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(3+1)-dimensional nonlinear propagation equation for ultrashort pulsed beam in left-handed material*

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In this paper a comprehensive framework for treating the nonlinear propagation of ultrashort pulse in metamaterial with dispersive dielectric susceptibility and magnetic permeability is presented. Under the slowly-evolving-wave approximation, a generalized (3+1)-dimensional wave equation first order in the propagation coordinate and suitable for both right-handed material (RHM) and left-handed material (LHM) is derived. By the commonly used Drude dispersive model for LHM, a (3+1)-dimensional nonlinear Schrödinger equation describing ultrashort pulsed beam propagation in LHM is obtained, and its difference from that for conventional RHM is discussed. Particularly, the self-steeping effect of ultrashort pulse is found to be anomalous in LHM.

Keywords: nonlinear propagation, left-handed material, ultrashort pulse, right-handed material

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1. Introduction

Left-handed material (LHM), also known as negative-refractive-index material (NIM), is a kind of metamaterial. In 1967, Veselago theoretically proved that the refractive index should be negative when both dielectric susceptibility and magnetic permeability are negative.[1] Though no natural LHM is reported as yet, many artificial LHMs are available.[2−7] In 1996, Pendry *et al* firstly demonstrated the negative magnetic permeability based on a periodic array of interspaced conducting nonmagnetic split ring resonators.[2] This attracted much interest to LHM in recent years. A kind of LHM in the microwave range is demonstrated by Smith *et al*;[3] Shvets showed that LHMs can be realized in the far-IR (infra red) spectrum;[4] Podolskiy *et al* designed LHMs in the near-IR and visible spectral ranges respectively;[5,6] Luo *et al* studied the effect of nonuniform-defect split ring resonators on LHMs.[7] At the same time, the manufacturing technology of LHM was improved constantly and some optical properties are discovered.[8−10] It is found that all the LHMs are dispersive in both dielectric susceptibility and magnetic permeability.

For the pulse propagation in conventional dispersive nonlinear medium, there are a lot of theoretical and experimental reports.[11−13] An important general three-dimensional nonlinear envelope equation was obtained by Brabec and Krausz, which provides an accurate description of the propagation of the envelope down to a pulse duration as short as one carrier oscillation cycle by using the slowly evolving wave approximation (SEWA).[11] For the propagation in LHM, some constructive works,[14−17] most of which are concerning monochromatic wave, have been done to validate the negative refraction characteristic. As for pulse propagation in LHM, Lazarides and Tsironis derived a system of coupled nonlinear Schrödinger equations in Ref.[18] for the envelopes of the propagating electric and magnetic fields in an isotropic, homogeneous, quasi-one-dimensional NIM. But the dispersive features of dielectric susceptibility and magnetic permeability were not embodied very clearly. Scalora *et al*[19] developed a simplified single nonlinear Schrödinger equation which describes the (1+1)-dimensional nonlinear propagation of ultrashort pulse in LHM. But their (3+1)-dimensional propagation need more rigorous derivation and systematical analysis for further investigation. Thus, the relation between the propagation equations for right-handed material (RHM) and for LHM is not clear. In

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this paper, we present a comprehensive framework for treating the nonlinear propagation of ultrashort pulse in metamaterial with dispersive dielectric susceptibility and magnetic permeability, and discuss the differences between the propagation equation for LHM and that for conventional RHM.

2. Derivation of the (3+1)-dimensional propagation equation in dispersive metamaterial

The dynamics of electromagnetic wave propagation in a medium is described by

$$\nabla \times \nabla \times \boldsymbol{E} = -\frac{\partial}{\partial t}\left(\mu \frac{\partial \boldsymbol{D}}{\partial t}\right), \tag{1}$$

where ∇ is the del operator, $\boldsymbol{E}$ is the electric field intensity vector in medium, t is time, μ is the absolute magnetic permeability, and $\boldsymbol{D}$ is the electric displacement vector and $\boldsymbol{D} = \varepsilon \boldsymbol{E} + \boldsymbol{P}_{\rm nl}$, with ε being the linear part of the absolute dielectric susceptibility and $\boldsymbol{P}_{\rm nl}$ the nonlinear polarization. Taking $\nabla \cdot \boldsymbol{E} = 0$ into account in the case of a homogeneous isotropic medium, we obtain from Eq.(1) that

$$\nabla^2 \boldsymbol{E} = \mu\varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \mu \frac{\partial^2 P_{\rm nl}}{\partial t^2}, \tag{2}$$

