac{M_{ks}}{D}\left(\dot{\xi}_l - \dot{q}_l\dot{\xi}_0\right)\left(Q'_l + \Lambda_l\right)\right] = 0. \end{aligned} \quad (40)$$

Then Noether symmetry is derived from the non-Noether symmetry (37), and the symmetry satisfies the Noether equation

$$(c_3 - c_2\dot{q}^1)(-\frac{q^2\dot{q}^2\dot{q}^1}{1+(q^2)^2}) + (c_6 - c_2\dot{q}^3)\frac{\dot{q}^2\dot{q}^1}{1+(q^2)^2} = -\dot{G}_N. \quad (41)$$

Nevertheless one cannot find the gauge function G_N from Eq.(41), and thus cannot obtain the Noether conservative quantity corresponding to the symmetry.

5. Conclusions

The present investigation is a generalization of the results in Refs.[2, 20] to nonholonomic systems. A condition under which exists the non-Noether symmetry with a conservative quantity is established for nonholonomic nonconservative dynamical systems. A criterion is also presented for deriving a classical Noether symmetry from the non-Noether symmetry. For systems without nonholonomic constraints, the results here reduce to those in Ref.[20]. For holonomic systems with potential generalized force only, the results here reduce to those in Ref.[2].

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