Vol 15 No 12, December 2006
1009-1963/2006/15(12)/3044-05

Chinese Physics

© 2006 Chin. Phys. Soc.
and IOP Publishing Ltd

The Casimir effect of Reissner–Nordström black hole*

Xiao Kui(肖　奎) and Liu Wen-Biao(刘文彪)†

Department of Physics, Institute of Theoretical Physics, Beijing Normal University, Beijing 100875, *China*

(Received 10 December 2005; revised manuscript received 14 July 2006)

The stress tensor of a massless scalar field satisfying a mixed boundary condition in a (1+1)-dimensional Reissner–Nordström black hole background is calculated by using Wald's axiom. We find that Dirichlet stress tensor and Neumann stress tensor can be deduced by changing the coefficients of the stress tensor calculated under a mixed boundary condition. The stress tensors satisfying Dirichlet and Neumann boundary conditions are discussed. In addition, we also find that the stress tensor in conformal flat spacetime background differs from that in flat spacetime only by a constant.

Keywords: Casimir effect, Reissner–Nordström black hole, stress tensor, scalar field

PACC: 9760L, 0470, 0370, 1110

1. Introduction

The Casimir effect originates from the name of the prominent Dutch physicist Casimir, who published a paper about the attraction between two neutral metallic plates in 1948.[1] This effect has been extensively studied since then, and finite vacuum energies have been computed for various fields that satisfy different boundary conditions in flat spacetime as well as in curved spacetime.[2] People have developed many methods of evaluating the effect, such as Green's function, the zeta function, dimensional regulation, and so on. Until now we have had a typical method of calculating the vacuum energy in a cavity in flat spacetime.[3]

In recent years, Setare and co-workers have studied the Casimir effect for a massless scalar field between two parallel plates in a two-dimensional black hole background with different boundary conditions.[4−7] Setare has discussed the stress tensor of a massless scalar field on two one-dimensional walls in a (1+1)-dimensional black hole background, and found that the two different boundary conditions have different contributions to the stress tensor. The relationship between the Casimir effect and the cosmological constant in a two-dimensional Achucarro–Ortiz black hole background is discussed in Ref.[6]. Gao has approximately calculated the stress tensor of a massless Dirac field in a two-dimensional Reissner–Nordström black hole background.[7]

The various properties of black hole and the field in curved spacetime have been studied.[8−11] In this paper, we discuss the Casimir effect with a mixed boundary condition for a massless scalar field in a Reissner–Nordström black hole background. In the process of calculating renormalized stress tensor, a general axiom, now is called Wald's axiom, which was instigated by Christensen[2,12] and pursued with great success by Wald [13,14] in the 1970s, is adopted. The axiom is as follows:

(1) The expectation values of the energy-momentum tensor are covariantly conserved.

(2) Causality holds.

(3) For off-diagonal elements, the standard results should be obtained.

(4) The standard results should be obtained in Minkowski spacetime.

(5) The energy–momentum tensor contains no local curvature tensor depending on derivatives of the metric higher than second order.

The Casimir effect with a mixed boundary condition for a massless scalar field in a Reissner–Nordström black hole background is calculated, and then the stress tensors with the Dirichlet boundary and Neumann boundary conditions are discussed separately. Comparing the two cases, we find that the tensor contributed by the boundary in curved space-

*Project supported by the National Basic Research Program of China (Grant No 2003CB716302) and the National Natural Science Foundation of China (Grant Nos 10373003 and 10475013).

†Corresponding author. E-mail: wbliu@bnu.edu.cn

time has the same form as that in flat spacetime, but has a different conformal factor from that in flat spacetime. And the relationship between the Casimir stress tensor in curved spacetime and that in flat spacetime is also discussed.

2. Vacuum expectation values of stress tensor

The line element of a Reissner–Nordström black hole[15] is

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)\mathrm{d}t^2+\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2+r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\right), \tag{1}$$

where m and Q are the mass and charge of the hole.

Now we reduce the dimension of spacetime to two dimensions

$$\mathrm{d}s^2 = -\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)\mathrm{d}t^2+\left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2, \tag{2}$$

and it can be written in the conformal form

$$\mathrm{d}s^2 = f(r)\left(-\mathrm{d}t^2+\mathrm{d}r^{*2}\right), \tag{3}$$

in which $f(r) = \left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)$, $\frac{\mathrm{d}r}{\mathrm{d}r^*} = \left(1-\frac{2m}{r}+\frac{Q^2}{r^2}\right)$.