where ∇^2 denotes the Laplacian. We do not substitute $1/c^2$ for $\varepsilon\mu$, because both ε and μ are functions of frequency and it will bring convenience for problems in LHM when they are discussed separately. The field can be written as $\boldsymbol{E} = \boldsymbol{A}(\boldsymbol{r}_\perp, z, t)\exp[\mathrm{i}(k_0 z - \omega_0 t)]$, where ω_0 and k_0 are circular frequency and wave vector respectively, and $\boldsymbol{r}_\perp = x\hat{e}_x + y\hat{e}_y$. The symbol $\boldsymbol{A}$ denotes the envelope of electric field. Fourier transforming Eq.(2) we have

$$\nabla_\perp^2 \tilde{\boldsymbol{E}} + \frac{\partial^2}{\partial z^2}\tilde{\boldsymbol{E}} = -\mu\varepsilon\omega^2 \tilde{\boldsymbol{E}} - \mu\omega^2 \tilde{\boldsymbol{P}}_{\rm nl}, \tag{3}$$

where $\nabla_\perp^2$ is the transversal Laplacian, $\tilde{\boldsymbol{E}}$ and $\tilde{\boldsymbol{P}}_{\rm nl}$ are the Fourier transforms of electric field and nonlinear polarization respectively, and ω is the circular frequency of electromagnetic wave. Since ε and μ are dispersive, $\omega\varepsilon$ and $\omega\mu$ in Eq.(3) can be expanded as

$$\omega\varepsilon(\omega) = \sum_{n=0}^{\infty}\left\{\frac{\partial^n[\omega\varepsilon(\omega)]}{\partial\omega^n}\Big|_{\omega=\omega_0}\frac{(\omega-\omega_0)^n}{n!}\right\}, \tag{4}$$

$$\omega\mu(\omega) = \sum_{n=0}^{\infty}\left\{\frac{\partial^n[\omega\mu(\omega)]}{\partial\omega^n}\Big|_{\omega=\omega_0}\frac{(\omega-\omega_0)^n}{n!}\right\}. \tag{5}$$

Substituting Eqs.(4) and (5) into Eq.(3), inverse Fourier transforming the consequent equation and ignoring the common exponential factor, we have

$$\begin{aligned}
&\nabla_\perp^2 \boldsymbol{A} + \left(\frac{\partial^2}{\partial z^2} + 2\mathrm{i}k_0\frac{\partial}{\partial z} - k_0^2\right)\boldsymbol{A} \\
= &-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\frac{\mathrm{i}^{m+n}}{m!n!}\frac{\partial^n[\omega\varepsilon(\omega)]}{\partial\omega^n}\Big|_{\omega=\omega_0}\frac{\partial^m[\omega\mu(\omega)]}{\partial\omega^m}\Big|_{\omega=\omega_0}\frac{\partial^{m+n}}{\partial t^{m+n}}\boldsymbol{A}\right\} \\
&-\mathrm{i}\sum_{n=0}^{\infty}\left\{\frac{\mathrm{i}^n}{n!}\frac{\partial^n[\omega\mu(\omega)]}{\partial\omega^n}\Big|_{\omega=\omega_0}\frac{\partial^{n+1}}{\partial t^{n+1}}\boldsymbol{P}_{\rm nl}\right\} - \omega_0\sum_{n=0}^{\infty}\left\{\frac{\mathrm{i}^n}{n!}\frac{\partial^n[\omega\mu(\omega)]}{\partial\omega^n}\Big|_{\omega=\omega_0}\frac{\partial^n}{\partial t^n}\boldsymbol{P}_{\rm nl}\right\}.
\end{aligned} \tag{6}$$

Defining

$$R_n = \sum_{m=0}^{n}\frac{\mathrm{i}^n}{m!(n-m)!}\frac{\partial^m[\omega\varepsilon(\omega)]}{\partial\omega^m}\Big|_{\omega=\omega_0}\frac{\partial^{n-m}[\omega\mu(\omega)]}{\partial\omega^{n-m}}\Big|_{\omega-\omega_0}, \tag{7}$$

$$S_n = \frac{\mathrm{i}^n}{n!}\frac{\partial^n[\omega\mu(\omega)]}{\partial\omega^n}\Big|_{\omega=\omega_0}, \tag{8}$$

and introducing the moving reference frame, $\xi = z$, $\tau = t - z/V_{\rm g}$, we can rewrite Eq.(6) as

$$\begin{aligned}
&\frac{1}{2\mathrm{i}k_0}\nabla_\perp^2\boldsymbol{A} + \frac{1}{2\mathrm{i}k_0}\frac{\partial^2}{\partial\xi^2}\boldsymbol{A} - \frac{1}{2\mathrm{i}k_0}2\frac{1}{V_{\rm g}}\frac{\partial^2}{\partial\tau\partial\xi}\boldsymbol{A} + \frac{\partial}{\partial\xi}\boldsymbol{A} \\
= &\frac{1}{2\mathrm{i}k_0}\left(k_0^2 - R_0\right)\boldsymbol{A} + \left(\frac{1}{V_{\rm g}} - \frac{1}{2\mathrm{i}k_0}R_1\right)\frac{\partial}{\partial\tau}\boldsymbol{A} - \frac{1}{2\mathrm{i}k_0}\left(\frac{1}{V_{\rm g}^2} + R_2\right)\frac{\partial^2}{\partial\tau^2}\boldsymbol{A} \\
&-\frac{1}{2\mathrm{i}k_0}\sum_{n=3}^{\infty}\left\{R_n\frac{\partial^n}{\partial\tau^n}\boldsymbol{A}\right\} - \frac{1}{2\mathrm{i}k_0}\left(\omega_0 + \mathrm{i}\frac{\partial}{\partial\tau}\right)\sum_{n=0}^{\infty}\left\{S_n\frac{\partial^n}{\partial\tau^n}\boldsymbol{P}_{\rm nl}\right\}.
\end{aligned} \tag{9}$$

With respect to ultrashort pulse whose duration is longer than a few optical oscillating cycles, the SEWA[11] is satisfied, thus

$$\left|\frac{\partial}{\partial\xi}\boldsymbol{A}(\boldsymbol{r}_\perp,\xi,\tau)\right| \ll |k_0\boldsymbol{A}(\boldsymbol{r}_\perp,\xi,\tau)|, \tag{10}$$

$$\left|\frac{\partial}{\partial\tau}\boldsymbol{A}(\boldsymbol{r}_\perp,\xi,\tau)\right| \ll \omega_0|\boldsymbol{A}(\boldsymbol{r}_\perp,\xi,\tau)|. \tag{11}$$

Then Eq.(9) becomes

$$\begin{aligned}\frac{\partial}{\partial\xi}\boldsymbol{A} = & -\frac{1}{2\mathrm{i}k_0}\left(1+\frac{\mathrm{i}}{\omega_0}\frac{\partial}{\partial\tau}\right)^{-1}\nabla_\perp^2\boldsymbol{A}+\left(1+\frac{\mathrm{i}}{\omega_0}\frac{\partial}{\partial\tau}\right)^{-1}\\ & \times\left[\frac{1}{2\mathrm{i}k_0}(k_0^2-R_0)\boldsymbol{A}+\left(\frac{1}{V_\mathrm{g}}-\frac{1}{2\mathrm{i}k_0}R_1\right)\frac{\partial}{\partial\tau}\boldsymbol{A}-\frac{1}{2\mathrm{i}k_0}\left(\frac{1}{V_\mathrm{g}^2}+R_2\right)\frac{\partial^2}{\partial\tau^2}\boldsymbol{A}-\frac{1}{2\mathrm{i}k_0}\sum_{n=3}^{\infty}\left(R_n\frac{\partial^n}{\partial\tau^n}\boldsymbol{A}\right)\right]\\ & -\frac{1}{2\mathrm{i}k_0}\omega_0\sum_{n=0}^{\infty}\left\{S_n\frac{\partial^n}{\partial\tau^n}\boldsymbol{P}_\mathrm{nl}\right\}.\end{aligned} \tag{12}$$

Equation (12) is a generalized (3+1)-dimensional propagation equation of pulse in nonlinear dispersive medium, and can be applied to describe the propagation of ultrashort pulse in dispersive LHM. The left hand side of Eq.(12) denotes the field propagation, while on the right hand side, the first term corresponds to conventional diffraction, the second, the third and the fourth terms correspond to dispersion terms of different order of the electric field, and the last term is the sum of all the dispersion terms of nonlinear polarization. R_n and S_n denote the nth-order dispersion coefficients of electric field and nonlinear polarization respectively. For simplicity, we make use of scalar approximation in the following part, i.e. substitute A for $\boldsymbol{A}$ and P_nl for $\boldsymbol{P}_\mathrm{nl}$ in Eq.(12).