Using Wald's first axiom and considering the anomalous trace in two dimensions[16] $\langle T^{\alpha}_{\alpha}\rangle_{\mathrm{ren}} = \frac{-R}{24}$, we can obtain the most general form of the stress tensor in the Reissner–Nordström black hole background[7]

$$\langle T^{\mu}_{\nu}\rangle = \begin{pmatrix} \langle T^{\alpha}_{\alpha}\rangle_{\mathrm{ren}} - f^{-1}(r)H(r) & 0 \\ 0 & f^{-1}(r)H(r) \end{pmatrix} + f^{-1}(r)\begin{pmatrix} -\beta & -\alpha \\ \alpha & \beta \end{pmatrix}, \tag{4}$$

in which

$$H(r) = -\frac{1}{24\pi}\left(\frac{m^2}{r^4}-\frac{2mQ^2}{r^5}+\frac{Q^2}{r^6}\right)+\frac{1}{24\pi}\left[\frac{m^2}{\left(m+\sqrt{m^2-Q^2}\right)^4} - \frac{2mQ^2}{\left(m+\sqrt{m^2-Q^2}\right)^5}+\frac{Q^4}{\left(m+\sqrt{m^2-Q^2}\right)^6}\right], \tag{5}$$

where α and β are two integral constants. We hope to determine these two constants by imposing Wald's axiom and treating the Reissner–Nordström black hole as a Casimir system. Two one-dimensional parallel plates are placed at r_1 and r_2 ($r_1 < r_2$) in a $(1+1)$-dimensional Reissner–Nordström black hole, then we consider the massless scalar field to evaluate the Casimir effect with a mixed boundary (Robin boundary) condition. There are two plates that are located at $r = r_i$ ($i = 1, 2$) satisfying the mixed boundary condition[17]

$$\left(C_i + n^{\mu}\cdot\nabla_{\mu}\right)\varphi(x) = 0, \quad r = r_i, \quad i = 1, 2, \tag{6}$$

where n^{μ} is normal to these surface and $n^{\mu}n_{\mu} = -1$, and C_i are constants. We can obtain the Dirichlet and Neumann boundary conditions by changing the constants C_i.

The $(1+1)$-dimensional Reissner–Nordström black hole spacetime is asymptotically flat, so the integral constants are evaluated by requiring the stress tensor given in Eq.(3) to coincide at infinity, i.e. $r_i \to \infty$, with the standard Casimir stress tensor in the $(1+1)$-Minkowski spacetime[17]

$$\langle T_{\nu}^{\mu}\rangle_{\rm ren}=-\frac{J_1\left(B_1,B_2\right)}{2\pi^{1/2}a^2\Gamma\left(\frac{3}{2}\right)}\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad r_1<r<r_2, \tag{7}$$

where

$$J_1\left(B_1,B_2\right)=\text{p.v.}\int_0^{\infty}\frac{t\mathrm{d}t}{\frac{\left(B_1t-1\right)\left(B_2t-1\right)}{\left(B_1t+1\right)\left(B_2t+1\right)}\mathrm{e}^{2t}-1}, \tag{8}$$

$$B_i=\frac{\bar{b}_i}{\bar{a}_ia},\qquad \frac{\bar{b}_i}{\bar{a}_i}=\frac{1}{C_i},\qquad i=1,2,\qquad a=\left(r_2-r_1\right). \tag{9}$$

Comparing Eq.(4) with Eq.(7), the two integral constants can be obtained

$$\alpha=0,\qquad \beta=c-\frac{1}{384\pi m^2}, \tag{10}$$

where

$$c=\frac{J_1\left(B_1,B_2\right)}{2\pi^{1/2}a^2\Gamma\left(\frac{3}{2}\right)}. \tag{11}$$

Now the stress tensor can be written as the sum of two terms, one is the contribution of the boundary, and the other is the vacuum polarization in a gravitational background without boundary.

$$\langle T_{\nu}^{\mu}\rangle=\begin{pmatrix}\langle T_{\alpha}^{\alpha}\rangle_{\rm ren}-f^{-1}\left(r\right)H\left(r\right)&0\\0&f^{-1}\left(r\right)H\left(r\right)\end{pmatrix}+f^{-1}\left(r\right)\begin{pmatrix}-\left(c-\frac{1}{384\pi m^2}\right)&0\\0&\left(c-\frac{1}{384\pi m^2}\right)\end{pmatrix}. \tag{12}$$

As in four dimensions[18] the boundary part is zero in the region $x<x_1$ and $x>x_2$ and only gravitational polarization part exists. Although the first gravitational polarization term contributes to the total stress tensor, the force acting on the plates is determined only by the second boundary part.