Comparing Eq.(12) with the propagation equation of ultrashort pulse in conventional medium[11], we find easily some significant differences between them resulted from the dispersive ε and μ in Eq.(12). First, the diffraction term is essentially different from that of conventional medium because the wave vector can be positive or negative. Second, each dispersion term of the electric field is affected by the dispersive dielectric susceptibility, the dispersive magnetic permeability and the space-time focusing effect. In the conventional medium, however, the magnetic permeability equals to the constant μ_0 (the magnetic permeability in vacuum) and the electric susceptibility is not affected by the space-time focusing effect. Third, the nonlinear part takes on a high-order dispersive character due to the dispersive permeability. Finally, the nonlinear part is not affected by the space–time focusing effect.

3. Application to right-handed and left-handed materials

3.1. Application to RHM

In deriving Eq.(12), we do not prescribe the sign of ε and μ, so this equation is also suitable for the ultrashort pulse propagation in RHM. We discuss this regime in this section by using the expressions for ε and μ in conventional RHMs. According to Brabec's (3+1)-dimensional nonlinear equation in RHM,[11] the field dispersion part is

$$\beta_0^2A+\mathrm{i}2\beta_0\beta_1\partial_tA+\mathrm{i}^2(\beta_1^2+\beta_0\beta_2)\partial_t^2A+\cdots, \tag{13a}$$

and the nonlinear polarization part is

$$\mu_0\omega_0^2\left(1+\frac{\mathrm{i}}{\omega_0}\frac{\partial}{\partial\tau}\right)^2B, \tag{13b}$$

where B denotes the amplitude of nonlinear envelope. In conventional medium, $k=\sqrt{\varepsilon}\omega/c$, so $\mu(\omega)=\mu_0$ and $\varepsilon=k^2c^2/\omega^2$.

According to Eq.(6), using a simple algebraic manipulation, we find that the field dispersion part is

$$k_0^2A+\mathrm{i}2k_0k'\frac{\partial}{\partial t}A+\mathrm{i}^2(k'^2+k_0k'')\frac{\partial^2}{\partial t^2}A+\cdots, \tag{14a}$$

and the nonlinear polarization part is

$$\mu_0\omega_0^2\left(1+\frac{\mathrm{i}}{\omega_0}\frac{\partial}{\partial\tau}\right)^2 P_{\mathrm{nl}}. \tag{14b}$$

In Eqs.(13) and (14), β_0 corresponds to k_0, β_1 corresponds to k', β_2 corresponds to k'' and B corresponds to P_{nl}. This shows that it is inter-convertible between Eq.(13) and Eq.(14). So Eq.(12) is suitable for the conventional positive-refractive-index media.

3.2. Application to LHM

All the coefficients in Eq.(12) depend on ε, μ, and the nonlinear property of the medium. For the dispersive property of the material, we use the typical Drude dispersive model, in which ε and μ are expressed as

$$\varepsilon(\omega)=\varepsilon(\bar{\omega})=\varepsilon_0\left(1-\frac{1}{\bar{\omega}(\bar{\omega}+\mathrm{i}\bar{\gamma})}\right),$$
$$\mu(\omega)=\mu(\bar{\omega})=\mu_0\left(1-\frac{\omega_{\mathrm{pm}}^2/\omega_{\mathrm{pe}}^2}{\bar{\omega}(\bar{\omega}+\mathrm{i}\bar{\gamma})}\right), \tag{15}$$

where $\bar{\omega}=\omega/\omega_{\mathrm{pe}}$, $\bar{\gamma}=\gamma/\omega_{\mathrm{pe}}\ll 1$, $\gamma=\gamma_{\mathrm{e}}=\gamma_{\mathrm{m}}$, ω_{pe} and ω_{pm} denote the plasma resonant circular frequencies of electric field and magnetic field respectively, and their magnitudes are near the frequency of electromagnetic wave, γ_{e} and γ_{m} denote the attenuation of electric field and magnetic field respectively and are relatively small, ε_0 and μ_0 are the absolute dielectric susceptibility and magnetic permeability in vacuum respectively. According to Eq.(15), $\varepsilon(\omega)<0$ when $\bar{\omega}<1$, and $\mu(\omega)<0$ when $\bar{\omega}<\omega_{\mathrm{pm}}/\omega_{\mathrm{pe}}$. So $n=-\sqrt{\varepsilon_{\mathrm{r}}\mu_{\mathrm{r}}}$ when $\bar{\omega}<\omega_{\mathrm{pm}}/\omega_{\mathrm{pe}}$. To verify our model and to make it convenient to compare, we choose $\bar{\gamma}=3.75\times10^{-4}$ and $\omega_{\mathrm{pm}}^2/\omega_{\mathrm{pe}}^2=0.64$. Thus, the real part of refractive index is negative and the imaginary part of refractive index is about zero for $\omega_0/\omega_{\mathrm{pe}}<0.8$, the real part of refractive index is zero for $0.8<\omega_0/\omega_{\mathrm{pe}}<1$, and the refractive index is positive for $\omega_0/\omega_{\mathrm{pe}}>1$. With this Drude model, Eqs.(7) and (8) can be rewritten as

$$R_n=\frac{\mathrm{i}^n}{\omega_{\mathrm{pe}}^{n-2}}\sum_{m=0}^{n}\frac{1}{(n-m)!m!}\frac{\partial^m[\bar{\omega}\varepsilon(\bar{\omega})]}{\partial\bar{\omega}^m}\Big|_{\omega=\omega_0}\frac{\partial^{n-m}[\bar{\omega}\mu(\bar{\omega})]}{\partial\bar{\omega}^{n-m}}\Big|_{\omega=\omega_0}, \tag{16}$$

$$S_n=\frac{\mathrm{i}^n}{n!}\frac{1}{\omega_{\mathrm{pe}}^{n-1}}\frac{\partial^n[\bar{\omega}\mu(\bar{\omega})]}{\partial\bar{\omega}^n}\Big|_{\omega=\omega_0}. \tag{17}$$

Defining the dimensionless dispersion $\tilde{R}_n=R_n\tau_0^{n-2}$ and nonlinear dispersion coefficients $\tilde{S}_n=S_n\tau_0^{n-1}$, we can prove that $|\tilde{R}_{n+1}/\tilde{R}_n|\sim 1/(\omega_{\mathrm{pe}}\tau_0)\ll 1$, and $|\tilde{S}_{n+1}/\tilde{S}_n|\sim 1/(\omega_{\mathrm{pe}}\tau_0)\ll 1$ in the negative refractive-index waveband for $n>1$ when the pulse width is broader than one cycle. We illustrate the variation of $|\tilde{R}_n/\tilde{R}_{n-1}|$ (n = 1,2) with $\bar{\omega}$ in Fig.1. It is obvious that $|\tilde{R}_1/\tilde{R}_0|$ is larger than $|\tilde{R}_2/\tilde{R}_1|$. At $\omega_0=\omega_{\mathrm{pm}}$, there are some special phenomenon according to later analysis. So it is enough to consider only several lower order terms for the pulse propagation in the negative-index bandwidth. In the following, we assume that the medium have Nth-order dispersion and ignore R_n and S_n with $n>N$.

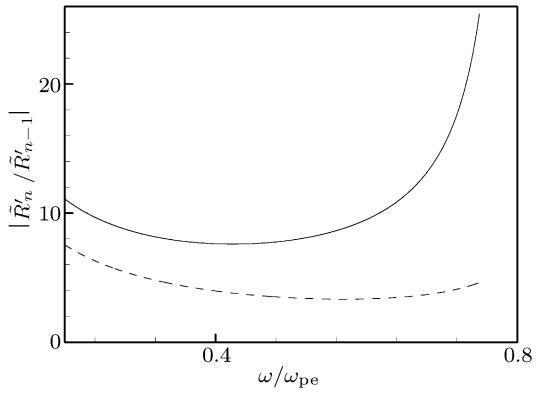


Fig.1. Variation of $|\tilde{R}_n/\tilde{R}_{n-1}|$ with $\omega/\omega_{\mathrm{pe}}$ for $n=1$ (solid curve) and $n=2$ (dashed curve).

We assume that the medium has a Kerr nonlinearity and an instantaneous response to the field, then $P_{\rm nl} = \chi^{(3)}|A|^2A$, where $\chi^{(3)}$ is the third order electric polarization. Substituting $P_{\rm nl}$ into Eq.(12) and using $k_0^2 = n^2\omega_0^2/c^2$ we have