The vacuum boundary part pressure acting on the plates is

$$\rho_{b_{1,2}}=\rho_b\left(r=r_{1,2}\right)=-f^{-1}\left(r\right)\frac{J_1\left(B_1,B_2\right)}{2\pi^{1/2}a^2\Gamma\left(\frac{3}{2}\right)}. \tag{13}$$

When the value of $J_1\left(B_1,B_2\right)$ changes, just as C_i $(i=1,2)$ changes, Eq.(13) corresponds to the attractive or repulsive force, and the force will disappear on the plates while $J_1\left(B_1,B_2\right)=0$. This result is valid for all vacuum states, because the effective forces created by the gravitational pressure or other vacuum states' pressures are the same on both sides of the plates.

3. The Dirichlet and Neumann boundary conditions

Equation (6) describes a mixed boundary condition, which is also called the Robin boundary condition. The Dirichlet and Neumann boundary conditions are particular cases of the mixed boundary condition. The first one corresponds to $C_i\to\infty$ $(i=1,2)$ under the assumption of $n^{\mu}\cdot\nabla_{\mu}$ being bounded, while the second one is obtained in the limit $C_i=0$ $(i=1,2)$. The Dirichlet boundary condition is a conformal invariance, and the Neumann boundary condition is a conformal invariance in the $(1+1)$-dimensional spacetime[14] too.

The contribution of Neumann boundary condition to the stress tensor can be given by letting $C_i=0$ $(i=1,2)$ and using the gamma function

$$\langle T_{\nu}^{(b)\mu}\rangle_{\rm NBC}=f^{-1}\left(r\right)\begin{pmatrix}-\frac{\pi}{24a^2}&0\\0&\frac{\pi}{24a^2}\end{pmatrix}, \tag{14}$$

and if we let $C_i \to \infty$ $(i = 1, 2)$ and consider

$$\lim_{B_i \to 0} \int_0^\infty \frac{t\mathrm{d}t}{\frac{(B_1 t - 1)(B_2 t - 1)}{(B_1 t + 1)(B_2 t + 1)} \mathrm{e}^{2t} - 1} = \frac{\pi^2}{24}, \tag{15}$$

we can obtain the stress tensor with the Dirichlet boundary condition

$$\langle T_\nu^{(b)\mu} \rangle_{\mathrm{DBC}} = f^{-1}(r) \begin{pmatrix} -\frac{\pi}{24a^2} & 0 \\ 0 & \frac{\pi}{24a^2} \end{pmatrix}. \tag{16}$$

This stress tensor is the same as that when we consider the same boundary condition for a massless Dirac field.[7] Comparing Eq.(14) with Eq.(16), we can have the same result when we consider the stress tensors with the Dirichlet boundary and Neumann boundary conditions, this result is the same as that in a flat spacetime background.[19] So there is the same vacuum boundary part pressure on the plates for the Dirichlet boundary condition and Neumann boundary condition

$$\rho_{b_{1,2}} = \rho_b(r = r_{1,2}) = \langle T_t^{(b)t} \rangle_{\mathrm{NBC}} = \langle T_t^{(b)t} \rangle_{\mathrm{DBC}} = -f^{-1}(r) \frac{\pi}{24a^2}. \tag{17}$$

The Casimir force on the unit area of the plates is equal to this vacuum pressure, so there is the same force for Dirichlet boundary and Neumann boundary.

The standard Casimir stress tensor of scalar field satisfying the Dirichlet boundary condition on one-dimensional wall is[20]

$$T_{\mu\nu} = \frac{-\pi^2}{24a^2} \delta_{\mu\nu}. \tag{18}$$

Comparing the second term on the right-hand side of Eq.(4) with the term on the right-hand side of Eq.(18), we can find that the contribution of Dirichlet boundary condition to the stress tensor is

$$\langle T_\nu^{(b)\mu} \rangle_{\mathrm{DBC}} = f^{-1}(r) \begin{pmatrix} -\frac{\pi}{24a^2} & 0 \\ 0 & \frac{\pi}{24a^2} \end{pmatrix}. \tag{19}$$

This stress tensor is the same as that described by Eq.(16) with $C_i \to \infty$. So we have the same result when we use Wald's axiom or just change the coefficients of the mixed boundary condition.