$$\begin{aligned}\frac{\partial}{\partial\xi}A = & \ \mathrm{i}\frac{1}{2k_0}\left(1+\frac{\mathrm{i}}{\omega_0}\frac{\partial}{\partial\tau}\right)^{-1}\nabla_\perp^2 A + \left(\frac{1}{V_{\rm g}} - \frac{1}{2k_0}\omega_0[\varepsilon(\omega_0)\gamma + \mu(\omega_0)\alpha]\right)\left(1+\frac{\mathrm{i}}{\omega_0}\frac{\partial}{\partial\tau}\right)^{-1}\frac{\partial}{\partial\tau}A \\ & -\frac{1}{2\mathrm{i}k_0}\left(\frac{1}{V_{\rm g}^2} - \left[\alpha\gamma + \frac{1}{2}\omega_0\mu(\omega_0)\alpha' + \frac{1}{2}\omega_0\varepsilon(\omega_0)\gamma'\right]\right)\left(1+\frac{\mathrm{i}}{\omega_0}\frac{\partial}{\partial\tau}\right)^{-1}\frac{\partial^2}{\partial\tau^2}A \\ & -\frac{1}{2\mathrm{i}k_0}\left(1+\frac{\mathrm{i}}{\omega_0}\frac{\partial}{\partial\tau}\right)^{-1}\sum_{n=3}^{N}R_n\frac{\partial^n}{\partial\tau^n}A + \mathrm{i}\frac{1}{2k_0}\omega_0^2\mu(\omega_0)\chi^{(3)}(|A|^2A) \\ & -\frac{1}{2k_0}\omega_0\gamma\chi^{(3)}\frac{\partial}{\partial\tau}(|A|^2A) - \frac{1}{2\mathrm{i}k_0}\omega_0\chi^{(3)}\sum_{n=2}^{N}\left\{S_n\frac{\partial^n}{\partial\tau^n}(|A|^2A)\right\}, \end{aligned} \tag{18}$$

where $\alpha = \left.\dfrac{\partial[\omega\varepsilon(\omega)]}{\partial\omega}\right|_{\omega=\omega_0}$, $\alpha' = \left.\dfrac{\partial^2[\omega\varepsilon(\omega)]}{\partial\omega^2}\right|_{\omega=\omega_0}$, $\gamma = \left.\dfrac{\partial[\omega\mu(\omega)]}{\partial\omega}\right|_{\omega=\omega_0}$, and $\gamma' = \left.\dfrac{\partial^2[\omega\mu(\omega)]}{\partial\omega^2}\right|_{\omega=\omega_0}$. $C_{\rm nl} = \dfrac{1}{2k_0}\omega_0^2\mu(\omega_0)\chi^{(3)}$ and $C_{\rm s} = \dfrac{1}{2k_0}\omega_0\gamma\chi^{(3)}$ are the nonlinear coefficient and self-steeping coefficient respectively, corresponding to those in conventional RHMs. Higher order terms with $n > N$ are ignored. According to the definition of group velocity, $V_{\rm g} = \left.\left(\dfrac{\partial k(\omega)}{\partial\omega}\right)^{-1}\right|_{\omega=\omega_0}$, and the definition of group velocity dispersion coefficient, $k'' = \left.\left(\dfrac{\partial^2 k(\omega)}{\partial\omega^2}\right)\right|_{\omega=\omega_0} = \left.\left(\dfrac{\partial}{\partial\omega}\dfrac{1}{V_{\rm g}}\right)\right|_{\omega=\omega_0}$, the expressions for group velocity and group velocity dispersion coefficient in LHM are $V_{\rm g} = \dfrac{2k_0}{\omega_0[\varepsilon(\omega_0)\gamma + \mu(\omega_0)\alpha]}$ and $k'' = \dfrac{1}{k_0}\left[\alpha\gamma + \dfrac{1}{2}\omega_0[\mu(\omega_0)\alpha' + \varepsilon(\omega_0)\gamma'] - \dfrac{1}{V^2}\right]$ respectively. With these definitions, Eq.(18) becomes

$$\begin{aligned}\frac{\partial}{\partial\xi}A = & \ \mathrm{i}\frac{1}{2k_0}\left(1+\frac{\mathrm{i}}{\omega_0}\frac{\partial}{\partial\tau}\right)^{-1}\nabla_\perp^2 A - \mathrm{i}\frac{k''}{2}\left(1+\frac{\mathrm{i}}{\omega_0}\frac{\partial}{\partial\tau}\right)^{-1}\frac{\partial^2}{\partial\tau^2}A - \frac{1}{2\mathrm{i}k_0}\left(1+\frac{\mathrm{i}}{\omega_0}\frac{\partial}{\partial\tau}\right)^{-1}\sum_{n=3}^{N}R_n\frac{\partial^n}{\partial\tau^n}A \\ & +\mathrm{i}\frac{1}{2k_0}\omega_0^2\mu(\omega_0)\left(\chi^{(3)}|A|^2A\right) - \frac{1}{2k_0}\omega_0\gamma\frac{\partial}{\partial\tau}\left(\chi^{(3)}|A|^2A\right) - \frac{1}{2\mathrm{i}k_0}\omega_0\sum_{n=2}^{N}\left\{S_n\frac{\partial^n}{\partial\tau^n}\left(\chi^{(3)}|A|^2A\right)\right\}. \end{aligned} \tag{19}$$

The physical meanings of each term in Eq.(19) are similar to those in Eq.(12). Note that the fourth term and the fifth term on the right hand side denotes nonlinearity effect and self-steeping effect corresponding to conventional RHMs respectively. Eq.(19) revealed some new features: (i) the diffraction effect and the self-steeping effect are different from that in conventional materials for the wave vector k_0 is negative; (ii) higher order dispersion of polarization will bring much more abundant nonlinear effects. If we only keep the leading term, the nonlinearity part of Eq.(19) will be identical to that of other simplified models.[19]

4. Variations of nonlinear and dispersion coefficients with centre frequency

Since the electromagnetic properties of LHM relate closely to the ultrashort pulse's centre frequency according to the above analysis, it is helpful to discuss the relation between the coefficients in Eq.(19) and the centre frequency according to the parameters given above. To analyse the response of LHM to frequency, it is useful to ignore the relatively small space–time fo-

cusing effect. We demonstrate the influence of pulse's centre frequency on group velocity, group velocity dispersion, nonlinear coefficient and self-steeping coefficient. The influence of centre frequency on V_g and β_2 is shown in Fig.2. Figure 2(a) illustrates that V_g has a maximum at $\omega_0/\omega_p \approx \sqrt{2}/2$ where V_g is about one third of the light speed in vacuum, and V_g is about zero in the range of $0.8 < \omega_0/\omega_p < 0.84$. Around $\omega_0 = \omega_{pm}$, the group velocity can even be negative. For the group velocity dispersion (GVD), Fig.2(b) shows that it is negative when $0 < \omega_0/\omega_p < \sqrt{2}/2$ and positive when $\sqrt{2}/2 < \omega_0/\omega_p < 0.8$, and its amplitude oscillates drastically as $\omega_0/\omega_p \to 0.8$. In short, the GVD may be positive or negative in the negative refractive index frequency range, and the evolution of pulse may be complex if the spectrum is broad enough (especially for ultrashort pulses which have a broad spectrum).

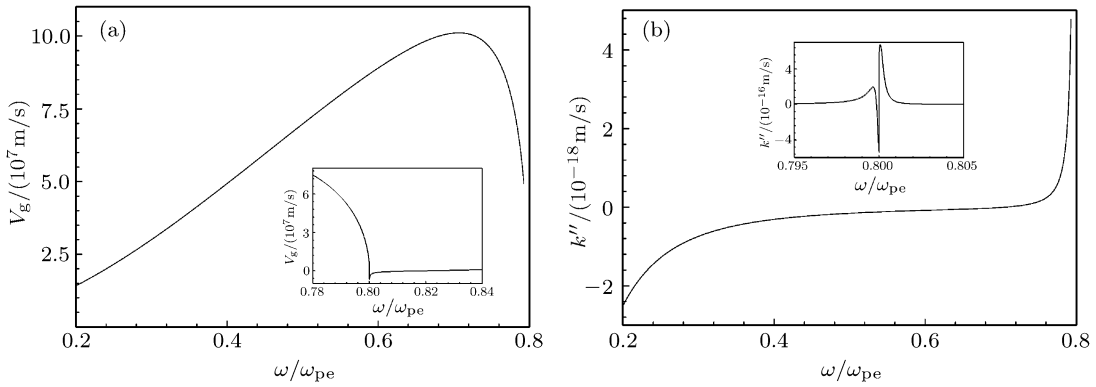


Fig.2. (a) Variation of group velocity with centre frequency. Inset figure shows the variation of group velocity for $\omega_0/\omega_p > 0.8$. (b) Variation of GVD with centre frequency. Inset figure shows the variation of GVD with centre frequency for $\omega_0/\omega_p > 0.8$. There is a singularity at $\omega_0/\omega_p = 0.8$.

The variation of C_{nl} and C_s with centre frequency is shown in Fig.3. The nonlinearity has a maximum around $\omega_0/\omega_p = 0.64$ and is almost zero for $\omega_0/\omega_p > 0.8$. Using a pulse whit a spectrum around $\omega_0/\omega_p \approx 0.64$ will be more advantageous to study the nonlinear effects in LHMs. As for self-steeping effect, it is negative in the whole range $\omega_0/\omega_p < 0.8$ and it oscillates drastically at $\omega_0/\omega_p = 0.8$. In addition, it's much smaller than nonlinearity.