Considering Wald's fourth axiom that is introduced in Section 2, and comparing the relationships between the second term on the right-hand side of Eq.(12) and the term on the right-hand side of Eq.(7), and between the term on the right-hand side of Eq.(19) (or Eq.(16)) and the term on the right-hand side of Eq.(18), we can find that the stress tensor of massless scalar field satisfying different boundary conditions in the curved (1+1)-dimensional spacetime can be written as:

$$\langle T_{(b)\nu}^{\mu} \rangle = \Omega^{-1}(r) \langle T_\nu^\mu \rangle_{\mathrm{ren}}, \tag{20}$$

where $\Omega(r)$ is the conformal factor, and $\langle T_\nu^\mu \rangle_{\mathrm{ren}}$ is the Casimir stress tensor with the mixed boundary condition. Equations (14) and (16) can also be written in this form. The Casimir stress tensor in the conformal flat spacetime background differs from that in the flat spacetime only by a constant.

4. Conclusions

The Casimir effect is one of the least intuitive consequences of quantum field theory. It is caused by fluctuations of vacuum energy in the quantum field theory when they are modified by the introduction of boundaries. The particular features of the resultant force depend on the nature of field and boundary condition imposed in some cases. This effect occurs in different fields of physics, including quantum electrodynamics, theory of hadrons, dark energy, entropy of black hole, and cosmology.

In this paper, we explicitly calculate the regularized stress tensor of a massless scalar field satisfying the mixed boundary condition on one-dimensional walls in a Reissner–Nordström black hole background, just according to the general properties of the renormalized stress tensor. The regularized stress tensor is also discussed when the field satisfies the Neumann boundary condition and the Dirichlet boundary condition. In all these boundary conditions, the pressures on the plates are obtained. We find that the stress tensor with Neumann boundary condition is the same as that with Dirichlet boundary condition, and there should be the same force on the plates under these two boundary conditions. As a result, we find the Casimir effect under different boundary conditions can be studied if we know the renormalized stress tensor in

Minkowski spacetime. In addition, the Casimir stress tensor in conformal flat spacetime background differs from that in flat spacetime only by a constant.

Acknowledgments

The authors would like to thank Zhang H B and Gao L for useful discussion.

References

[1] Casimir H B G 1948 *Proc. Kon. Ned. Aked. Wet.* **51** 793
[2] Birrell N D and Davies P C W 1982 *Quantum Fields in Curved Spacetime* (Cambridge: Cambridge University Press) p89, p229
[3] Ford L H 1988 *Phys. Rev.* D **38** 528
[4] Setare M R 2001 *Class. Quantum. Grav.* **18** 2097
Saharian A A and Setare M R 2003 *Phys. Lett.* B **552**119
Christodoulakis T and Diamandis G A 2001 *Phys. Rev.* D **64** 124022
[5] Setare M R 2005 *Mod. Phys. Lett.* A **20** 441
[6] Vagenas E C 2003 *Phys. Rev.* D **68** 024015
[7] Gao X-Q and Zhu J Y 2005 gr-qc/0501049
[8] Jing J L and Chen S B 2005 *Chin. Phys.* **14** 683
[9] Huang X J and Wang Y J 2004 *Chin. Phys.* **13** 1588
[10] Zhao Z, Liu W B and Jiang Y L 2000 *Acta Phys. Sin.* **49** 586 (in Chinese)
[11] Zhang J Y 2005 *Acta Phys. Sin.* **54** 2453 (in Chinese)
[12] Christensen S M 1975 *Bull. Am. Phys. Soc.* **20** 99
[13] Wald R M 1977 *Commun. Math. Phys.* **54** 1
[14] Wald R M 1978 *Phys. Rev.* D **17** 1477
Wald R M 1978 *Ann. Phys.* (N.Y.) **110** 472
[15] Yang S Z 2004 *Acta Phys. Sin.* **53** 4007 (in Chinese)
[16] Davies P C W 1977 *Proc. R. Soc. Lond.* A **354** 529
[17] Romeo A and Saharian A A 2004 *J. Phys. A*: *Math. Gen.* **35** 1297
[18] Setare M R and Saharian A 2000 hep-th/0002179
[19] Cooke J H 1997 *Am. J. Phys.* **66** 569
[20] Saharian A A 2000 hep-th/0002239