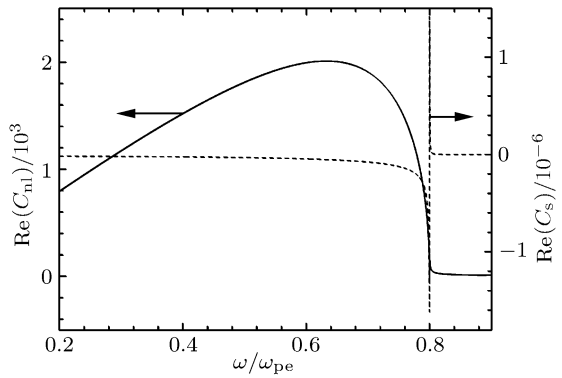


Fig.3. Variation of nonlinear coefficient (solid curve, left vertical axis) and self-steeping coefficient (dash curve, right vertical axis) with centre frequency.

5. Nonlinear propagation of ultrashort pulse in LHM

In this section, we discuss the pulse propagation in nonlinear LHM with effects of GVD, self-phase modulation and self-steeping. Set $\omega_0/\omega_{pe} = 0.55$, and neglect the higher order terms for $n > 2$ in Eq.(19). It is clear from the above analysis that the refractive index, the magnetic permeability and the GVD are negative and γ is positive at $\omega_0/\omega_{pe} = 0.55$, so the dispersion is anomalous. For simplicity, we normalize Eq.(19) by using the transforms: $t = \tau/\tau_0$, $L_D = \tau_0^2/|\beta_2|$, $z = \dfrac{\xi}{L_D}$, $L_{nl} = \dfrac{2|k_0|}{\omega_0^2|\mu(\omega_0)|\chi^{(3)}P_0}$, $N^2 = \dfrac{L_d}{L_{nl}}$, and $s = \dfrac{\gamma}{\tau_0\omega_0\mu(\omega_0)}$. The simulation result of propagation of a hyperbolic secant pulse in RHM and LHM for $N^2 = 1$ is shown in Fig.4. We find that the envelope in LHM does not change evidently. This is the same as the soliton in conventional materials. It suggests that a soliton can also be formed in LHMs. Another important phenomenon is that the pulse in LHM moves oppositely in the moving coordinate system, which is aroused by the negative self-steepening effect.

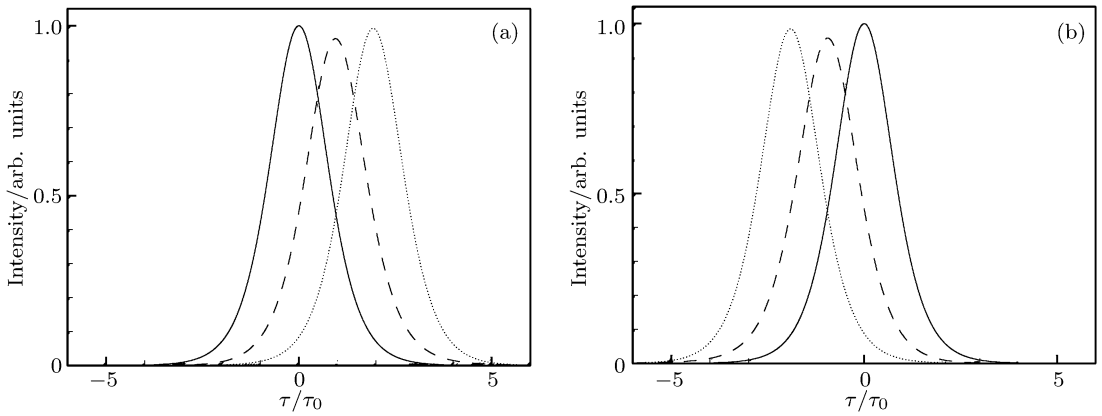


Fig.4. Pulse shape at $z = 0$ (solid curve), $z = 6$ (dashed curve) and $z = 10$ (dotted curve) in (a) RHM for $s = 0.2$, and (b) LHM for $s = -0.2$.

6. Conclusion

In conclusion, a (3+1)-dimensional nonlinear propagation equation is derived, which models the propagation of ultrashort pulse in nonlinear electromagnetic dispersive medium characterized by the dispersive permittivity and permeability. Based on the Drude dispersive model, a generalized (3+1)-dimensional nonlinear Schrödinger equation for ultrashort pulse propagation in LHM is obtained. By theoretical and numerical analysis, it is found that the dispersive magnetic permeability makes the nonlinear polarization dispersive, the GVD can be normal or abnormal in the negative refractive index regime and it plays the same role as it does in conventional medium. Finally, numerical results show that the self-steeping effect makes the pulse propagating in LHM evolve in the opposite direction to that in RHM.

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